IPhO 2018
Lisbon, Portugal

Solutions to Theory Problem 1

LIGO-GW150914

(V. Cardoso, C. Herdeiro)

July 15, 2018

v6.0

GW150914 (10 points)

Part A. Newtonian (conservative) orbits (3.0 points)

A.1 Apply Newton's law to mass M_1:

$$M_1 \frac{\mathrm{d}^2 \vec{r}_1}{\mathrm{d}t^2} = G \frac{M_1 M_2}{|\vec{r}_2 - \vec{r}_1|^2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} . \tag{1}$$

Use, from eq. (1) of the question sheet

$$\vec{r}_2 = -\frac{M_1}{M_2} \vec{r}_1 , \tag{2}$$

in eq. (1) above, to obtain

$$\frac{\mathrm{d}^2 \vec{r}_1}{\mathrm{d}t^2} = -\frac{G M_2^3}{(M_1 + M_2)^2 r_1^2} \frac{\vec{r}_1}{r_1} . \tag{3}$$

A.1	1.0pt
$n = 3, \qquad \alpha = \frac{G M_2^3}{(M_1 + M_2)^2} .$	

A.2 The total energy of the system is the sum of the two kinetic energies plus the gravitational potential energy. For circular motions, the linear velocity of each of the masses reads

$$|\vec{v}_1| = r_1 \Omega , \qquad |\vec{v}_2| = r_2 \Omega , \tag{4}$$

Thus, the total energy is

$$E = \frac{1}{2}(M_1 r_1^2 + M_2 r_2^2)\Omega^2 - \frac{G M_1 M_2}{L} , \tag{5}$$

Now,

$$(M_1 r_1 - M_2 r_2)^2 = 0 \qquad \Rightarrow \qquad M_1 r_1^2 + M_2 r_2^2 = \mu L^2 . \tag{6}$$

Thus,

$$E = \frac{1}{2}\mu L^2 \Omega^2 - G\frac{M\mu}{L} . \tag{7}$$

A.2	1.0pt
$A(\mu, \Omega, L) = \frac{1}{2}\mu L^2 \Omega^2 .$	

A.3 Energy (3) of the question sheet can be interpreted as describing a system of a mass μ in a circular orbit with angular velocity Ω, radius L, around a mass M (at rest). Equating the gravitational acceleration to the centripetal acceleration:

$$G\frac{M}{L^2} = \Omega^2 L . \tag{8}$$

This is indeed Kepler's third law (for circular orbits). Then, from (7),

$$E = -\frac{1}{2} G \frac{M\mu}{L} . \tag{9}$$

A.3	1.0pt
$\beta = -\frac{1}{2} .$	

Part B - Introducing relativistic dissipation (7.0 points)

B.1 Some simple trigonometry for the x, y motion of the masses (in an appropriate Cartesian system) yields:

$$(x_1, y_1) = r_1(\cos(\Omega t), \sin(\Omega t)) , \qquad (x_2, y_2) = -r_2(\cos(\Omega t), \sin(\Omega t)) . \tag{10}$$

Then,

$$Q_{ij} = \frac{M_1 r_1^2 + M_2 r_2^2}{2} \begin{pmatrix} \frac{4}{3}\cos^2(\Omega t) - \frac{2}{3}\sin^2(\Omega t) & 2\sin(\Omega t)\cos(\Omega t) & 0 \\ 2\sin(\Omega t)\cos(\Omega t) & \frac{4}{3}\sin^2(\Omega t) - \frac{2}{3}\cos^2(\Omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} , \tag{11}$$

or, using some simple trigonometry and (6),

$$Q_{ij} = \frac{\mu L^2}{2} \begin{pmatrix} \frac{1}{3} + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & \frac{1}{3} - \cos 2\Omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} . \tag{12}$$

B.1 1.0pt

$$k = 2\Omega , \quad a_1 = a_2 = \frac{1}{3} , a_3 = -\frac{2}{3} , \quad b_1 = 1, b_2 = -1, b_3 = 0 , c_{12} = c_{21} = 1, c_{ij} \overset{\text{otherwise}}{=} 0 .$$

B.2 First take the derivatives:

$$\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} = 4\Omega^3 \mu L^2 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{13}$$

Then perform the sum:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}(4\Omega^3 \mu L^2)^2 [2\sin^2(2\Omega t) + 2\cos^2(2\Omega t)] = \frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{14}$$

B.2 1.0pt

$$\xi = \frac{32}{5} .$$

B.3 Now we assume a sequency of Keplerian orbits, with decreasing energy, which is being taken from the system by the GWs.

First, from (9), differentiating with respect to time,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} , \tag{15}$$

Since this loss of energy is due to GWs, we equate it with (minus) the luminosity of GWs, given by (14)

$$\frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{16}$$

We can eliminate the L and $\mathrm{d}L/\mathrm{d}t$ dependence in this equation in terms of Ω and $\mathrm{d}\Omega/\mathrm{d}t$, by using Kepler's third law (8), which relates:

$$L^3 = G\frac{M}{\Omega^2} , \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{2}{3}\frac{L}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}t} . \tag{17}$$

Substituting in (16), we obtain:

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} G^5 \mu^3 M^2 \equiv \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} (GM_\mathrm{c})^5 \ . \tag{18}$$

B.3	1.0pt
$M_\mathrm{c} = (\mu^3 M^2)^{1/5}$.	

B.4 Angular and cycle frequencies are related as $\Omega = 2\pi f$. From the information provided above: *GWs have a frequency which is twice as large as the orbital frequency*, we have

$$\frac{\Omega}{2\pi} = \frac{f_\mathrm{GW}}{2} \ . \tag{19}$$

Formula (10) of the question sheet has the form

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \chi \Omega^{11/3} \ , \qquad \chi \equiv \frac{96}{5}\frac{(GM_\mathrm{c})^{5/3}}{c^5} \ . \tag{20}$$

Thus, from (11) of the question sheet

$$\Omega(t)^{-8/3} = \frac{8}{3}\chi(t_0 - t) \ , \tag{21}$$

or, using (20) and the definition of χ gives

$$f_\mathrm{GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{GM_\mathrm{c}}{c^3}\right)^{5/3}(t_0 - t) \ . \tag{22}$$

B.4	2.0pt
$p = 1$.	

B.5 From the figure, we consider the two Δt's as half periods. Thus, the (cycle) GW frequency is $f_\mathrm{GW} = 1/(2\Delta t)$. Then, the four given points allow us to compute the frequency at the mean time of the two intervals as

	t_AB	t_CD
t (s)	0.0045	0.037
f_GW (Hz)	$(2 \times 0.009)^{-1}$	$(2 \times 0.006)^{-1}$

Now, using (22) we have two pairs of (f_GW,t) values for two unknowns (t_0,M_c). Expressing (22) for both t_AB and t_CD and dividing the two equations we obtain:

$$t_0 = \frac{At_\mathrm{CD} - t_\mathrm{AB}}{A - 1} \ , \qquad A \equiv \left(\frac{f_\mathrm{GW}(t_\mathrm{AB})}{f_\mathrm{GW}(t_\mathrm{CD})}\right)^{-8/3} \ . \tag{23}$$

Replacing by the numerical values, $A \simeq 2.95$ and $t_0 \simeq 0.054$ s. Now we can use (22) for either of the two values t_AB or t_CD and determine M_c. One obtains for the chirp mass

$$M_\mathrm{c} \simeq 6 \times 10^{31}\ \mathrm{kg} \simeq 30 \times M_\odot \ . \tag{24}$$

Thus, the total mass M is

$$M = 4^{3/5} M_\mathrm{c} \simeq 69 \times M_\odot \ . \tag{25}$$

This result is actually remarkably close to the best estimates using the full theory of General Relativity! [Even though the actual objects do not have precisely equal masses and the theory we have just used is not valid very close to the collision.]

B.5 1.0pt

$$M_{\text{c}} \simeq 30 \times M_{\odot} \,, \qquad M \simeq 69 \times M_{\odot} \,.$$

B.6 From (8), Kepler's law states that $L = (GM/\Omega^2)^{1/3}$. The second pair of points highlighted in the plot correspond to the cycle prior to merger. Thus, we use (19) to obtain the orbital angular velocity at $t_{\overline{\text{CD}}}$:

$$\Omega_{t_{\overline{\text{CD}}}} \sim 2.6 \times 10^2 \text{ rad/s} \,. \tag{26}$$

Then, using the total mass (25) we find

$$L \sim 5 \times 10^2 \text{ km} \,. \tag{27}$$

Thus, these objects have a maximum radius of $R_{\text{max}} \sim 250$ km. Hence they have over 30 times more mass and,

$$\frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3 \tag{28}$$

they are 3000 times smaller than the Sun and!

Their linear velocity is

$$v_{\text{col}} = \frac{L}{2}\Omega \simeq 7 \times 10^4 \text{ km/s} \,. \tag{29}$$

They are moving at over 20% of the velocity of light!

B.6 1.0pt

$$L_{\text{collision}} \sim 5 \times 10^2 \text{ km} \,, \qquad \frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3 \,, \qquad \frac{v_{\text{col}}}{c} \sim 0.2 \,.$$

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Solutions to Theory Problem 2

Where is the neutrino?

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v1.2

Where is the neutrino? (10 points)

Part A. ATLAS Detector physics (4.0 points)

A.1

The magnetic force is the centripetal force:

$$m\frac{v^2}{r} = evB \Rightarrow r = \frac{mv}{eB}\,.$$

First express the velocity in terms of the kinetic energy,

$$K = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}}\,,$$

and then insert it in the expression above for the radius to get

A.1	0.5pt
$$r = \frac{\sqrt{2Km}}{eB}\,.$$	

A.2

The radius of the circular motion of a charged particle in the presence of a uniform magnetic field is given by,

$$r = \frac{mv}{eB}\,.$$

This formula is valid in the relativistic scenario if the mass correction, $m \to \gamma m$ is included:

$$r = \frac{\gamma mv}{eB} = \frac{p}{eB} \Rightarrow p = reB\,.$$

Note that the radius of the circular motion is half the radius of the inner part of the detector. One obtains [1 MeV/$c = 5.34 \times 10^{-22}$ m kg s^{-1}]

A.2	0.5pt
$$p = 330 \text{ MeV}/c\,.$$	

A.3

The acceleration for the particle is $a = \frac{evB}{\gamma m} \sim \frac{ecB}{\gamma m}$, in the ultrarelativistic limit. Then,

$$P = \frac{e^4c^2\gamma^4B^2}{6\pi\epsilon_0c^3\gamma^2m^2} = \frac{e^4\gamma^2c^4B^2}{6\pi\epsilon_0c^5m^2}.$$

Since $E = \gamma mc^2$ we can obtain $\gamma^2c^4 = \frac{E^2}{m^2}$ and, finally,

$$P = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2 .$$

Therefore,

A.3 1.0pt

$$\xi = \frac{1}{6\pi}\,, \quad n = 5 \text{ and } k = 4.$$

A.4

The power emitted by the particle is given by,

$$P = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2 \,.$$

The energy of the particle as a function of time can be calculated from

$$\int_{E_0}^{E(t)} \frac{1}{E^2} \mathrm{d}E = -\int_0^t \frac{e^4}{6\pi\epsilon_0 m^4 c^5} B^2 \mathrm{d}t \,,$$

where $E(0) = E_0$. This leads to,

$$\frac{1}{E(t)} - \frac{1}{E_0} = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}\, t \quad \Rightarrow \quad E(t) = \frac{E_0}{1 + \alpha E_0 t}\,,$$

with

A.4 1.0pt

$$\alpha = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}\,.$$

A.5

If the initial energy of the electron is 100 GeV, the radius of curvature is extremely large ($r = \frac{E}{eBc} \approx 167$ m). Therefore, in approximation, one can consider the electron is moving in the inner part of the ATLAS detector along a straight line. The time of flight of the electron is $t = R/c$, where $R = 1.1$ m is the radius of the inner part of the detector. The total energy lost due to synchrotron radiation is,

$$\Delta E = E(R/c) - E_0 = \frac{E_0}{1 + \alpha E_0 \frac{R}{c}} - E_0 \approx -\alpha E_0^2 \frac{R}{c}$$

and

A.5 0.5pt

$$\Delta E = -56 \text{ MeV}\,.$$

A.6

In the ultrarelativistic limit, $v \approx c$ and $E \approx pc$. The cyclotron frequency is,

$$\omega(t) = \frac{c}{r(t)} = \frac{ecB}{p(t)} = \frac{ec^2B}{E(t)}$$

A.6 0.5pt

$$\omega(t) = \frac{ec^2B}{E_0}\left(1 + \frac{e^4B^2}{6\pi\epsilon_0 m^4c^5}E_0 t\right) .$$

Part B. Finding the neutrino (6.0 points)

B.1

Since the W^+ boson decays into an anti-muon and a neutrino, one can use principles of conservation of energy and linear momentum to calculate the unknown $p_z^{(\nu)}$ of the neutrino. Moreover, the anti-muon and the neutrino can be considered massless, which implies that the magnitude of their momenta (times c) and their energies are the same. Therefore, the conservation of linear momentum can be expressed as

$$\vec{p}^{\,(W)} = \vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)},$$

and the conservation of energy as,

$$E^{(W)} = cp^{(\mu)} + cp^{(\nu)}.$$

In addition, one can also relate the energy and the momentum of the W^+ boson through its mass,

$$m_W^2 = (E^{(W)})^2/c^4 - (p^{(W)})^2/c^2$$

which leads to a quadratic equation on $p_z^{(\nu)}$,

$$\begin{aligned} m_W^2 &= \left[(p^{(\mu)} + p^{(\nu)})^2 - (\vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)})^2\right]/c^2 \\ &= \left(2p^{(\mu)}p^{(\nu)} - 2\vec{p}^{\,(\mu)} \cdot \vec{p}^{\,(\nu)}\right)/c^2 \end{aligned}$$

B.1 1.5pt

$$m_W^2 = \frac{1}{c^2}\left(2p^{(\mu)}\sqrt{(p_{\mathrm{T}}^{(\nu)})^2 + (p_z^{(\nu)})^2} - 2\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} - 2p_z^{(\mu)}p_z^{(\nu)}\right).$$

B.2

The numerical substitution directly in the answer of B.1, using

$$p^{(\mu)} = 37.2\,\mathrm{GeV}/c \qquad m_W^2c^2 = 6464.2\,(\mathrm{GeV}/c)^2 \qquad p_{\mathrm{T}}^{(\nu)\,2} = 10\,864.9\,(\mathrm{GeV}/c)^2$$

$$\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} = 2439.3(\mathrm{GeV}/c)^2 \qquad p_z^{(\mu)} = -12.4\,\mathrm{GeV}/c,$$

leads to

$$6464.2 = 74.4\sqrt{10\,864.9 + p_z^{(\nu)\,2}} - 4878.6 + 24.8p^{(\nu)}.$$

This is a quadratic equation, equivalent to

$$0.88889\,p_z^{(\nu)\,2} + 101.64\,p_z^{(\nu)} - 12378 = 0$$

whose solutions are:

B.2	1.5pt
$p_z^{(\nu)} = 74.0$ GeV/c or $p_z^{(\nu)} = -188.3$ GeV/c.	

The general solution of the equation above in B.1 leads to

$$p_z^{(\nu)} = \frac{2\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} p_z^{(\mu)} + m_W^2 c^2 p_z^{(\mu)}}{2(p_{\mathrm{T}}^{(\mu)})^2} \pm \frac{p^{(\mu)} \sqrt{-4(p_{\mathrm{T}}^{(\mu)})^2 (p_{\mathrm{T}}^{(\nu)})^2 + 4(\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)})^2 + 4\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} m_W^2 c^2 + m_W^4 c^4}}{2(p_{\mathrm{T}}^{(\mu)})^2}$$

Numerical substitution leads to the above mentioned values for $p_z^{(\nu)}$.

B.3

The final state particles of the top quark decay are the anti-muon, the neutrino and jet 1. Since the neutrino is now fully reconstructed the energy and linear momentum of the top quark can be calculated as,

$$E^{(\mathrm{t})} = cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)}$$
$$\vec{p}^{\,(\mathrm{t})} = \vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)} + \vec{p}^{\,(j_1)} \,.$$

The top quark mass is,

$$m_{\mathrm{t}} = \sqrt{(E^{(\mathrm{t})})^2/c^4 - (\vec{p}^{\,(\mathrm{t})})^2/c^2} = c^{-1}\sqrt{\left(p^{(\mu)} + p^{(\nu)} + p^{(j_1)}\right)^2 - \left(\vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)} + \vec{p}^{\,(j_1)}\right)^2} \,.$$

The substitution of values leads to two possible masses:

B.3	1.0pt
$m_{\mathrm{t}} = 169.3$ GeV/c^2 or $m_{\mathrm{t}} = 311.2$ GeV/c^2	

B.4

According to the frequency distribution for signal (dashed line), the probability of the $m_{\mathrm{t}} = 169.3$ GeV/c^2 solution is roughly 0.1 while the probability of the $m_{\mathrm{t}} = 311.2$ GeV/c^2 solution is below 0.01. Therefore,

B.4	The most likely candidate is the $m_{\mathrm{t}} = 169.3$ GeV/c^2 solution.	1.0pt

B.5

The top quark energy for the most likely candidate is $E^{(\mathrm{t})} = cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)} = 272.6$ GeV .

$$d = vt = v\gamma t_0 = \frac{p^{(\mathrm{t})}}{m_\mathrm{t}} t_0 = ct_0 \sqrt{\frac{{E^{(\mathrm{t})}}^2}{m_\mathrm{t}^2 c^4} - 1} \,.$$

B.5	1.0pt
$d = 2 \times 10^{-16}$ m .	

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Solutions to Theory Problem 3

Physics of Live Systems

(Rui Travasso, Lucília Brito)

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v1.0

Physics of Live Systems (10 points)

Part A. The physics of blood flow (4.5 points)

A.1

Since the vessel network is symmetrical, the flow in a vessel of level $i+1$ is half the flow in a vessel of level i.

In this way, we can sum the pressure differences in all levels:

$$\Delta P = \sum_{i=0}^{N-1} Q_i R_i = Q_0 \sum_{i=0}^{N-1} \frac{R_i}{2^i}.$$

Introducing the radii dependences yields

$$\Delta P = Q_0 \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = Q_0 \frac{8\ell_0 \eta}{\pi r_0^4} \sum_{i=0}^{N-1} \frac{2^{4i/3}}{2^i 2^{i/3}} = Q_0 N \frac{8\ell_0 \eta}{\pi r_0^4}.$$

Therefore

$$Q_0 = \Delta P \frac{\pi r_0^4}{8N\ell_0 \eta}.$$

Hence, the flow rate for a vessel network in level i is

A.1	1.3pt
$Q_i = \Delta P \frac{\pi r_0^4}{2^{i+3} N \ell_0 \eta}.$	

A.2

Replace values in the formula and change units appropriately

$$Q_0 = \frac{\Delta P \pi r_0^4}{8N\ell_0 \eta} =$$
$$= \frac{(55-30) \times 1.013 \times 10^5 \times 3.1415 \times (6.0 \times 10^{-5})^4}{760 \times 48 \times 2.0 \times 10^{-3} \times 3.5 \times 10^{-3}} = 4.0 \times 10^{-10} \text{ m}^3/\text{s}$$

to obtain the final value in the requested unites:

A.2	0.5pt
$Q_0 \simeq 1.5 \text{ m}\ell/\text{h}.$	

A.3

The current is given by

$$I = \frac{P_{\text{in}} \mathrm{e}^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \,.$$

The pressure difference in the capacitor is

$$P_{\text{out}} \mathrm{e}^{i(\omega t + \phi)} = \frac{P_{\text{in}} \mathrm{e}^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \frac{1}{i\omega C} = \frac{P_{\text{in}} \mathrm{e}^{i\omega t}}{i\omega C R - \omega^2 L C + 1} \,.$$

The amplitude is

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$

To be smaller than P_{in}, for $\omega \to 0$:

$$(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2 > 1 \iff -2CL + C^2 R^2 > 0 \,.$$

Replacing the expressions for L, C, and R we get: $\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1$.

A.3		2.0pt
	$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$	
Condition:	$$\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1 \,.$$	

Alternative way to obtain P_{out}:

The amplitude of the current in the equivalent circuit is $I_0 = \frac{P_{\text{in}}}{Z}$, where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is the modulus of the impedance. Hence, the voltage amplitude in the capacitor is

$$P_{\text{out}} = \frac{1}{\omega C} \times I_0 = \frac{P_{\text{in}}}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}} .$$

A.4

The previous condition can also be expressed as

$$h < \frac{64\eta^2 \ell^2}{3Er^3\rho} \,.$$

For the network referred to in **A.2**

$$h < \frac{64\eta^2 \ell_0^2 \times 2^i}{3 \times 2^{2i/3} E r_0^3 \rho} = \frac{64 \times (3.5 \times 10^{-3})^2 \times (2.0 \times 10^{-3})^2}{3 \times 0.06 \times 10^6 \times (6.0 \times 10^{-5})^3 \times 1.05 \times 10^3} \times 2^{i/3} = 7.7 \times 10^{-5} \times 2^{i/3} \,.$$

For $i = 0$, in the worse case scenario,

$$h_{\text{max}} = 7.7 \times 10^{-5} \times 2^0 = 7.7 \times 10^{-5} \text{ m}$$

This value is certainly observed in these vessels since their radius range from 18 μm to 60 μm. A wall width smaller than 80 μm is certainly reasonable.

A.4	Maximum $h = 8 \times 10^{-5}$ m	0.7pt

Part B. Tumor growth (5.5 points)

B.1

The expressions for the masses of tumour and normal tissue are written as:

$$\begin{cases} M_{\text{T}} = V_{\text{T}}\rho_{\text{T}} = V_{\text{T}}\rho_0(1 + \frac{p}{K_{\text{T}}}) \\ \\ M_{\text{N}} = V\rho_0 = (V - V_{\text{T}})\rho_0(1 + \frac{p}{K_{\text{N}}}) \end{cases}$$

The pressure, p, can be expressed as

$$p = \frac{M_{\text{T}}\, K_{\text{T}}}{V_{\text{T}}\, \rho_0} - K_{\text{T}}$$

and, then, used in the equation for M_{N}:

$$M_{\text{N}} = (V - V_{\text{T}})\, \frac{M_{\text{N}}}{V}\, \left[\left(1 - \frac{K_{\text{T}}}{K_{\text{N}}}\right) + \frac{M_{\text{T}}\, V K_{\text{T}}}{V_{\text{T}}\, M_{\text{N}}\, K_{\text{N}}}\right]$$

Simplifying and rearranging the terms, the equation for v becomes

$$(1-\kappa)\, v^2 - (1+\mu)\, v + \mu = 0\,,$$

for which the solution is (the other solution of the quadratic equation is not physically relevant since does not lead to $v = 0$ for $\mu = 0$)

B.1	$$v = \frac{1 + \mu - \sqrt{(1+\mu)^2 - 4\mu\,(1-\kappa)}}{2(1-\kappa)}\,.$$	1.0pt

B.2

For $r < R_{\text{T}}$, the conservation of energy implies that

$$4\pi r^2 (-k) \frac{\text{d}T}{\text{d}r} = \mathcal{P}\frac{4}{3}\pi r^3\,.$$

Therefore, the temperature difference to 37 °C = 310.15 K, $\Delta T(r)$, is given by

$$\Delta T(r) = -\frac{\mathcal{P} r^2}{6k} + C\,,$$

where C is a constant.

For $r > R_\mathrm{T}$, the conservation of energy implies that

$$4\pi r^2(-k)\frac{\mathrm{d}T}{\mathrm{d}r} = \mathcal{P}\frac{4}{3}\pi R_\mathrm{T}^3\,.$$

Therefore, the temperature difference to 37 °C is

$$\Delta T(r) = \frac{\mathcal{P} R_\mathrm{T}^3}{3kr}\,.$$

In this case there is no constant, since very far away the increase in temperature is zero.

Matching the two solutions at $r = R_\mathrm{T}$ gives

$$C = \frac{\mathcal{P} R_\mathrm{T}^2}{2k}\,.$$

Therefore the temperature at the centre of the tumour, in SI units, is

B.2	Temperature: $310.15 + \frac{\mathcal{P} R_\mathrm{T}^2}{2k}$.	1.7pt

B.3

The increase in temperature at the tumour surface (the lower temperature in the tumour) is

$$\Delta T(R_\mathrm{T}) = \frac{\mathcal{P} R_\mathrm{T}^2}{3k}\,.$$

This increase should be equal to 6.0 K. Therefore,

$$\mathcal{P} = \frac{3\Delta T k}{R_\mathrm{T}^2} = \frac{3 \times 6 \times 0.6}{0.05^2} = 4.3 \text{ kW/m}^3.$$

B.3	$\mathcal{P}_\mathrm{min} = 4.3$ kW/m^3.	0.5pt

B.4

We can relate δr with the pressure in the tumour, using the relation given in the text up to leading order in $p - P_\mathrm{cap}$: $\delta r = \frac{p - P_\mathrm{cap}}{2(p_\mathrm{c} - P_\mathrm{cap})}\,\delta r_\mathrm{c}$. Therefore, if $p - P_\mathrm{cap}$ is very small, also it is δr.

The pressure can be related with the volume. We know that

$$\frac{M_\mathrm{N}}{V_\mathrm{N}} = \frac{\rho_0 V}{V - V_\mathrm{T}} = \frac{\rho_0}{1 - v} = \rho_0\left(1 + \frac{p}{K_\mathrm{N}}\right)\,.$$

And so $p = \frac{K_\mathsf{N} v}{1-v}$.

When the thinner vessels are narrower, the flow rate in the main vessel is altered:

$$\Delta P = (Q_0 + \delta Q_0) \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = (Q_0 + \delta Q_0) \frac{8\ell_0 \eta}{\pi r_0^4} \left(\sum_{i=0}^{N-2} \frac{2^{4i/3}}{2^i 2^{i/3}} + \frac{2^{4(N-1)/3}}{2^{N-1} 2^{(N-1)/3} \left(1 - \frac{\delta r}{r_0/2^{(N-1)/3}}\right)^4} \right)$$

$$\implies \Delta P \simeq (Q_0 + \delta Q_0) \frac{\Delta P}{N Q_0} \left(N - 1 + 1 + \frac{4\,\delta r}{r_{N-1}} \right)$$

Noting that $\frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0}$, we obtain

$$1 + \frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{1}{1 + \frac{4\,\delta r}{N r_{N-1}}} \simeq 1 - \frac{4\,\delta r}{N r_{N-1}} \, .$$

And so:

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{4}{N} \frac{\delta r}{r_{N-1}} \, .$$

Putting all together

B.4 2.3pt

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{2}{N} \frac{K_\mathsf{N} v - (1-v) P_\mathsf{cap}}{(1-v)(p_\mathsf{c} - P_\mathsf{cap})} \frac{\delta r_\mathsf{c}}{r_{N-1}} \, .$$