Dark Matter

The first formal inference for the existence of dark matter was provided by Fritz Zwicky based on his observation on the dynamics of the Coma galaxy cluster, a cluster of galaxies that consist of about a thousand of galaxies. Zwicky used the Virial theorem to estimate the mass of the galaxy cluster. In a simple sun-planet system, where the planet revolves around the sun in a circular orbit, the Virial theorem states that the kinetic energy of the planet is exactly related to its gravitational potential energy. While in a general case for a system of many particles bounded by some interaction, the Virial theorem, relates the time average total kinetic energy with its time average total potential energy.

In 1933, based on his observation on the velocity of the galaxies near the edge of the Coma galaxy cluster, Zwicky estimated that the cluster has more mass than what was visually observed (i.e. the galaxies). The gravitational attraction from the observable matter (the galaxies) was too small to account for the velocities of the galaxies. Thus there must be some hidden masses that account for such a large velocity. That hidden mass is the dark matter mass. In what follows, assume that the mass of each galaxy is the sum of its visible mass and the mass of the dark matter which moves together with that galaxy, and dark matter interacts with visible matter only gravitationally.

A. Cluster of Galaxies

Consider a cluster of galaxies consist of a large number N of galaxies and dark matter that are distributed homogeneously in a sphere of radius R with the total mass (galaxies and dark matters) of the cluster M. Assume that the average total mass (visible and dark matter) of a galaxy is m.

A.1	Assuming a continuous distribution of matter in the cluster, find the total gravitational potential energy of the cluster, in terms of M and R.	1.0 pt.

Due to cosmological expansion, any distance object will be moving away from an observer on Earth with a speed that depends on the distance from the observer to the object. A certain Lyman (a hydrogen emission spectral line) frequency from a type IA supernova on the i-th galaxy in the galaxy cluster is observed to be f_i , with i = 1,...,N, while the same corresponding Lyman frequency on Earth is f_0.

A.2	Determine the average speed V_{cr} of the whole galaxy cluster moving away from the Earth in terms of f_i (with $i = 1, \ldots, N$.), f_0 and N. Note that a galaxy speed is very small compared to the speed of light c.	0.5 pt.

A.3	By assuming that the galaxies velocities with respect to the center of the cluster are isotropic (the same in all direction), determine the root-mean square speed v_{rms} of the galaxies with respect to the center of the cluster in terms of N, f_i (with $i = 1, \ldots, N$.), and f_0. From this result determine the mean kinetic energy of a galaxy with respect to the center of the cluster in terms of v_{rms} and m.	1.5 pt.

To find the total mass of the cluster, one can use Virial theorem. The theorem stated that for a system of particles bounded by their conservative force,

$$\langle K \rangle_t = -\gamma \langle U \rangle_t,$$

where $\langle K \rangle_t$ is the time average total kinetic energy, $\langle U \rangle_t$ is the time average total potential energy, and γ is a constant. This theorem can be derived by assuming that for a system of particles bounded by its own interaction, the magnitudes of the position and momentum of each particle are finite, and thus the following quantity

$$\Gamma = \sum_i \vec{p_i} \cdot \vec{r_i}$$

is finite.

A.4	Using the fact that the time average over a long period of time of $d\Gamma/dt$ vanishes, $\langle \frac{d\Gamma}{dt} \rangle_t = 0$, determine γ in the Virial theorem above for the case of gravitational interaction. (Hint: Try to do the problem with the summation in Γ for a small finite number of galaxies).	1.7 pt.
A.5	From the previous results determine the total dark matter mass of the cluster in terms of N, m_g, R and v_{rms}, where m_g is the average total visible mass of a galaxy. Note that the root-mean square speed of the dark matter is the same as that of the galaxies.	0.5 pt.

B. Dark Matter in a Galaxy

Dark matter also exists inside and around a galaxy. Consider a spherical galaxy with a visible edge radius R_g (an approximate outermost distance where a large number of stars still visible, but note that a very small number of stars may still be distributed in the region beyond R_g). Assume that the stars in the galaxy are point particles with an average mass m_s. The stars in the galaxy, distributed homogeneously with a number density n, are assumed to move in circular orbits.

B.1	If the galaxy consists only of stars, find the speed $v(r)$ of a star as a function of its distance to the center of the galaxy and sketch $v(r)$ for $r < R_g$ and $r \geq R_g$.	0.8 pt.

The existence of dark matter can be inferred from the galaxy rotation curve, which is a plot of $v(r)$ obtained from observations. The figure below shows a common pattern of the galaxy rotation curve. You may assume simplifyingly that $v(r)$ is a linear function for $r \leq R_g$ and a constant v_0 for $r > R_g$.

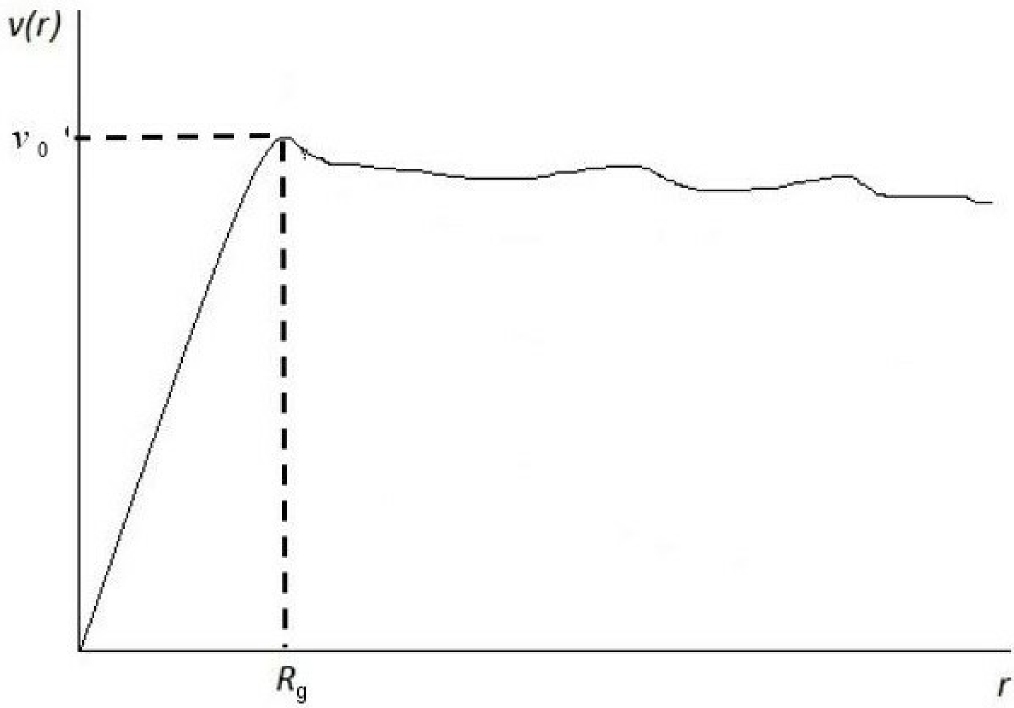


Fig. 1 Plot of galaxy rotation curve in a galaxy.

B.2	Find the total mass m_R of that part of the galaxy which lies within the sphere of radius R_g in terms of v_0 and R_g.	0.5 pt.

The discrepancy between the figure in B.2 and the plot obtained in B.1 indicates the existence of dark matter.

B.3	Determine the dark matter mass density as a function of r, R_g, v_0, n, and m_s for $r < R_g$ and $r \geq R_g$.	1.5 pt.

C. Interstellar Gas and Dark Matter

Now consider a young galaxy whose mass is dominated by interstellar gas and dark matters (neglect the mass of the stars). The interstellar gas is assumed to consist of identical particles of mass m_p. The number density $n(r)$ and temperature $T(r)$ of the gas depend on the distance from the center of the galaxy r. Although many physical processes happen in the gas, we can assume the gas is in a hydrostatic equilibrium due to its pressure and the galaxy gravitational attraction.

C.1	Find the pressure gradient of the gas dP/dr, in terms of $m'(r)$, r and $n(r)$. Here, $m'(r)$ is the total mass of gas and dark matters within a sphere of radius r from the galaxy center.	0.5 pt.

C.2	Assuming the interstellar gas as an ideal gas, find $m'(r)$ in terms of $n(r)$, $T(r)$ and their derivatives with respect to r.	0.5 pt.

Next for simplicity assume that the gas is in isothermal distribution at temperature T_0 and the interstellar gas number density is given by

$$n(r) = \frac{\alpha}{r(\beta + r)^2}$$

,

where α and β are some constants.

C.3	Find the dark matter mass density as a function of r inside the galaxy.	1.0 pt.

Earthquake, Volcano and Tsunami

Indonesia is the supermarket of natural hazards. Almost all of the natural hazards have occurred in Indonesia, such as volcano eruptions, earthquakes and tsunami.

A. Merapi Volcano Eruption

(**Fig. 1**: Pyroclastic cloud during Merapi eruption, Courtesy of Volcanological Office of Yogyakarta, BPPTKG)

Merapi volcano in Yogyakarta is one of the most active volcano in Java. Pyroclastic flows are well-known eruption characteristics of the volcano. The pyroclastic flow is a hot mixture of gas and rock which travels away from a volcano. In October 26th, 2010, Merapi showed his explosive character by producing an ash plume that reached 12 km altitude (Fig. 1) and pyroclastic currents displacing more than 20,000 people around the volcano.

Let us look at the causes of the largest eruption of Merapi in 2010. It is known by geophysicists that the influence of the external water into the magma plays an important role to the explosive behavior of volcanic eruptions (hydro magmatic eruptions). Let's assume that we dealt with a volcano as a system that consists of mixture of magmatic particles and water. The volcano vents structures and atmosphere are being boundary of the system. The explosive eruption is considered to be happening in two stages, (1) an instantaneous magma-water interaction, and (2) a system expansion. In the first stage, a mass of magma (m_m) at an absolute temperature (T_m) mixes with a mass of external water (m_w) at an absolute temperature (T_w). The thermal equilibrium is reached almost instantaneously. This interaction can be perceived as a nearly-constant volume process. Latent heat of evaporation of water and latent heat of melting of magma can be neglected.

A.1	Find the equilibrium temperature at the first stage in terms of the masses and heat capacity per unit mass of water c_{Vw} and magma c_{Vm}.	0.5 pt.
A.2	Determine its equilibrium pressure at the first stage by assuming that the mixture can be modeled as ideal gas. Assume that the volume per unit mole of the mixture is v_e.	0.3 pt.

The system expansion (the second stage) can occur through several possibilities, one of which is thermal detonation. Although such process is quite complicated, we can empirically measure the relative velocity of the erupted mixture. The velocity of gas during the eruption depend on the pressure p, the total mass m and the volume V of the mixture in the conduit of a volcano.

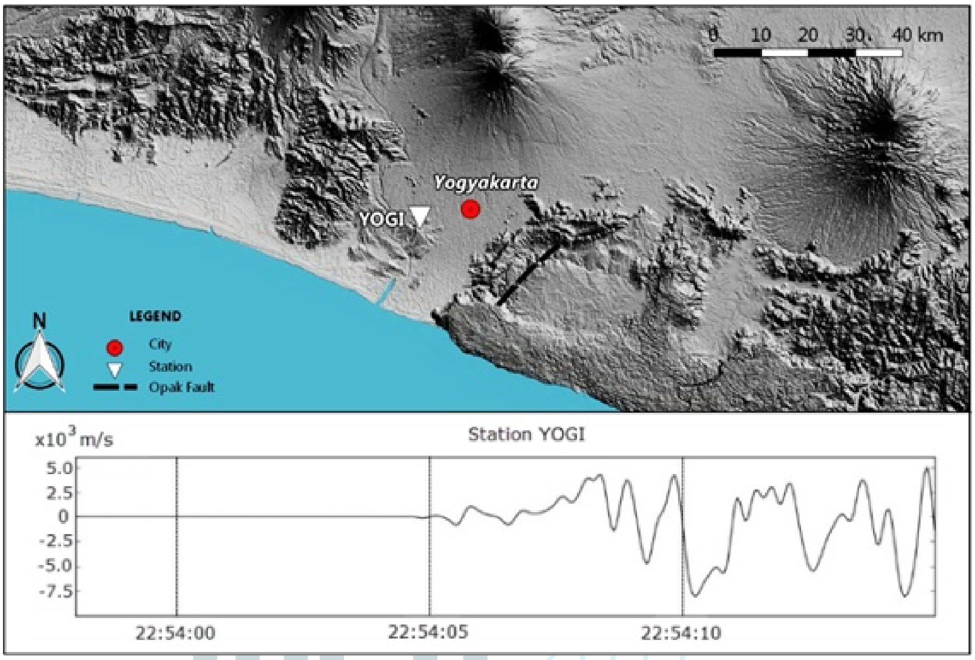
A.3	Express the velocity of gas during the eruption in terms of p, m, and V up to a proportional constant κ.	0.5 pt.

The observed pressure is around the order of 100 MPa. This makes the eruption (relative) velocity can be as large as ballistic velocity.

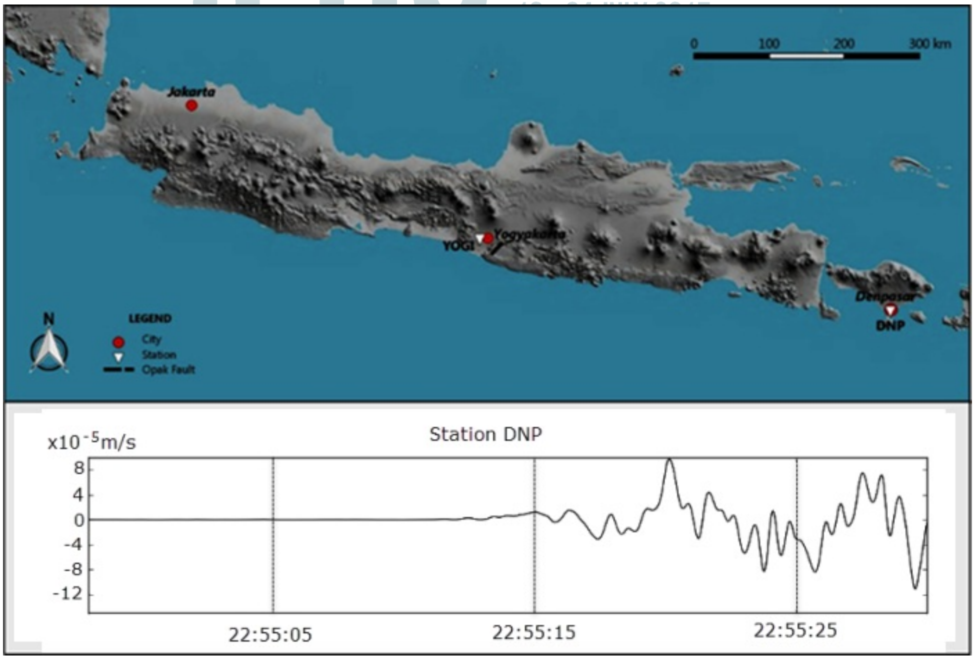
B. The Yogyakarta Earthquake

The 2006 Yogyakarta earthquake of magnitude M_w = 6.4, which destroyed many buildings in the Bantul and Yogyakarta area, occured at 05:54:00.00 local time or 22:54:00.00 UTC. The earthquake was caused by a sudden displacement of the Opak fault segment (see Fig. 2). The hypocenter was located 15 km below the surface.

The seismic wave that propagates on the earth crust can be recorded using seismometer. The diagram from seismometer is called seismogram (Fig. 2 and 3, Lower graph). The seismograms represent the vertical ground velocity as a function of time recorded by the seismic station at Gamping Station Yogyakarta (YOGI) (Fig. 2) and Denpasar, Bali (DNP) (Fig. 3). In general, seismic wave consists of three wave types: the longitudinal or primary (P-wave), the transversal or secondary (S-wave), and the surface wave. The P-wave and S-wave travel in the subsurface while the surface wave travels along the Earth surface. Seismic waves traveling through subsurfaces to the stations can be divided into those that propagate in a straight line, those that are reflected by a layer's boundary, and those that are refracted into the next layer. The longitudinal wave or the primary wave has the highest velocity, while the surface wave has the lowest velocity, around 60% of the P-wave.



(**Fig. 2**: The maps location of YOGI)



(**Fig. 3**: The maps location of DNP (Denpasar))

The distance between the epicenter (the projection of hypocenter on the Earth surface) and the YOGI and DNP stations respectively are 22.5 km and 500 km. The depth of the Earth crust layer in Java, Indonesia, is 30 km. Beneath the Earth's crust is the Earth's mantle layer. Just like other wave phenomena, seismic wave also satisfies the Snell's law. The seismic wave can also be reflected by the mantle layer. In this problem we assume that the earth curvature is neglected.

B.1	Fig. 2 shows the seismogram at the YOGI station. Use the data to find the velocity of the P-wave in the Earth crust.	0.5 pt.
B.2	Find the travel time of the direct P wave and reflected wave due to the Yogyakarta earthquake that arrived at the DNP station in Denpasar.	0.6 pt.

By assuming that the Earth is composed of only two layers: the crust and the mantle, the primary wave propagates in the crust and in the mantle with different constant velocities. The velocity in the mantle is faster than in the crust. Note that P-wave refracted into the mantle at the right angle (90^o) is being partly refracted back into the crust along its entire path of propagation along the crust-mantle boundary.

B.3	Find the velocity of the P-wave in the mantle.	1.2 pt.

For a more realistic Earth structure, the crust can be divided into a number of thin layers so that the velocity of the seismic wave is a function of the depth z according to $v(z) = v_0 + az$ where a is a constant and the hypocenter is approximated on the surface. In this model, the wave ray is curving.

B.4	Let us define the ray parameter $p = \sin\theta(z)/v(z)$, where $\theta(z)$ is the angle between the ray and the normal. Suppose a seismic wave arrives to the station with ray parameter p; express the distance to the hypocenter in terms of p, v_0, and a. Assume that the hypocenter is very close to the ground surface.	1.4 pt.
B.5	Find the travel time T from hypocenter to any station, in form of integral over z.	1.0 pt.

The earth consists of a stack of homogeneous layers with the velocity of each layer is v_i and the thickness of δz_i.

B.6	From the result of the previous problem, approximate the travel time (T) from the hypocenter to DNP station by assuming that the crust consists of only three discretized layers, (i = 1, 2, 3), characterized by v_1 = 6.65 km/sec, v_2 = 6.97 km/sec, v_3 = 6.99 km/sec, p = 0.143 sec/km, δz_1 = 6.0 km, δz_2 = 9.0 km, δz_3 = 15 km.	1.0 pt.

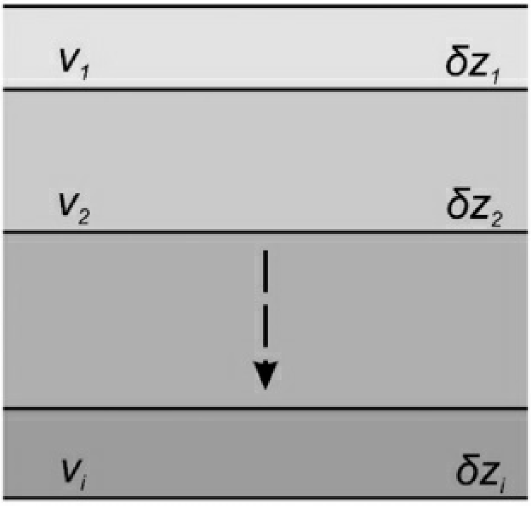


Fig. 4: A simplified model of earth's layers.

C. Java Tsunami

The 2006 Pangandaran earthquake and tsunami occurred on July 17 at 15:19:27 local time off the coast of west and central Java. During the earthquake where the epicenter fault is on the ocean floor, the fault may be displaced producing a remarkably large water wave called tsunami. In other words, a tsunami is a shallow-water wave which is initiated by a tiny amplitude, but with an extremely large wavelength. Consider a sudden fault causing a lifting of some ocean floor as shown in Fig. 5. Assume that the energy of the earthquake is transformed to the potential energy of this raised ocean water. For simple model we approximate that the raised water has a geometry of a box with its area of $\lambda L/2$ (where $L \gg \lambda$) and height of h.

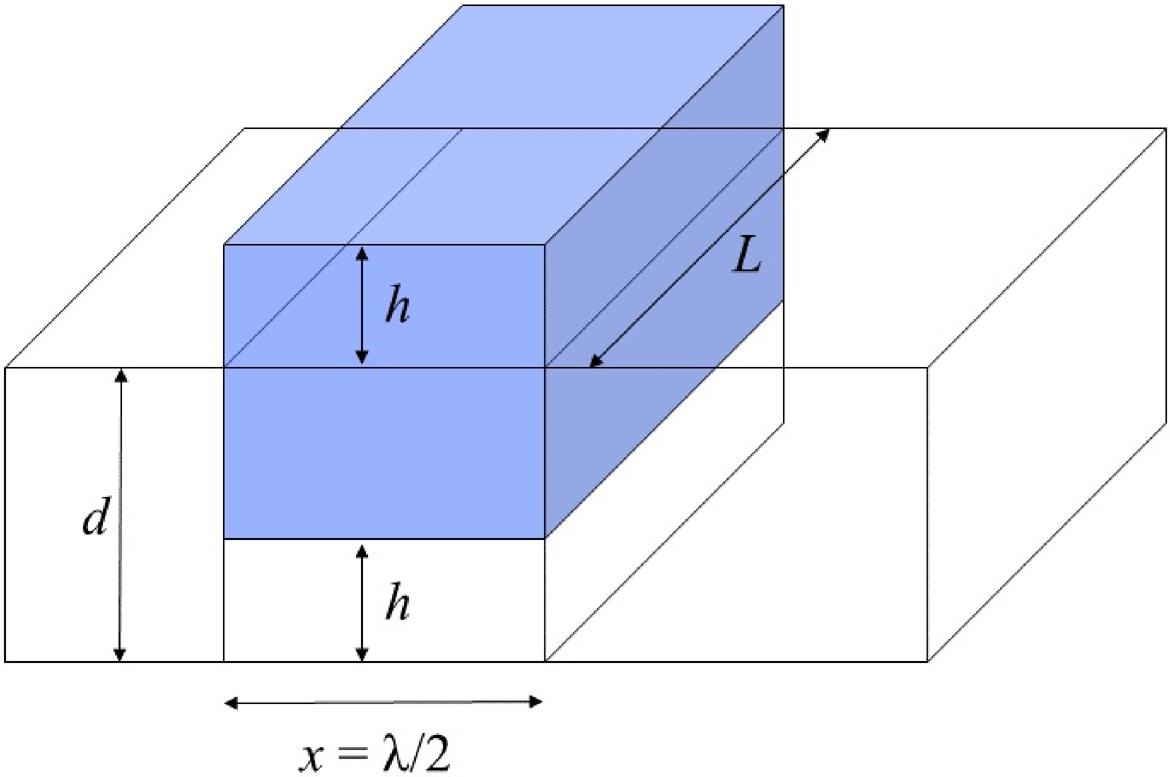


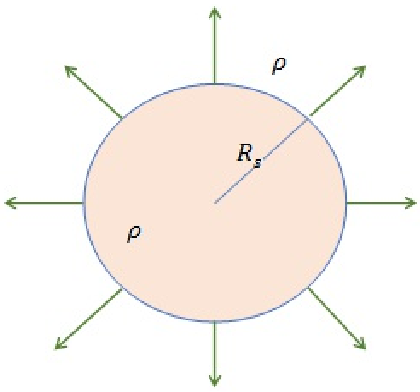
Fig. 5: Illustration for the tsunami wave d is the depth of the ocean.

C.1	Find the potential energy stored in the raised ocean water due to the earthquake with respect to ocean surface. Assume that the density of sea water is ρ.	0.5 pt.
C.2	Find the speed of tsunami wave up to dimensionless factor.	1.2 pt.
C.3	Using energy argument, determine the amplitude of the tsunami wave as a function of the depth, assuming that the depth varies slowly and also knowing that at a depth of d_0 the amplitude is A_0.	1.3 pt.

Cosmic Inflation

Due to the relative movement of galaxies observed from the earth, the wavelength of visible spectrum of a particular galaxy differs from its original wavelength, which is known as the electromagnetic Doppler effect. One expects, for a collection of galaxies, to a random distributions of wavelength shifts: some positive (red shift) and some negative (blue shift). However, observations show that all, expect for a nearby group of galaxies, are red shifted. This must be true even if the observation take place on different point in the universe. As a conclusion, our universe must be expanding. On the other hand local irregularity of the universe can be neglected on scales of more than 100 Mpc, in which 1 pc = 3.26 light-years. Averaged over large scales, the clumpy distribution of galaxies becomes more and more isotropic (independent of direction) and homogeneous (independent of position). Therefore we can assume the universe as a matter having a uniform mass density ρ and is expanding.

A. Expansion of Universe



For a simple model of our universe, let us consider an expanding uniform-density sphere embedded in a medium of a much larger sphere with the same density. Let say at some time, the radius of the smaller sphere is R_s. To express the expansion of the sphere, the time dependency of the radius $R(t)$ can be expressed by scale factor $a(t)$, that is $R(t) = a(t)R_s$.

Using Newton's law of gravity to evaluate velocity of a mass element on the sphere boundary according the model of our universe, one can obtain a set of Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = A_1\rho(t) - \frac{kc^2}{R_s^2 a^2(t)} \qquad (1)$$

where k a dimensionless constant, and c is velocity of light.

A.1	Determine the constant A_1 in the equation (1)	1.3 pt.

The discussion so far is non-relativistic. But in fact, it can be extended to a relativistic system by reinterpreting $\rho(t)c^2$ as total energy density (excluding the gravitational potential energy). In this relativistic system derives the 2nd Friedmann equation:

$$\dot{\rho} + A_2 \left(\rho + \left(\frac{p}{c^2}\right)\right) \frac{\dot{a}}{a} = 0 \qquad (2)$$

using the 1st law thermodynamics of an adiabatic system, where p denotes the pressure on the sphere.

A.2	Determine the constants A_2 in the equation (2)	0.9 pt.

To solve Eqs. (1) and (2), one should assume a relation $p = p(\rho)$, such as $p(t)/c^2 = w\rho(t)$, where w is a constant. There is also a factor $H = \dot{a}/a$ being called Hubble parameter. The present values of parameters are usually symbolized by subscript 0 such as t_0, ρ_0, H_0, a_0 and so on. For simplicity, we take $a_0 = 1$.

Universe is believed to start from a big explosion called Big-Bang that produces radiation of relativistic particles. During its expansion, the universe is cooling down and the particles in it become non-relativistic. However, the recent observations clarify that the present universe is dominated by cosmological constant energy density. For the case of photon, as the universe is expanding, the photon's wavelength expands proportionally to the scale factor.

A.3	For each of the following three cases determine the resulting value of w: (i) a universe filled only with radiation (i.e. photon energy), (ii) a universe filled only with non-relativistic matter and (iii) a universe with constant energy density.	1.2 pt.
A.4	In the case of $k = 0$, find $a(t)$ for each case of (i) to (iii) being mentioned in A.3. Use the initial condition $a(t = 0) = 0$ for case (i) and (ii), and use the condition $a_0 = 1$ for case (iii).	1.2 pt.

Constant k in Eq. (1) refers to classification of spatial geometry of the universe. Its value can be $k = +1$ for positive-curvature universe (closed), $k = 0$ for flat universe (infinite), and $k = -1$ for negative-curvature universe (open, infinite). Let define a density ratio $\Omega = \rho/\rho_c$, where $\rho_c c^2 = H^2/A_1$ is critical energy density. Note that A_1 is obtained from problem A.1.

A.5	Express k in Eq.(1) in terms of Ω, H, a, and R_0.	0.1 pt.
A.6	Find a range for Ω that corresponds to each value of $k = +1$, $k = 0$ and $k = -1$.	0.3 pt.

B. Motivation To Introduce Inflation Phase and Its General Conditions

The observation of cosmic microwave background radiation (CMB) suggests that our present universe is approximately flat. The problem is that if this is true then the present universe should start from exactly flat early universe, otherwise any deviation from the flatness will eventually grow over time and spoil the present flatness.

B.1	Find $(\Omega(t) - 1)$ as a function of time for the universe when it is dominated by radiation or non-relativistic matter (see problem A.3).	0.5 pt.

To solve the problem, at some early time in its history, the universe should undergo a constant energy density domination period which leads to an exponential expansion so called inflation period.

B.2	For this constant energy density domination period, find $(\Omega(t) - 1)$ as a function of time. Assume that $(\Omega(t) - 1) \ll 1$.	0.3 pt.
B.3	Show that condition for inflation implies several following conditions: negative pressure, accelerated expansion $(\ddot{a} > 0)$, and decreasing Hubble radius $(d(aH)^{-1}/dt < 0)$.	0.9 pt.
B.4	Show that the condition of decreasing Hubble radius can be expressed in terms of parameter $\epsilon = -\dot{H}/H^2$ as $\epsilon < 1$.	0.2 pt.

Inflation occurs as long as $\epsilon < 1$ and then ends when $\epsilon = 1$. We can define e-folding number N, such that $dN = d\ln a = Hdt$ and $N = 0$ at the end of inflation.

C. Inflation Generated by Homogenously Distributed Matter

As an example of simple physical system that can generate period of inflation is a universe dominated by homogenously distributed matter. This kind of matter is called inflaton and can be characterized by a function $\phi(t)$.

The dynamical equation of the matter can be expressed as

$$\ddot{\phi} + 3H\dot{\phi} = -V', \tag{3}$$

where $V = V(\phi)$ is a potential function and $V' = \frac{\partial V}{\partial \phi}$. The Hubble parameter satisfies

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]. \tag{4}$$

with M_{pl} is a constant called the reduced Planck mass. Inflation phase occurs during domination of potential energy V over kinetic energy $\dot{\phi}^2/2$ for sufficient time such that $\ddot{\phi}$ term in equation (3) can be neglected. This condition is called slow-roll approximation.

The quantities ϵ and $\eta_V = \delta + \epsilon$, where $\delta = -\ddot{\phi}/(H\dot{\phi})$, are called 'slow-roll' parameters.

C.1	Estimate parameter ϵ, parameter η_V, $dN/d\phi$ in terms of potential $V(\phi)$ and its first and second derivative (V' and V'').	1.7 pt.

D. Inflation with A Simple Potential

Predictions of any inflation model should be compared to observational constraints from CMB as follow n_s = 0.968 ± 0.006 and r < 0.12, where $r = 16\epsilon$ and $n_s = 1 + 2\eta_V - 6\epsilon$ are evaluated at $\phi = \phi_{start}$ for inflation model being generated by a dominant matter. Assume that potential of matter takes a simple form $V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{pl}}\right)^n$ where n is any integer and Λ is a constant.

D.1	Calculate ϕ_{end} at the end of inflation.	0.5 pt.
D.2	Express r and n_s in terms of e-folding number N and integer n. Estimate the value of n that is closest to observational values r and n_s. Take N = 60 in your calculation.	0.9 pt.