Two Problems in Mechanics (10 points)

Please read the general instructions in the separate envelope before you start this problem.

Part A. The Hidden Disk (3.5 points)

We consider a solid wooden cylinder of radius r_1 and thickness h_1. Somewhere inside the wooden cylinder, the wood has been replaced by a metal disk of radius r_2 and thickness h_2. The metal disk is placed in such a way that its symmetry axis B is parallel to the symmetry axis S of the wooden cylinder, and is placed at the same distance from the top and bottom face of the wooden cylinder. We denote the distance between S and B by d. The density of wood is ρ_1, the density of the metal is $\rho_2 > \rho_1$. The total mass of the wooden cylinder and the metal disk inside is M.

In this task, we place the wooden cylinder on the ground so that it can freely roll to the left and right. See Fig. 1 for a side view and a view from the top of the setup.

The goal of this task is to determine the size and the position of the metal disk.

In what follows, when asked to express the result in terms of known quantities, you may always assume that the following are known:

$$r_1,\ h_1,\ \rho_1,\ \rho_2, M\ . \tag{1}$$

The goal is to determine r_2, h_2 and d, through indirect measurements.

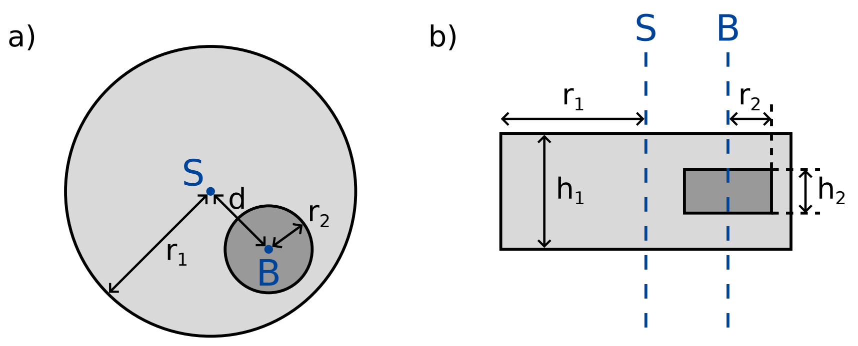


Figure 1: a) side view b) view from above

We denote b as the distance between the centre of mass C of the whole system and the symmetry axis S of the wooden cylinder. In order to determine this distance, we design the following experiment: We place the wooden cylinder on a horizontal base in such a way that it is in a stable equilibrium. Let us now slowly incline the base by an angle Θ (see Fig. 2). As a result of the static friction, the wooden cylinder can roll freely without sliding. It will roll down the incline a little bit, but then come to rest in a stable equilibrium after rotating by an angle ϕ which we measure.

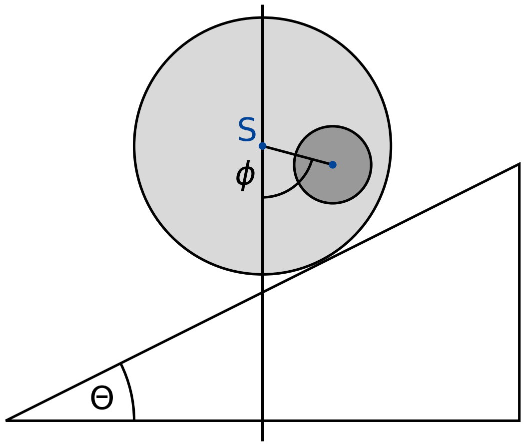


Figure 2: Cylinder on an inclined base.

A.1	Find an expression for b as a function of the quantities (1), the angle ϕ and the tilting angle Θ of the base.	0.8pt

From now on, we can assume that the value of b is known.

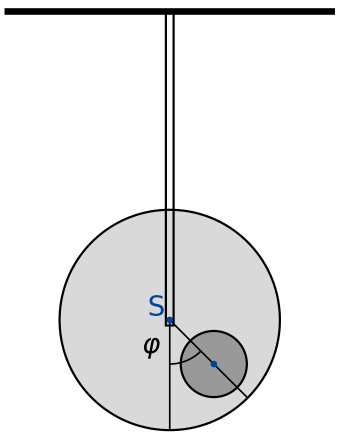


Figure 3: Suspended system.

Next we want to measure the moment of inertia I_S of the system with respect to the symmetry axis S. To this end, we suspend the wooden cylinder at its symmetry axis from a rigid rod. We then turn it away from its equilibrium position by a small angle φ, and let it go. See figure 3 for the setup. We find that φ describes a periodic motion with period T.

A.2	Find the equation of motion for φ. Express the moment of inertia I_S of the system around its symmetry axis S in terms of T, b and the known quantities (1). You may assume that we are only disturbing the equilibrium position by a small amount so that φ is always very small.	0.5pt

From the measurements in questions **A.1** and **A.2**, we now want to determine the geometry and the position of the metal disk inside the wooden cylinder.

A.3	Find an expression for the distance d as a function of b and the quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.	0.4pt

A.4	Find an expression for the moment of inertia I_S in terms of b and the known quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.	0.7pt

A.5	Using all the above results, write down an expression for h_2 and r_2 in terms of b, T and the known quantities (1). You may express h_2 as a function of r_2.	1.1pt

Part B. Rotating Space Station (6.5 points)

Alice is an astronaut living on a space station. The space station is a gigantic wheel of radius R rotating around its axis, thereby providing artificial gravity for the astronauts. The astronauts live on the inner side of the rim of the wheel. The gravitational attraction of the space station and the curvature of the floor can be ignored.

B.1	At what angular frequency ω_{ss} does the space station rotate so that the astronauts experience the same gravity g_E as on the Earth's surface?	0.5pt

Alice and her astronaut friend Bob have an argument. Bob does not believe that they are in fact living in a space station and claims that they are on Earth. Alice wants to prove to Bob that they are living on a rotating space station by using physics. To this end, she attaches a mass m to a spring with spring constant k and lets it oscillate. The mass oscillates only in the vertical direction, and cannot move in the horizontal direction.

B.2	Assuming that on Earth gravity is constant with acceleration g_E, what would be the angular oscillation frequency ω_E that a person on Earth would measure?	0.2pt

B.3	What angular oscillation frequency ω does Alice measure on the space station?	0.6pt

Alice is convinced that her experiment proves that they are on a rotating space station. Bob remains sceptical. He claims that when taking into account the change in gravity above the surface of the Earth, one finds a similar effect. In the following tasks we investigate whether Bob is right.

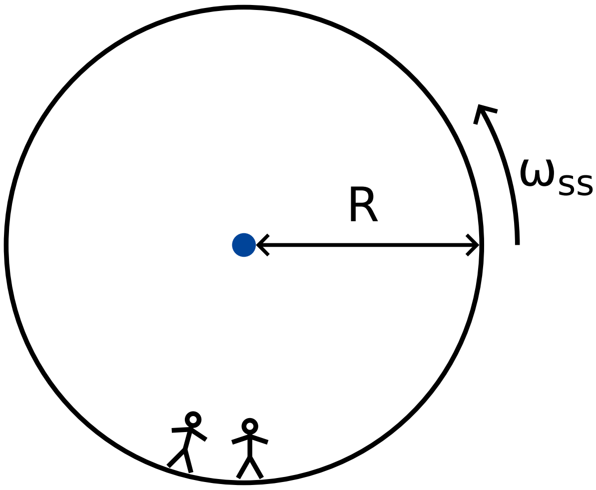


Figure 4: Space station

B.4	Derive an expression of the gravity $g_E(h)$ for small heights h above the surface of the Earth and compute the oscillation frequency $\tilde{\omega}_E$ of the oscillating mass (linear approximation is enough). Denote the radius of the Earth by R_E. Neglect the rotation of Earth.	0.8pt

Indeed, for this space station, Alice does find that the spring pendulum oscillates with the frequency that Bob predicted.

B.5	For what radius R of the space station does the oscillation frequency ω match the oscillation frequency $\tilde{\omega}_E$ on the Earth? Express your answer in terms of R_E.	0.3pt

Exasperated with Bob's stubbornness, Alice comes up with an experiment to prove her point. To this end she climbs on a tower of height H over the floor of the space station and drops a mass. This experiment can be understood in the rotating reference frame as well as in an inertial reference frame.

In a uniformly rotating reference frame, the astronauts perceive a fictitious force $\vec{F}_C$ called the Coriolis force. The force $\vec{F}_C$ acting on an object of mass m moving at velocity $\vec{v}$ in a rotating frame with constant angular frequency $\vec{\omega}_{ss}$ is given by

$$\vec{F}_C = 2m\vec{v} \times \vec{\omega}_{ss} \, . \tag{2}$$

In terms of the scalar quantities you may use

$$F_C = 2mv\omega_{ss} \sin\phi \, , \tag{3}$$

where ϕ is the angle between the velocity and the axis of rotation. The force is perpendicular to both the velocity v and the axis of rotation. The sign of the force can be determined from the right-hand rule, but in what follows you may choose it freely.

B.6	Calculate the horizontal velocity v_x and the horizontal displacement d_x (relative to the base of the tower, in the direction perpendicular to the tower) of the mass at the moment it hits the floor. You may assume that the height H of the tower is small, so that the acceleration as measured by the astronauts is constant during the fall. Also, you may assume that $d_x \ll H$.	1.1pt

To get a good result, Alice decides to conduct this experiment from a much taller tower than before. To her surprise, the mass hits the floor at the base of the tower, so that $d_x = 0$.

B.7	Find a lower bound for the height of the tower for which it can happen that $d_x = 0$.	1.3pt

Alice is willing to make one last attempt at convincing Bob. She wants to use her spring oscillator to show the effect of the Coriolis force. To this end she changes the original setup: She attaches her spring to a ring which can slide freely on a horizontal rod in the x direction without any friction. The spring itself oscillates in the y direction. The rod is parallel to the floor and perpendicular to the axis of rotation of the space station. The xy plane is thus perpendicular to the axis of rotation, with the y direction pointing straight towards the center of rotation of the station.

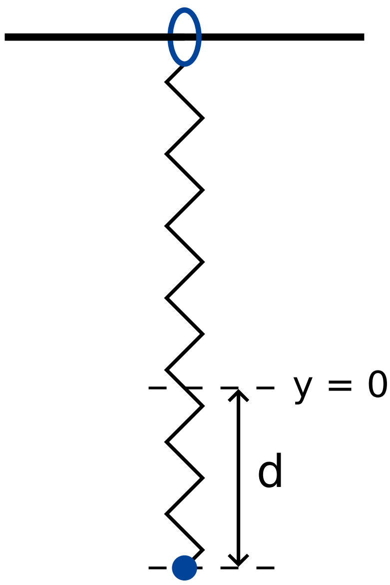


Figure 5: Setup.

B.8	Alice pulls the mass a distance d downwards from the equilibrium point $x = 0$, $y = 0$, and then lets it go (see figure 5). • Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss} d$ is small, and neglect the Coriolis force for motion along the y-axis. • Sketch the trajectory $(x(t), y(t))$, marking all important features such as amplitude.	1.7pt

Alice and Bob continue to argue.

Nonlinear Dynamics in Electric Circuits (10 points)

Please read the general instructions in the separate envelope before you start this problem.

Introduction

Bistable non-linear semiconducting elements (e.g. thyristors) are widely used in electronics as switches and generators of electromagnetic oscillations. The primary field of applications of thyristors is controlling alternating currents in power electronics, for instance rectification of AC current to DC at the megawatt scale. Bistable elements may also serve as model systems for self-organization phenomena in physics (this topic is covered in part B of the problem), biology (see part C) and other fields of modern nonlinear science.

Goals

To study instabilities and nontrivial dynamics of circuits including elements with non-linear $I-V$ characteristics. To discover possible applications of such circuits in engineering and in modeling of biological systems.

Part A. Stationary states and instabilities (3 points)

Fig. 1 shows the so-called **S-shaped** $I-V$ characteristics of a non-linear element X. In the voltage range between $U_\mathrm{h} = 4.00\ \mathrm{V}$ (the holding voltage) and $U_\mathrm{th} = 10.0\ \mathrm{V}$ (the threshold voltage) this $I-V$ characteristics is multivalued. For simplicity, the graph on Fig. 1 is chosen to be piece-wise linear (each branch is a segment of a straight line). In particular, the line in the upper branch touches the origin if it is extended. This approximation gives a good description of real thyristors.

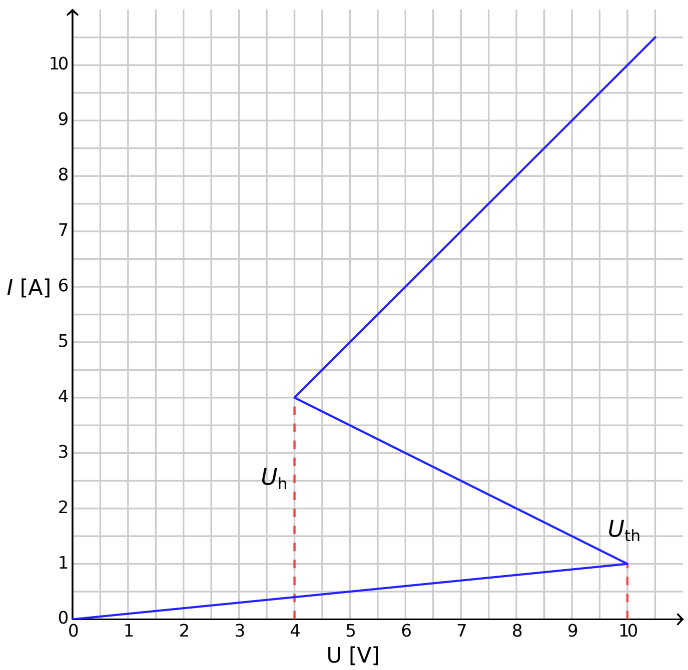


Figure 1: $I-V$ characteristics of the non-linear element X.

A.1	Using the graph, determine the resistance R_{on} of the element X on the upper branch of the $I-V$ characteristics, and R_{off} on the lower branch, respectively. The middle branch is described by the equation $$I = I_0 - \frac{U}{R_{\mathrm{int}}}. \quad (1)$$ Find the values of the parameters I_0 and R_{int}.	0.4pt

The element X is connected in series (see Fig.2) with a resistor R, an inductor L and an ideal voltage source $\mathcal{E}$. One says that the circuit is in a stationary state if the current is constant in time, $I(t) = \mathrm{const}$.

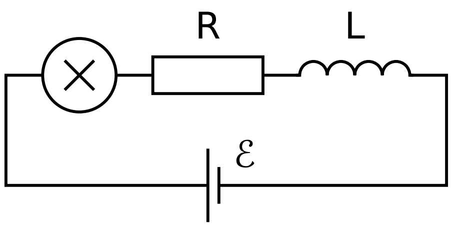


Figure 2: Circuit with element X, resistor R, inductor L and voltage source $\mathcal{E}$.

A.2	What are the possible numbers of stationary states that the circuit of Fig. 2 may have for a fixed value of $\mathcal{E}$ and for $R = 3.00\ \Omega$? How does the answer change for $R = 1.00\ \Omega$?	1pt

A.3	Let $R = 3.00\ \Omega, L = 1.00\ \mu\mathrm{H}$ and $\mathcal{E} = 15.0\ \mathrm{V}$ in the circuit shown in Fig. 2. Determine the values of the current $I_{\mathrm{stationary}}$ and the voltage $V_{\mathrm{stationary}}$ on the non-linear element X in the stationary state.	0.6pt

The circuit in Fig. 2 is in the stationary state with $I(t) = I_{\mathrm{stationary}}$. This stationary state is said to be stable if after a small displacement (increase or decrease in the current), the current returns towards the stationary state. And if the system keeps moving away from the stationary state, it is said to be unstable.

A.4	Use numerical values of the question **A.3** and study the stability of the stationary state with $I(t) = I_{\mathrm{stationary}}$. Is it stable or unstable?	1pt

Part B. Bistable non-linear elements in physics: radio transmitter (5 points)

We now investigate a new circuit configuration (see Fig. 3). This time, the non-linear element X is connected in parallel to a capacitor of capacitance $C = 1.00\ \mu\mathrm{F}$. This block is then connected in series to a resistor of resistance $R = 3.00\ \Omega$ and an ideal constant voltage source of voltage $\mathcal{E} = 15.0\ \mathrm{V}$. It turns out that this circuit undergoes oscillations with the non-linear element X jumping from one branch of the $I-V$ characteristics to another over the course of one cycle.

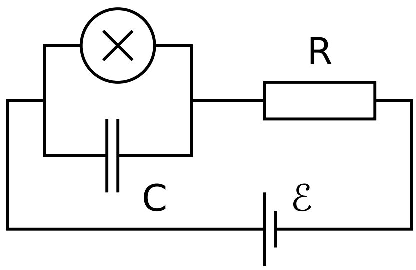


Figure 3: Circuit with element X, capacitor C, resistor R and voltage source $\mathcal{E}$.

B.1	Draw the oscillation cycle on the $I - V$ graph, including its direction (clockwise or anticlockwise). Justify your answer with equations and sketches.	1.8pt

B.2	Find expressions for the times t_1 and t_2 that the system spends on each branch of the $I - V$ graph during the oscillation cycle. Determine their numerical values. Find the numerical value of the oscillation period T assuming that the time needed for jumps between the branches of the $I - V$ graph is negligible.	1.9pt

B.3	Estimate the average power P dissipated by the non-linear element over the course of one oscillation. An order of magnitude is sufficient.	0.7pt

The circuit in Fig. 3 is used to build a radio transmitter. For this purpose, the element X is attached to one end of a linear antenna (a long straight wire) of length s. The other end of the wire is free. In the antenna, an electromagnetic standing wave is formed. The speed of electromagnetic waves along the antenna is the same as in vacuum. The transmitter is using the main harmonic of the system, which has period T of question **B.2**.

B.4	What is the optimal value of s assuming that it cannot exceed $1\ \mathrm{km}$?	0.6pt

Part C. Bistable non-linear elements in biology: neuristor (2 points)

In this part of the problem, we consider an application of bistable non-linear elements to modeling of biological processes. A neuron in a human brain has the following property: when excited by an external signal, it makes one single oscillation and then returns to its initial state. This feature is called excitability. Due to this property, pulses can propagate in the network of coupled neurons constituting the nerve systems. A semiconductor chip designed to mimic excitability and pulse propagation is called a *neuristor* (from neuron and transistor).

We attempt to model a simple neuristor using a circuit that includes the non-linear element X that we investigated previously. To this end, the voltage $\mathcal{E}$ in the circuit of Fig. 3 is decreased to the value $\mathcal{E}' = 12.0\ \mathrm{V}$. The oscillations stop, and the system reaches its stationary state. Then, the voltage is rapidly increased back to the value $\mathcal{E} = 15.0\ \mathrm{V}$, and after a period of time τ (with $\tau < T$) is set again to the value $\mathcal{E}'$ (see Fig. 4). It turns out that there is a certain critical value $\tau_{\mathrm{crit.}}$, and the system shows qualitatively different behavior for $\tau < \tau_{\mathrm{crit}}$ and for $\tau > \tau_{\mathrm{crit}}$.

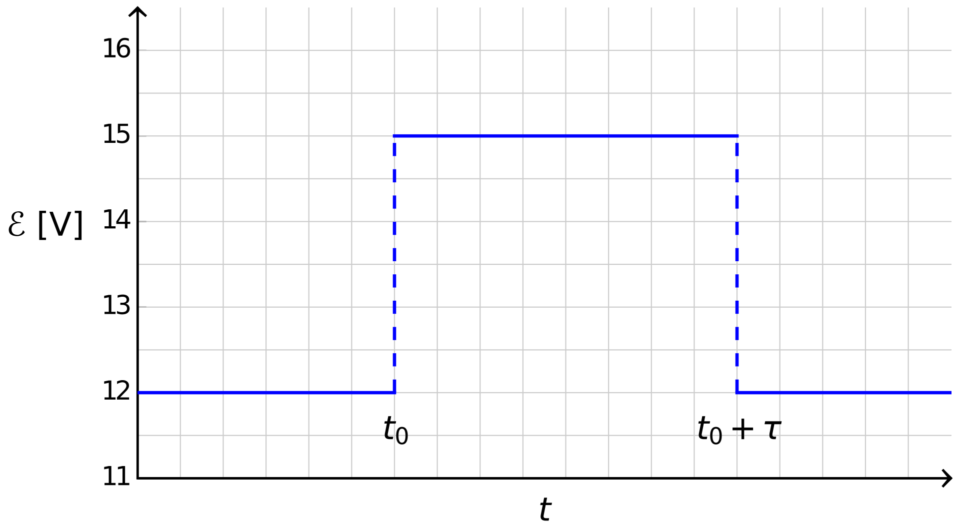


Figure 4: Voltage of the voltage source as a function of time.

C.1	Sketch the graphs of the time dependence of the current $I_X(t)$ on the non-linear element X for $\tau < \tau_{\mathrm{crit}}$ and for $\tau > \tau_{\mathrm{crit}}$.	1.2pt

C.2	Find the expression and the numerical value of the critical time τ_{crit} for which the scenario switches.	0.6pt

C.3	Is the circuit with $\tau = 1.00 \times 10^{-6}$ s a neuristor?	0.2pt

Large Hadron Collider (10 points)

Please read the general instructions in the separate envelope before you start this problem.

In this task, the physics of the particle accelerator LHC (Large Hadron Collider) at CERN is discussed. CERN is the world's largest particle physics laboratory. Its main goal is to get insight into the fundamental laws of nature. Two beams of particles are accelerated to high energies, guided around the accelerator ring by a strong magnetic field and then made to collide with each other. The protons are not spread uniformly around the circumference of the accelerator, but they are clustered in so-called bunches. The resulting particles generated by collisions are observed with large detectors. Some parameters of the LHC can be found in table 1.

LHC ring	
Circumference of ring	$26659\ \mathrm{m}$
Number of bunches per proton beam	2808
Number of protons per bunch	1.15×10^{11}
Proton beams	
Energy of protons	$7.00\ \mathrm{TeV}$
Centre of mass energy	$14.0\ \mathrm{TeV}$

Table 1: Typical numerical values of relevant LHC parameters.

Particle physicists use convenient units for the energy, momentum and mass: The energy is measured in electron volts $[\mathrm{eV}]$. By definition, $1\ \mathrm{eV}$ is the amount of energy gained by a particle with elementary charge, e, moved through a potential difference of one volt ($1\ \mathrm{eV} = 1.602 \cdot 10^{-19}\ \mathrm{kg\,m^2 s^{-2}}$).

The momentum is measured in units of eV/c and the mass in units of eV/c^2, where c is the speed of light in vacuum. Since $1\ \mathrm{eV}$ is a very small quantity of energy, particle physicists often use MeV ($1\ \mathrm{MeV} = 10^6\ \mathrm{eV}$), GeV ($1\ \mathrm{GeV} = 10^9\ \mathrm{eV}$) or TeV ($1\ \mathrm{TeV} = 10^{12}\ \mathrm{eV}$).

Part A deals with the acceleration of protons or electrons. Part B is concerned with the identification of particles produced in the collisions at CERN.

Part A. LHC accelerator (6 points)

Acceleration:

Assume that the protons have been accelerated by a voltage V such that their velocity is very close to the speed of light and neglect any energy loss due to radiation or collisions with other particles.

A.1	Find the exact expression for the final velocity v of the protons as a function of the accelerating voltage V, and physical constants.	0.7pt

A design for a future experiment at CERN plans to use the protons from the LHC and to collide them with electrons which have an energy of $60.0\ \mathrm{GeV}$.

A.2	For particles with high energy and low mass the relative deviation $\Delta = (c - v)/c$ of the final velocity v from the speed of light is very small. Find a first order approximation for Δ and calculate Δ for electrons with an energy of $60.0\ \mathrm{GeV}$using the accelerating voltage V and physical constants.	0.8pt

We now return to the protons in the LHC. Assume that the beam pipe has a circular shape.

A.3	Derive an expression for the uniform magnetic flux density B necessary to keep the proton beam on a circular track. The expression should only contain the energy of the protons E, the circumference L, fundamental constants and numbers. You may use suitable approximations if their effect is smaller than precision given by the least number of significant digits. Calculate the magnetic flux density B for a proton energy of $E = 7.00\ \mathrm{TeV}$, neglecting interactions between the protons.	1.0pt

Radiated Power:

An accelerated charged particle radiates energy in the form of electromagnetic waves. The radiated power P_{rad} of a charged particle that circulates with a constant angular velocity depends only on its acceleration a, its charge q, the speed of light c and the permittivity of free space ε_0.

A.4	Use dimensional analysis to find an expression for the radiated power P_{rad}.	1.0pt

The real formula for the radiated power contains a factor $1/(6\pi)$; moreover, a full relativistic derivation gives an additional multiplicative factor γ^4, with $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$.

A.5	Calculate P_{tot}, the total radiated power of the LHC, for a proton energy of $E = 7.00\ \mathrm{TeV}$ (Note table 1). You may use suitable approximations.	1.0pt

Linear Acceleration:

At CERN, protons at rest are accelerated by a linear accelerator of length $d = 30.0\ \mathrm{m}$ through a potential difference of $V = 500\ \mathrm{MV}$. Assume that the electrical field is homogeneous. A linear accelerator consists of two plates as sketched in Figure 1.

A.6	Determine the time T that the protons take to pass through this field.	1.5pt

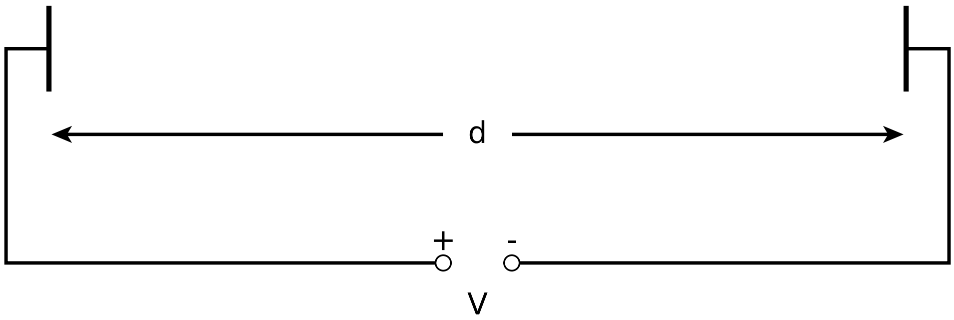


Figure 1: Sketch of an accelerator module.

Part B. Particle Identification (4 points)

Time of flight:

It is important to identify the high energy particles that are generated in the collision in order to interpret the interaction process. A simple method is to measure the time (t) that a particle with known momentum needs to pass a length l in a so-called Time-of-Flight (ToF) detector. Typical particles which are identified in the detector, together with their masses, are listed in table 2.

Particle	**Mass** [MeV/c^2]
Deuteron	1876
Proton	938
charged Kaon	494
charged Pion	140
Electron	0.511

Table 2: Particles and their masses.

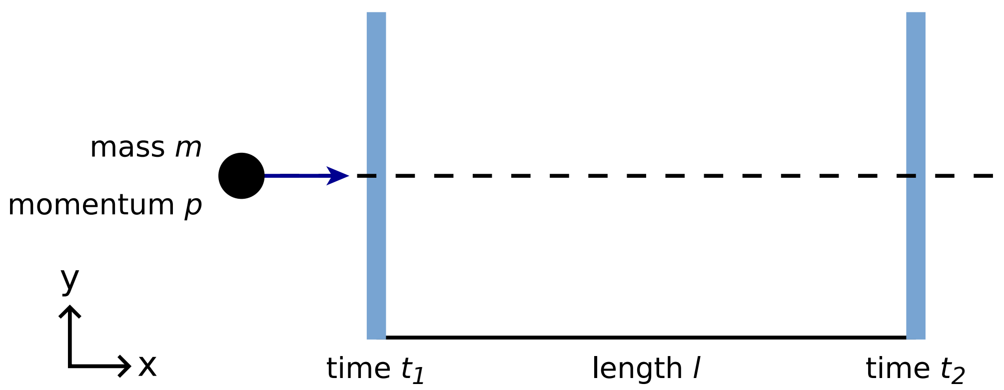
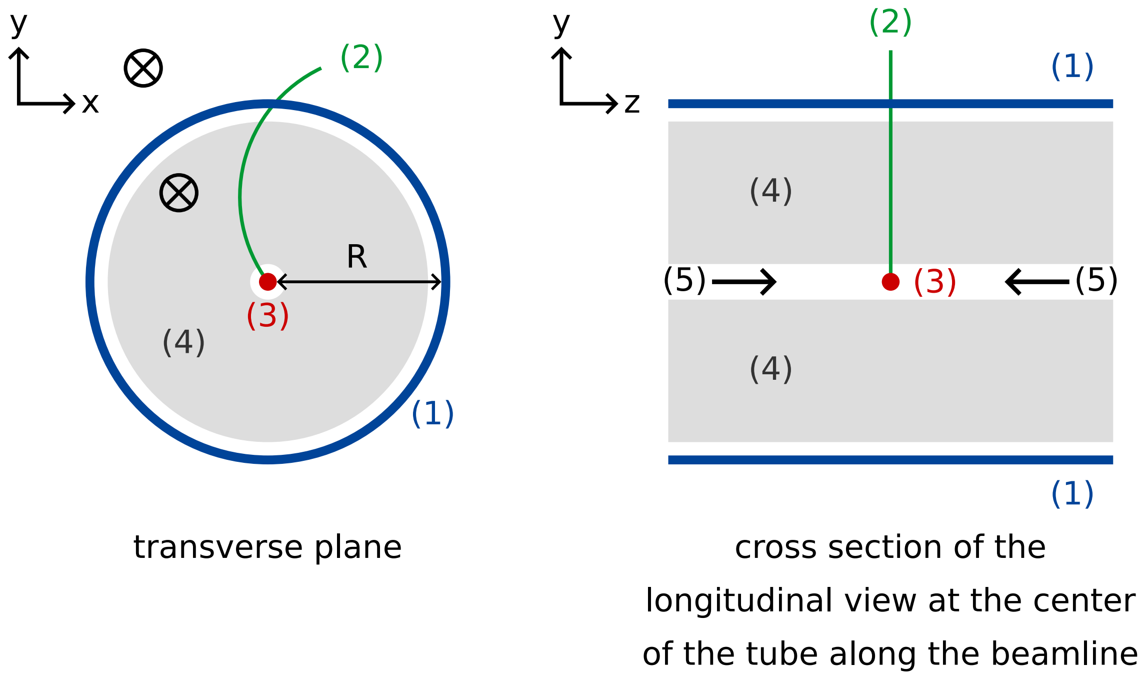


Figure 2: Schematic view of a time-of-flight detector.

B.1	Express the particle mass m in terms of of the momentum p, the flight length l and the flight time t, assuming that particles have elementary charge e and travel with velocity close to c on straight tracks in the ToF detector and that they travel perpendicular to the two detection planes (see figure 2).	0.8pt

B.2	Calculate the minimal length l of a ToF detector that allows to safely distinguish a charged kaon from a charged pion, given both their momenta are measured to be $1.00\ \mathrm{GeV/c}$. For a good separation it is required that the difference in the time-of-flight is larger than three times the time resolution of the detector. The typical resolution of a ToF detector is $150\ \mathrm{ps}\ (1\ \mathrm{ps} = 10^{-12}\ \mathrm{s})$.	0.7pt

In the following, particles produced in a typical LHC detector are identified in a two stage detector consisting of a tracking detector and a ToF detector. Figure 3 shows the setup in the plane transverse and longitudinal to the proton beams. Both detectors are tubes surrounding the interaction region with the beam passing in the middle of the tubes. The tracking detector measures the trajectory of a charged particle which passes through a magnetic field whose direction is parallel to the proton beams. The radius r of the trajectory allows one to determine the transverse momentum $\mathrm{p_T}$ of the particle. Since the collision time is known the ToF detector only needs one tube to measure the flight time (time between the collision and the detection in the ToF tube). This ToF tube is situated just outside the tracking chamber. For this task you may assume that all particles created by the collision travel perpendicular to the proton beams, which means that the created particles have no momentum along the direction of the proton beams.



(1) - ToF tube
(2) - track
(3) - collision point
(4) - tracking tube
(5) - proton beams
$\otimes$ - magnetic field

Figure 3 : Experimental setup for particle identification with a tracking chamber and a ToF detector. Both detectors are tubes surrounding the collision point in the middle. Left : transverse view perpendicular to the beamline. Right : longitudinal view parallel to the beam line. The particle is travelling perpendicular to the beam line.

B.3	Express the particle mass in terms of the magnetic flux density B, the radius R of the ToF tube, fundamental constants and the measured quantities: radius r of the track and time-of-flight t.	1.7pt

We detected four particles and want to identify them. The magnetic flux density in the tracking detector was $B = 0.500\ \mathrm{T}$. The radius R of the ToF tube was $3.70\ \mathrm{m}$. Here are the measurements ($1\ \mathrm{ns} = 10^{-9}\ \mathrm{s}$):

Particle	*Radius of the trajectory* r $[\mathrm{m}]$	*Time of flight* t $[\mathrm{ns}]$
A	5.10	20
B	2.94	14
C	6.06	18
D	2.31	25

B.4	Identify the four particles by calculating their mass.	0.8pt