Problem 1 : Solution/marking scheme – Two Problems in Mechanics *(10 points)*

Part A. The Hidden Disk (3.5 points)

A1 (0.8 pt) Find an expression for b as a function of the quantities (1), the angle ϕ and the tilting angle Θ of the base.

Solution A1: **[0.8]**

Geometric solution: use that torque with respect to point of contact is 0 $\Rightarrow$ center of gravity has to be vertically above point of contact.

$$\sin\phi = \frac{D}{b}$$ **0.3**

$$\sin\Theta = \frac{D}{r_1}$$ **0.3**

Here D may be called another name. Solve this:

$$\sin\phi = \frac{r_1}{b}\sin\Theta \;\Rightarrow\; b = \frac{r_1 \sin\Theta}{\sin\phi}$$ **0.2**

Alternative: Torque and forces with respect to another point: **[0.8]**
Correct equation for torque **0.3**
Correct equation for force **0.3**
Correct solution **0.2**

A2 (0.5 pt) Find the equation of motion for φ. Express the moment of inertia I_S of the cylinder around its symmetry axis S in terms of T, b and the known quantities (1). You may assume that we are only disturbing the equilibrium position by a small amount so that φ is always very small.

Solution A2: **[0.5]**

Write some equation of the form $\ddot{\varphi} = -\omega^2\varphi$ **0.1**

Writing an equation of the form $\varphi = A\cos\omega t$ is also correct.
Two solutions:

1. Kinetic energy: $\frac{1}{2}I_S\dot{\varphi}^2$ and potential energy: $-bMg\cos\varphi$. Total energy is conserved, and differentiation w.r.t. time gives the equation of motion.
2. Angular equation of motion from torque, $\tau = I_S\ddot{\varphi} = -Mgb\sin\varphi$.

Correct equation (either energy conservation or torque equation of motion) **0.3**
Final answer

$$T = 2\pi\sqrt{\frac{I_S}{Mgb}} \;\Rightarrow\; I_S = \frac{MgbT^2}{4\pi^2}$$ **0.1**

(Derivation:

$$\Rightarrow \ddot{\varphi} = -\frac{bMg}{I_S}\sin\varphi \simeq -\frac{bgM}{I_S}\varphi$$

so that

$$\omega^2 = \frac{bgM}{I_S}$$

)

A3 (0.4 pt) Find an expression for the distance d as a function of b and the quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

Solution A3: **[0.4]**

Some version of the center of mass equation, e.g.

$$b = \frac{dM_2}{M_1 + M_2}$$ **0.2**

correct solution:

$$d = \frac{bM}{\pi h_2 r_2^2(\rho_2 - \rho_1)}$$ **0.2**

A4 (0.7 pt) Find an expression for the moment of inertia I_S in terms of b and the known quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

Solution A4: **[0.7]**

correct answer for moment of inertia of homogeneous disk

$$I_1 = \frac{1}{2}\pi h_1 \rho_1 r_1^4$$ **0.2**

Mass wrong **-0.1**
Factor 1/2 wrong in formula for moment of inertia of a disk **-0.1**
Correct answer for moment of inertia of 'excess' disk:

$$I_2 = \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4$$ **0.2**

Using Steiner's theorem:

$$I_S = I_1 + I_2 + d^2 \pi r_2^2 h_2(\rho_2 - \rho_1)$$ **0.1**

correct solution:

$$I_S = \frac{1}{2}\pi h_1 \rho_1 r_1^4 + \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4 + \frac{b^2 M^2}{\pi r_2^2 h_2(\rho_2 - \rho_1)}$$ **0.2**

In terms of d rather than b gives 0.1pts rather than 0.2pts for the final answer: **0.1**

$$I_S = \frac{1}{2}\pi h_1 \rho_1 r_1^4 + \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4 + d^2 \pi r_2^2 h_2(\rho_2 - \rho_1)$$

A5 (1.1 pt) Using all the above results, write down an expression for h_2 and r_2 in terms of b, T and the quantities (1). You may express h_2 as a function of r_2.

Solution A5: **[1.1]**

It is not clear how exactly students will attempt to solve this system of equations. It is likely that they will use the following equation:

$$M = \pi r_1^2 h_1 \rho_1 + \pi r_2^2 h_2(\rho_2 - \rho_1) \ .$$ **0.3**

solve I_S for r_2^2:

$$r_2^2 = \frac{2}{M - \pi r_1^2 h_1 \rho_1}\left(I_S - \frac{1}{2}\pi h_1 \rho_1 r_1^4 - b^2 \frac{M^2}{M - \pi r_1^2 h_1 \rho_1}\right)$$ **0.4**

replace I_S by T:

$$I_S = \frac{MgbT^2}{4\pi^2}$$ **0.1**

solve correctly for r_2:

$$r_2 = \sqrt{\frac{2}{M - \pi r_1^2 h_1 \rho_1}\left(M\frac{bgT^2}{4\pi^2} - \frac{1}{2}\pi h_1 \rho_1 r_1^4 - b^2 \frac{M^2}{M - \pi r_1^2 h_1 \rho_1}\right)}$$ **0.1**

write down an equation for h_2 along the lines of $M = \pi r_1^2 \rho_1 h_1 + \pi r_2^2(\rho_2 - \rho_1) h_2$ and solve it correctly:

$$h_2 = \frac{M - \pi r_1^2 \rho_1 h_1}{\pi r_2^2(\rho_2 - \rho_1)}$$ **0.2**

Part B. Rotating Space Station (6.5 points)

B1 (0.5 pt) At what angular frequency ω_{ss} does the space station rotate so that the astronauts experience the same gravity g_E as on the Earth's surface?

Solution B1: **[0.5]**

An equation for the centrifugal force along the lines of

$$F_{ce} = m\omega^2 r$$ **0.1**

Balancing the forces, correct equation

$$g_E = \omega_{ss}^2 R$$ **0.2**

Correct solution

$$\omega_{ss} = \sqrt{g_E/R}$$ **0.2**

B2 (0.2 pt) Assuming that on Earth gravity is constant with acceleration g_E, what would be the angular oscillation frequency ω_E that a person on Earth would measure?

Solution B2: **[0.2]**

Realize that result is independent of g_E **0.1**

Correct result:

$$\omega_E = \sqrt{k/m}$$ **0.1**

B3 (0.6 pt) What angular oscillation frequency ω does Alice measure on the space station?

Solution B3: **[0.6]**

some version of the correct equation for force

$$F = -kx \pm m\omega_{ss}^2 x$$ **0.2**

getting the sign right

$$F = -kx + m\omega_{ss}^2 x$$ **0.2**

Find correct differential equation

$$m\ddot{x} + (k - m\omega_{ss}^2)x = 0$$ **0.1**

Derive correct result

$$\omega = \sqrt{k/m - \omega_{ss}^2}$$ **0.1**

Using g_E/R instead of ω_{ss}^2 is also correct.

B4 (0.8 pt) Derive an expression of the gravity $g_E(h)$ for small heights h above the surface of the Earth and compute the oscillation frequency $\tilde{\omega}_E$ (linear approximation is enough). The radius of the Earth is given by R_E.

Solution B4: **[0.8]**

$$g_E(h) = -GM/(R_E + h)^2 \quad \textbf{0.1}$$

linear approximation of gravity:

$$g_E(h) = -\frac{GM}{R_E^2} + 2h\frac{GM}{R_E^3} + \dots \quad \textbf{0.2}$$

Realize that $g_E = GM/R_E^2$:

$$g_E(h) = -g_E + 2hg_E/R_E + \dots \quad \textbf{0.1}$$

Opposite sign is also correct, as long as it is opposite in both terms.
Realize what this means for force, i.e. that the constant term can be eliminated by shifting the equilibrium point:

$$F = -kx + 2xmg_E/R_E \quad \textbf{0.2}$$

Find correct differential equation

$$m\ddot{x} + (k - 2mg_E/R_E)x = 0 \quad \textbf{0.1}$$

correct result

$$\tilde{\omega}_E = \sqrt{k/m - 2g_E/R_E} \quad \textbf{0.1}$$

No points are deducted if student answers with $\tilde{\omega}_E/(2\pi)$ because "oscillation frequency" might also be interpreted as inverse period.

B5 (0.3 pt) For what radius R of the space station does the oscillation frequency ω match the oscillation frequency $\tilde{\omega}_E$ on the surface of the Earth? Express your answer in terms of R_E.

Solution B5: **[0.3]**

Write down equation

$$\omega_{ss}^2 = 2g_E/R_E \quad \textbf{0.1}$$

Solve

$$R = R_E/2 \quad \textbf{0.2}$$

If GM/R_E^2 rather than g_E is used, give only *0.1pt*.

B6 (1.1 pt) Calculate the horizontal velocity v_x and the horizontal displacement d_x (relative to the base of the tower, in the direction perpendicular to the tower) of the mass at the moment it hits the floor. You may assume that the height H of the tower is small, so that the acceleration as measured by the astronauts is constant during the fall. Also, you may assume that $d_x \ll H$.

Solution B6: [1.1]

There are several possible solutions.

Solution one – Using Coriolis force

- Velocity v_x

  Equation for Coriolis force with correct velocity:

  $$F_C(t) = 2m\omega_{ss}^2 Rt\omega_{ss} = 2m\omega_{ss}^3 Rt$$ **0.1**

  Integrate this, or realize that it is like uniform acceleration for the velocity:

  $$v_x(t) = \omega_{ss}^3 Rt^2$$ **0.2**

  plug in correct value for

  $$t = \sqrt{2H/\omega_{ss}^2 R}$$ **0.2**

  overall correct result

  $$v_x = 2H\omega_{ss}$$ **0.1**

- The displacement d_x:

  Integrate $v_x(t)$:

  $$d_x = \frac{1}{3}R\omega_{ss}^3 t^3$$ **0.3**

  Instead of integrating, students may simply 'average' by taking $\frac{1}{2}$ of the final velocity. This gives a factor of $\frac{1}{2}$ instead of $\frac{1}{3}$. *Deduct a total of 0.1 pts for this.* **-0.1**

  Plug in value for t

  $$d_x = \frac{1}{3}R\omega_{ss}^3(2H/\omega_{ss}^2 R)^{3/2} = \frac{1}{3}2^{3/2}H^{3/2}R^{-1/2} = \frac{1}{3}\sqrt{\frac{8H^3}{R}}$$ **0.2**

Solution two – Using inertial frame This solution is similar to the way to solve B7, but needs more complicated approximations than Solution one.

- v_x

  Here ϕ denotes the angle swept by the mass and α the angle the astronauts (and tower) has rotated when the mass lands on the floor, see

  Initially the velocity of the mass in an inertial frame is $v_x = \omega_{ss}(R - H)$. **0.1**

  When the mass lands, the x-direction has been rotated by ϕ so the new horizontal velocity component is then

  $$\omega_{ss}(R - H)\cos\phi$$ **0.1**

  (Student may also write $\cos\alpha$ instead of $\cos\phi$, since $d_x \ll H$.)

  $$\cos\phi = \frac{R - H}{R} = 1 - \frac{H}{R}$$ **0.1**

  Transforming to the rotating reference frame, one needs to subtract $\omega_{ss}R$. **0.1**

  Finally in the reference frame of the astronauts

  $$v_x = \omega_{ss}R\left(1 - \frac{H}{R}\right)^2 - \omega_{ss}R \approx \omega_{ss}R\left(1 - 2\frac{H}{R}\right) - \omega_{ss}R = -2\omega_{ss}H$$ **0.2**

  The sign of the velocity depend on the choice of reference direction, so a positive sign is also correct.

- d_x

  With the notation from the calculation of v_x

$$d_x = (\alpha - \phi)R$$ **0.1**

$$\phi = \arccos\left(1 - \frac{H}{R}\right)$$

$$\alpha = \omega_{ss} t$$

where t is the fall time of the mass, which is given by

$$t = \frac{\sqrt{R^2 - (R-H)^2}}{\omega_{ss}(R-H)}$$ **0.1**

(see solution to B7)

Writing $\xi \equiv H/R$ this means

$$d_x = \left[\frac{\sqrt{1-(1-\xi)^2}}{1-\xi} - \arccos(1-\xi)\right] R$$ **0.3**

which is a valid end answer to the problem. It is possible, but not necessary, to approximate this for small ξ:

$$\arccos(1-\xi) \approx \sqrt{2\xi}\left(1 + \frac{\xi}{12}\right)$$

which after insertion into the equation for d_x and approximation of small ξ yields the same result as in Solution one:

$$d_x = \frac{2}{3}\sqrt{\frac{2H^3}{R}}$$

If this end answer misses the factor 2/3, deduct 0.1 points. **-0.1**

Solution three – Inertial frame with geometry trick

This is an alternative solution to obtain d_x

The mass travels the distance l, and during the fall the space station rotates by ϕ, see Figure 2. According to the intersecting chord theorem,

$$l^2 = H(2R - H)$$ **0.1**

The rotated angle is $\phi = \omega_{ss} t$ where

$$t = \frac{l}{R-H}$$ **0.1**

is the fall time. Thus

$$\phi = \frac{\sqrt{H(2R-H)}}{R-H}$$ **0.1**

$$\frac{d}{R} = \phi - \arcsin\frac{l}{R} = \frac{\sqrt{H(2R-H)}}{R-H} - \arcsin\sqrt{x(2-x)}$$ **0.1**

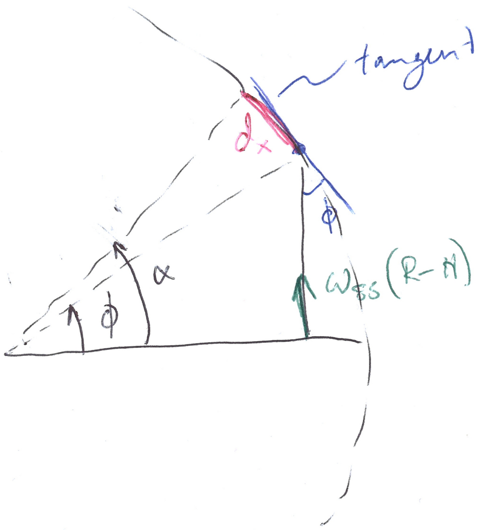


Figure 1: Notation for solution two

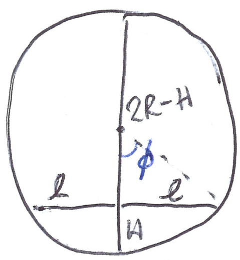


Figure 2: Notation for solution three.

Denote $x \equiv H/R$ and $y \equiv \sqrt{x(2-x)}$. Since

$$\arcsin y \approx y + \frac{y^3}{6}$$

one gets

$$\frac{d}{R} \approx y(1+x) - y - y^3/6 = y(x - y^2/6) \approx 2xy/3 \approx 2x\sqrt{2x}/3 = \frac{2}{3}\sqrt{\frac{2H^3}{R}}$$

Final answer **0.1**

B7 (1.3 pt) Find a lower bound for the height of the tower for which it can happen that $d_x = 0$.

Solution B7: **[1.3]**

The key is to use a non-rotating frame of reference. If the mass is released close enough

to the center, its linear velocity will be small enough for the space station to rotate more than 2π before it hits the ground.
The velocity is given by

$$v = \omega_{ss}(R - H)$$ **0.1**

distance d that the mass flies before hitting the space station

$$d^2 = R^2 - (R - H)^2$$ **0.1**

use non-rotating frame of reference to obtain time t until impact

$$t = d/v = \frac{\sqrt{R^2 - (R-H)^2}}{\omega_{ss}(R - H)}$$ **0.1**

Now there are several possible ways to relate H and the rotated angle ϕ of the space station:
Solution one

$$t = \frac{R\sin\phi}{\omega_{ss} R\cos\phi}$$ **0.2**

This time must match $t = \phi/\omega_{ss}$. Obtain the equation

$$\phi = \tan\phi$$ **0.2**

Realizing that there is an infinite number of solutions. **0.2**
This equation has one trivial solution $\phi = 0$, next solution is slightly less than $3\pi/2$ which corresponds to the case $H > R$ (and is thus not correct). The one that gives a lower bound for H is the third solution

$$\phi \approx 5\pi/2$$

The equation $\phi = \tan\phi$ can be solved graphically or numerically to obtain a close value ($\phi = 7.725$ rad) which means

$$H/R = (1 - \cos\phi) \approx 0.871$$

Give points if the method is correct, depending on the value of H/R found, according to these intervals: **0.4**
$0.85 \leq H/R \leq 0.88$: *0.4 pts*
$0.5 \leq H/R < 0.85$: *0.3 pts*
$0 < H/R < 0.5$ or $H > 0.88$: *0.2 pts*
$H = 0$ or method is incorrect: *0 pts*

Solution two
relation between H and rotated angle ϕ

$$\frac{R - H}{R} = \cos\phi$$ **0.2**

obtain equation of the form

$$\frac{H}{R} = 1 - \cos\left(\frac{\sqrt{1 - (1 - H/R)^2}}{1 - H/R}\right)$$ **0.2**

Figure 3 gives a plot of $f(x) = 1 - \cos\left(\frac{\sqrt{1-(1-x)^2}}{1-x}\right)$. The goal is to find an approximate solution for the second intersection. The first intersection is discarded – it is introduced because of $\cos\phi = \cos(-\phi)$ and corresponds to a situation with $H > R$.
Realizing that there is an infinite number of solutions. **0.2**

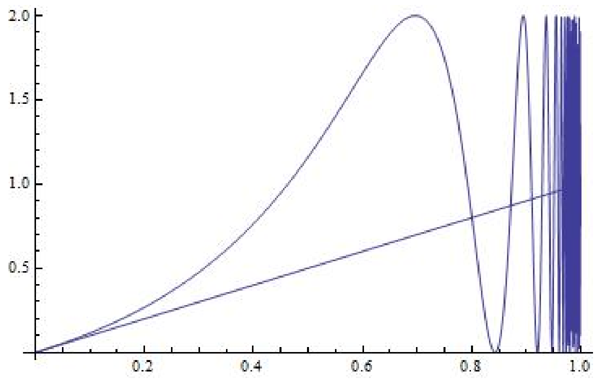


Figure 3: Plot of $f(H/R)$ and H/R

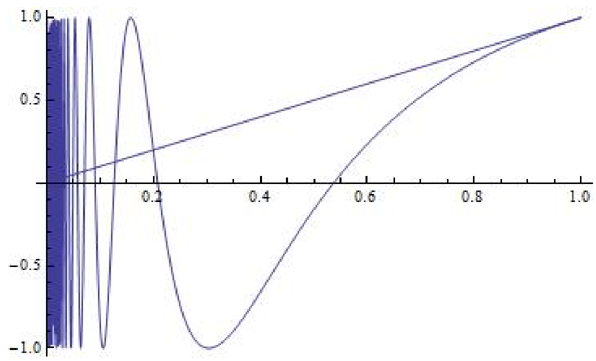


Figure 4: Plot of $g(x)$ and x

- introduce new variable $x := 1 - H/R$, so that the equation becomes

$$x = \cos(\sqrt{1-x^2}/x) =: g(x)$$

- $g(x)$ is then smaller than x up to the first solution. In particular it is negative in some region (see figure 4). Finding the third zero thus gives a lower bound for the solution:

$$\frac{\sqrt{1-x^2}}{x} = 5\pi/2$$

- give lower bound

$$x = 1/\sqrt{25\pi^2/4+1} \Rightarrow H = R(1 - 1/\sqrt{25\pi^2/4+1}) \approx 0.874$$

Note: the actual result is $H/R = 0.871\ldots$.

Use the same points for the numerical answer as was mentioned in solution one. **0.4**

If the student plots f rather than g, find solution to $f = 1$: is equivalent to the solution above. *Give same number of points.*

It is also possible to use $\cos\left(\frac{\sqrt{1-x^2}}{x}\right) = \sin(1/x)$.

B8 (1.7 pt) Alice pulls the mass a distance d downwards from the equilibrium point $x = 0$, $y = 0$, and then lets it go (see figure 4).

- Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss}d$ is small.
- Sketch the trajectory $(x(t), y(t))$, marking all important features such as amplitude.

Solution B8: **[1.7]**

Note: we did not specify the overall sign of the Coriolis force. Give same amount of points if using opposite convention, but it has to be consistent! Otherwise: subtract *0.1pt* for each instance of inconsistency. **-0.1**
Students are allowed to express everything in terms of ω, they don't need to write $\sqrt{k/m - \omega_{ss}^2}$ explicitly. Deduct *0.1pt* however if they use k/m instead of ω.. **-0.1**

Realize that $y(t)$ is standard harmonic oscillation:

$$y(t) = A\cos\omega t + B$$ **0.1**

Give correct constants from initial conditions

$$y(t) = -d\cos\omega t$$ **0.2**

Correct expression for $v_y(t)$:

$$v_y(t) = -d\omega\sin\omega t$$ **0.1**

Coriolis force in x-direction

$$F_x(t) = 2m\omega_{ss}v_y(t) = -2m\omega_{ss}d\omega\sin\omega t$$ **0.2**

Realize that this implies that $x(t)$ is also a harmonic oscillation... **0.1**

...but with a constant movement term superimposed: vt **0.1**

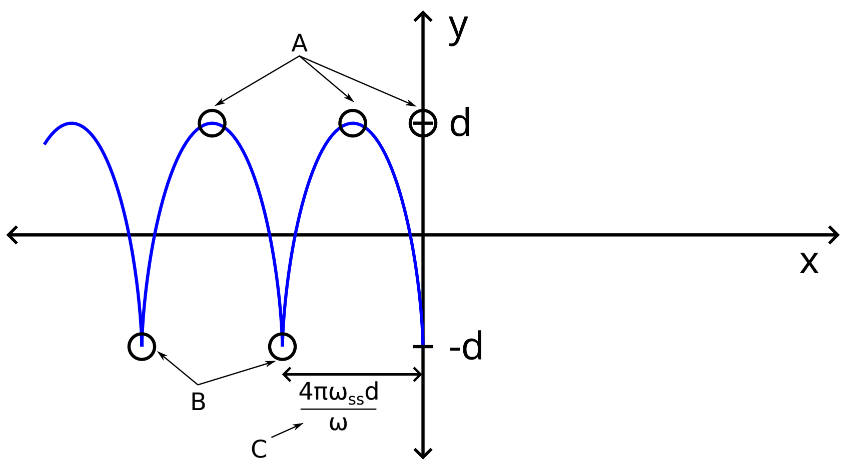
getting the correct amplitude:

$$A = \frac{2\omega_{ss}d}{\omega}$$ **0.1**

Correct answer with correct initial conditions:

$$x(t) = \frac{2\omega_{ss}d}{\omega}\sin\omega t - 2\omega_{ss}dt$$ **0.2**

Sketch:



Correct qualitative sketch:	
periodic motion	**0.1**
overall constant movement	**0.1**
B): cusps	**0.1**
And additionally correct quantitative sketch:	
A)+B): peaks and cusps are at $y = \pm d$	**0.1**
C): cusps are at distance $\Delta x = \dfrac{4\pi\omega_{ss} d}{\omega}$ from each other	**0.2**

Problem 2 : Solution/marking scheme – Nonlinear Dynamics in Electric Circuits *(10 points)*

Part A. Stationary states and instabilities (3 points)

Solution A1: **[0.4]**

By looking at the $I-V$ graph, we obtain

$$R_{\text{off}} = 10.0\,\Omega,$$ **0.1**

$$R_{\text{on}} = 1.00\,\Omega,$$ **0.1**

$$R_{\text{int}} = 2.00\,\Omega,$$ **0.1**

$$I_0 = 6.00\,A.$$ **0.1**

Note: No penalty for the number of digits in this question

Solution A2: **[1]**

Kirchoff law for the circuit (U is the voltage of the bistable element):

$$\mathcal{E} = IR + U$$ **0.1**

This yields

$$I = \frac{\mathcal{E} - U}{R}$$ **0.1**

Hence, stationary states of the circuit are intersections of the line defined by this equation and the $I-V$ graph of X. **0.2**

For $R = 3.00\,\Omega$, one always gets exactly one intersection. **0.2**
For $R = 1.00\,\Omega$, one gets 1, 2 or 3 intersections depending on the value of $\mathcal{E}$. **0.4**
The following table summarizes the number of points granted for possible answers to the last subquestion with $R = 1.00\,\Omega$:

Possible answer	1	2	3	1,3	1,2	2,3	1,2,3
Points	0	0	0.2	0.3	0	0.2	0.4

Solution A3: **[0.6]**

The stationary state is on the intermediate branch, one can thus use the corresponding equation: **0.2**

$$I_{\text{stationary}} = \frac{\mathcal{E} - R_{\text{int}} I_0}{R - R_{\text{int}}} \quad \textbf{0.1}$$

$$= 3.00\,A \quad \textbf{0.1}$$

$$U_{\text{stationary}} = R_{\text{int}}(I_0 - I) \quad \textbf{0.1}$$

$$= 6.00\,V \quad \textbf{0.1}$$

Extra (non-physical) stationary states on the switched on and/or switched off branches lead to a penalty of 0.2 point.

Solution A4: **[1]**

Any correct modeling such as the following: **0.5**
The Kirchoff law for the circuit reads

$$\mathcal{E} = IR + U_X + L\frac{dI}{dt} = IR + (I_0 - I)R_{\text{int}} + L\frac{dI}{dt}$$

This implies

$$L\frac{dI}{dt} = \mathcal{E} - I_0 R_{\text{int}} - (R - R_{\text{int}})I$$

The separation between two cases is of importance, especially because of the relative sign of dI/dt:
If $I > I_{\text{stationary}}$, we have $dI/dt < 0$ and I decreases. **0.2**
If $I < I_{\text{stationary}}$, we have $dI/dt > 0$ and I increases. **0.2**
Note: Formulas with time derivatives are not essential, any other correct justification is accepted.
We conclude that the stationary state is stable. **0.1**

Note: The checkbox gives 0.1 points if "stable" is checked, regardless of the previous reasoning (also if there is nothing). A wrong reasoning leading to check the "unstable" option doesn't however give any point for the checkbox.

Part B. Bistable non-linear elements in physics and engineering: radio transmitter (5 points)

Solution B1: **[1.8]**

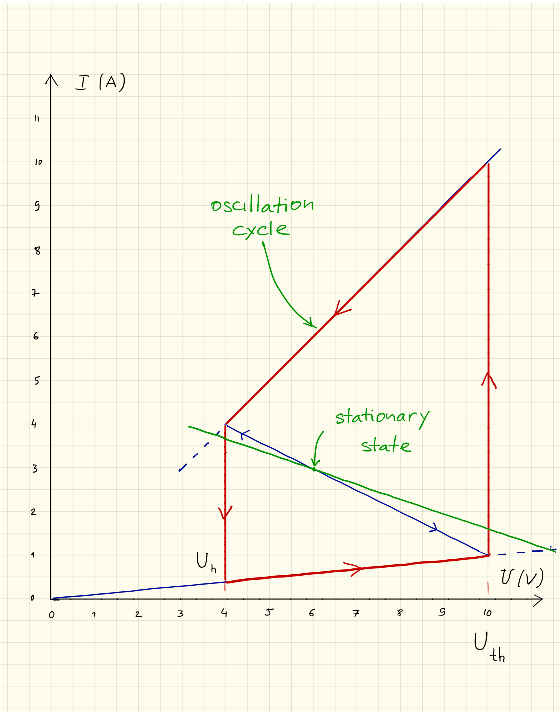
A correctly drawn cycle gives 1.2 points, distributed as follows:

- Switched on branch is part of the cycle **0.2**
- Switched off branch is part of the cycle **0.2**
- Jumps are vertical (constant U) **0.2**
- Jumps are positioned at U_{h} and U_{th} **0.2**
- The system moves to the left on the switched on branch **0.2**

- The system moves to the right on the switched off branch **0.2**

Each of the following observations individually gives up to 0.2 points, but their total cannot exceed 0.6:

- U constant during jumps because the charge on the capacitor cannot change instantaneously **0.2**
- The intermediate branch cannot be part of the cycle because there is a stationary state on it **0.2**
- Jumps occur at corners of the IV graph because at those points the system has nowhere else to go **0.2**
- The system moves moves to the left on the switched on branch because it approaches the stable stationary state (which is located outside the IV graph), or argument with the Kirchoff law **0.2**
- The system moves moves to the right on the switched off branch because it approaches the stable stationary state (which is located outside the IV graph), or argument with the Kirchoff law **0.2**



Solution B2: **[1.9]**

Since the non-linear element is oscillating between the switched on and switched off branches we can put $U_X = R_{\text{on/off}} I_X$. On either of the branches, the circuit behaves as a standard RC-circuit with conductance C and resistance $R_{\text{on/off}}R/(R_{\text{on/off}} + R)$ (the resistor and the element X being connected in parallel). Another way to express it is to **0.5** write the Kirchhoff law for the switched on and switched off branches

$$R_{\text{on/off}} RC \frac{dI_X}{dt} = \mathcal{E} - (R_{\text{on/off}} + R) I_X$$

The time constant of the circuit is

$$\frac{R_{\text{on/off}} R}{R_{\text{on/off}} + R} C.$$

If the branch in question (switched on or switched off) extended indefinitely, after a long time the system would have landed in a stationary state with the voltage

$$U_{\text{on/off}} = \frac{R_{\text{on/off}}}{R_{\text{on/off}} + R} \mathcal{E}.$$

Then, the time dependence of the voltage drop on the non-linear element is a sum of the constant term $U_{\text{on/off}}$ and of the exponentially decaying term:

$$U_X(t) = \frac{R_{\text{on/off}}}{R_{\text{on/off}} + R} \mathcal{E} + \left(U_{\text{on/off}} - \frac{R_{\text{on/off}}}{R_{\text{on/off}} + R} \mathcal{E} \right) e^{-\frac{R_{\text{on/off}} + R}{R_{\text{on/off}} RC} t}$$

There are 0.5 points distributed as follow for $U_X(t)$:

- Correct exponential **0.2**
- Correct constant term $(t \to \infty)$ **0.1**
- Correct coefficient in front of the exponential **0.1**
- Correct equation for $U_X(t)$ **0.1**

Time spent by the system on the switched on branch during one cycle:

$$t_{\text{on}} = \frac{R_{\text{on}} R}{R_{\text{on}} + R} C \log \left(\frac{U_{\text{th}} - U_{\text{on}}}{U_{\text{h}} - U_{\text{on}}} \right) = 2.41 \cdot 10^{-6}\, s,$$ **0.4**

Time spent by the system on the switched off branch during one cycle:

$$t_{\text{off}} = \frac{R_{\text{off}} R}{R_{\text{off}} + R} C \log \left(\frac{U_{\text{off}} - U_{\text{h}}}{U_{\text{off}} - U_{\text{th}}} \right) = 3.71 \cdot 10^{-6}\, s.$$ **0.4**

The total period of oscillations:

$$T = t_{\text{on}} + t_{\text{off}} = 6.12 \cdot 10^{-6}\, s$$ **0.1**

Note: Correct final answers give full points. One may earn points for intermediate steps (see above) for partial answers.

Solution B3: **[0.7]**

Neglect the energy consumed on the switched off branch. The energy consumed on the switched on branch during the cycle is estimated by

$$E = \frac{1}{R_{\text{on}}} \left(\frac{U_h + U_{th}}{2} \right)^2 t_{\text{on}} = 1.18 \cdot 10^{-4}\,\text{J}.$$ **0.4**

For the power, this gives an estimate of

$$P \sim \frac{E}{T} = 19.3\,\text{W}.$$ **0.3**

Note:

- *Formula + answer inside* $5\,\text{W} \le P \le 50\,\text{W}$ *give full points*
- *Formula + answer outside the range above but inside* $1\,\text{W} \le P \le 100\,\text{W}$ *give 0.5 points*
- *answer outside range but good formula gives 0.4 points*

Also, the proposed formula is only an example, any other reasonable approximation of the integral of the upper branch should be accepted.

Solution B4: **[0.6]**

The wave length of the radio signal is given by $\lambda = cT = 1.82 \cdot 10^3\,\text{m}$. **0.2**
The optimal length of the antenna is $\lambda/4$ (or $3\lambda/4, 5\lambda/4$ *etc.*) **0.3**
The only choice which is below 1 km is $s = \lambda/4 = 459\,\text{m}$. **0.1**
Note: The correct answer $s = \lambda/4 = 459\,\text{m}$ *gives full points, and the mistake* $s = \lambda/2 = 918\,\text{m}$ *only 0.4 pts.*

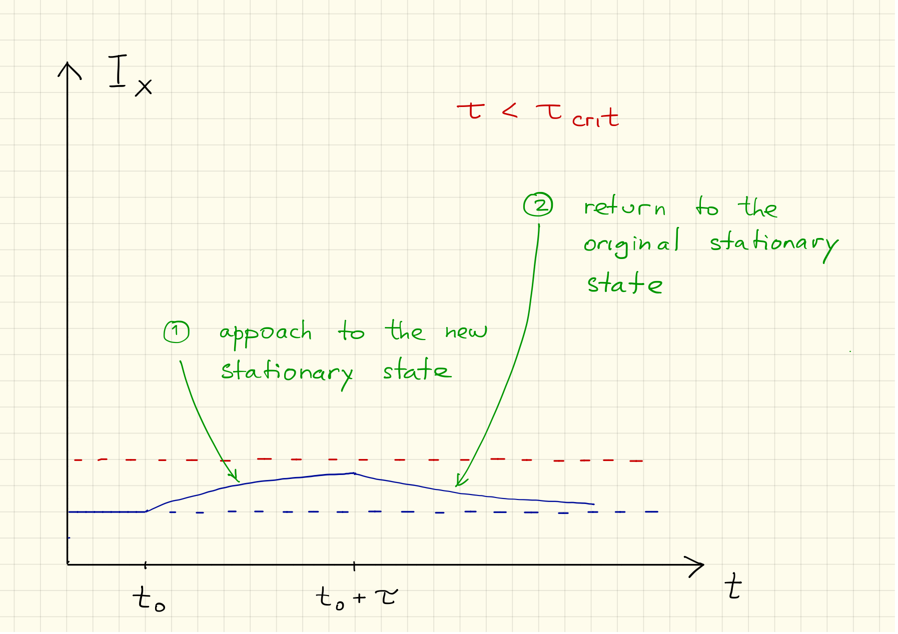
Part C. Bistable non-linear elements in biology: neuristor (2 points)

Solution C1: **[1.2]**

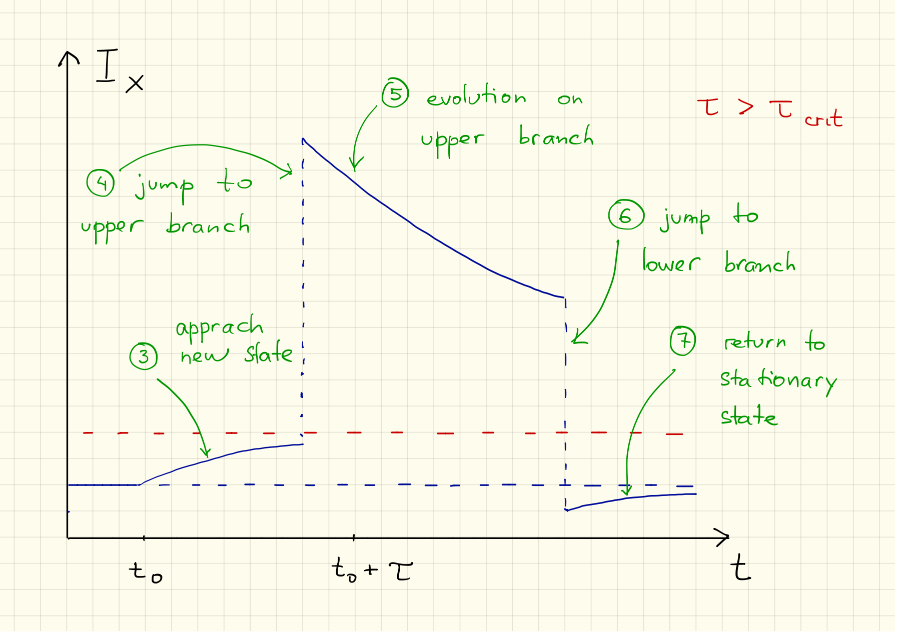
For $\tilde{\mathcal{E}} = 12.0\,\text{V}$, the steady state of the system is located on the switched off branch:

$$\tilde{U} = \frac{R_{\text{off}}}{R + R_{\text{off}}} \tilde{\mathcal{E}} = 9.23\,\text{V}.$$

When the voltage is increased to $\mathcal{E} = 15.0\,\text{V}$, the system starts moving to the right along the switched off branch (in the same way it did in task **B**).
If the voltage drops again before the system reaches the threshold voltage, it will simply return to the stationary state.
If system reaches the threshold voltage, it will jump to the switched on branch, and it will make one oscillations (since $\tau < T$) before the voltage drops again and it returns to the stationary state.



1. Approach to the new stationary state **0.2**
2. Return to the old stationary state **0.2**



3. Approach to the new stationary state **0.1**
4. Jump to the upper branche before $t_0 + \tau$ **0.2**
5. Evolution on the upper branch **0.2**
6. Jump to the lower branche below the old stationary state **0.1**
7. Return to the old stationary state (from below) **0.2**

Solution C2: **[0.6]**

The time needed to reach the threshold voltage is given by

$$\tau_{\text{crit}} = \frac{R_{\text{off}} R}{R_{\text{off}} + R} C \log\left(\frac{U_{\text{off}} - \tilde{U}}{U_{\text{off}} - U_{\text{th}}}\right) = 9.36 \cdot 10^{-7}\, s.$$

Note: This is the same formula as for t_{off} in task **B2**, with U_h replaced by $\tilde{U}$.

- Correct time constant **0.2**
- Correct choice of voltages **0.2**
- Correct final formula **0.1**
- Correct numerical value **0.1**

Note: Correct final answers give full points. One may earn points for intermediate steps (see above) for partial answers.

Solution C3: **[0.2]**

Since $\tau > \tau_{\text{crit}}$, the system will make one oscillation. We conclude that the system is a neuristor. **0.2**

Note: 0.2 are given only if "Yes" is checked, regardless of the development of the other tasks.

Problem 3 : Solution/marking scheme – Large Hadron Collider *(10 points)*

Part A. LHC Accelerator (6 points)

A1 (0.7 pt) Find the exact expression for the final velocity v of the protons as a function of the accelerating voltage V, and fundamental constants.

Solution A1: **[0.7]**

Conservation of energy:

$$m_p \cdot c^2 + V \cdot e = m_p \cdot c^2 \cdot \gamma = \frac{m_p \cdot c^2}{\sqrt{1 - v^2/c^2}}$$ **0.5**

Penalties

No or incorrect total energy **-0.3**

Missing rest mass **-0.2**

Solve for velocity:

$$v = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + V \cdot e}\right)^2}$$ **0.2**

---

without proton rest mass: **[0.5]**

$$V \cdot e \simeq m_p \cdot c^2 \cdot \gamma = \frac{m_p \cdot c^2}{\sqrt{1 - v^2/c^2}}$$ **0.3**

Solve for velocity:

$$v = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{V \cdot e}\right)^2}$$ **0.2**

Classical solution: **[0.2]**

$$v = \sqrt{\frac{2 \cdot e \cdot V}{m_p}}$$ **0.2**

A2 (0.8 pt) For particles with high energy and low rest mass the relative deviation $\Delta = (c - v)/c$ of the final velocity v from the speed of light is very small. Find a suitable approximation for Δ and calculate Δ for electrons with an energy of 60.0 GeV.

Solution A2: **[0.8]**

velocity (from previous question):

$$v = c \cdot \sqrt{1 - \left(\frac{m_e \cdot c^2}{m_e \cdot c^2 + V \cdot e}\right)^2} \quad \text{or} \quad c \cdot \sqrt{1 - \left(\frac{m_e \cdot c^2}{V \cdot e}\right)^2}$$ **0.1**

relative difference:

$$\Delta = \frac{c - v}{c} = 1 - \frac{v}{c}$$ **0.1**

$$\rightarrow \Delta \simeq \frac{1}{2}\left(\frac{m_e \cdot c^2}{m_e \cdot c^2 + V \cdot e}\right)^2 \quad \text{or} \quad \frac{1}{2}\left(\frac{m_e \cdot c^2}{V \cdot e}\right)^2$$ **0.4**

relative difference

$$\Delta = 3.63 \cdot 10^{-11}$$ **0.2**

---

classical solution gives no points **0.0**

A3 (1.0 pt) Derive an expression for the uniform magnetic flux density B necessary to keep the proton beam on a circular track. The expression should only contain the energy of the protons E, the circumference L, fundamental constants and numbers. You may use suitable approximations if their effect is smaller than the precision given by the least number of significant digits. Calculate the magnetic flux density B for a proton energy of $E = 7.00$ TeV.

Solution A3: **[1.0]**

Balance of forces:

$$\frac{\gamma \cdot m_p \cdot v^2}{r} = \frac{m_p \cdot v^2}{r \cdot \sqrt{1 - \frac{v^2}{c^2}}} = e \cdot v \cdot B \qquad \mathbf{0.3}$$

In case of a mistake, partial points can be given for intermediate steps (up to max 0.2). Examples:

Example:	Lorentz force	**0.1**
Example:	$\frac{\gamma \cdot m_p \cdot v^2}{r}$	**0.1**

Energy:

$$E = (\gamma - 1) \cdot m_p \cdot c^2 \simeq \gamma \cdot m_p \cdot c^2 \rightarrow \gamma = \frac{E}{m_p c^2}$$

Therefore:

$$\frac{E \cdot v}{c^2 \cdot r} = e \cdot B \qquad \mathbf{0.3}$$

With

$$v \simeq c \text{ and } r = \frac{L}{2\pi}$$

follows:

$$\rightarrow B = \frac{2\pi \cdot E}{e \cdot c \cdot L} \qquad \mathbf{0.2}$$

Solution:

$$B = 5.50\text{T} \qquad \mathbf{0.2}$$

Penalty for < 2 or > 4 significant digits **-0.1**

Calculation without approximations is also correct but does not give more points

$$B = \frac{2\pi \cdot m_p \cdot c}{e \cdot L} \cdot \sqrt{\left(\frac{E}{m_p \cdot c^2}\right)^2 - \left(1 + \frac{m \cdot c^2}{E}\right)^2} \qquad \mathbf{0.5}$$

Penalty for each algebraic mistake **-0.1**

Classical calulation gives completely wrong result and maximum 0.3 pt **[0.3]**

$$\frac{m_p \cdot v^2}{r} = e \cdot v \cdot B \qquad \mathbf{0.1}$$

$$B = \frac{2\pi}{L \cdot e}\sqrt{2 \cdot m_p \cdot E} \qquad \mathbf{0.1}$$

$$B = 0.0901\text{T}$$	**0.1**
Penalty for < 2 or > 4 significant digits	**-0.1**

A4 (1.0 pt) An accelerated charged particle radiates energy in the form of electromagnetic waves. The radiated power P_{rad} of a charged particle that circulates with a constant angular velocity depends only on its acceleration a, its charge q, the speed of light c and the permittivity of free space ϵ_0. Use a dimensional analysis to find an expression for the radiated power P_{rad}.

Solution A4: **[1.0]**

Ansatz:

$$P_{rad} = a^{\alpha} \cdot q^{\beta} \cdot c^{\gamma} \cdot \epsilon_0^{\delta}$$ **0.2**

Dimensions: [a]=$\mathrm{ms^{-2}}$, [q]=C=As, [c]=$\mathrm{ms^{-1}}$,[ϵ_0]=As(Vm)$^{-1}$=A^{2}s^2(Nm^2)$^{-1}$=A^{2}s^4(kgm^3)$^{-1}$

All dimensions correct **0.3**

Three dimensions correct **0.2**

Two dimensions correct **0.1**

---

if dimensions: N and Coulomb [ϵ_0]= C^2(Nm^2)$^{-1}$

$$\frac{\mathrm{m}^{\alpha}}{\mathrm{s}^{2\alpha}} \cdot \mathrm{C}^{\beta} \cdot \frac{\mathrm{m}^{\gamma}}{\mathrm{s}^{\gamma}} \cdot \frac{\mathrm{C}^{2\delta}}{\mathrm{N}^{\delta} \cdot \mathrm{m}^{2\delta}} = \frac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{s}}$$ **0.1**

From this follows:

$$\mathrm{N} :\to \delta = -1, \quad \mathrm{C} :\to \beta + 2 \cdot \delta = 0, \quad \mathrm{m} :\to \alpha + \gamma - 2\delta = 1, \quad \mathrm{s} :\to 2 \cdot \alpha + \gamma = 1$$ **0.2**

Two equations correct **0.1**

And therefore:

$$\to \alpha = 2, \beta = 2, \gamma = -3, \delta = -1$$ **0.1**

---

if dimensions: N and As [ϵ_0]=A^{2}s^2(Nm^2)$^{-1}$

$$\frac{\mathrm{m}^{\alpha}}{\mathrm{s}^{2\alpha}} \cdot \mathrm{A}^{\beta} \cdot \mathrm{s}^{\beta} \cdot \frac{\mathrm{m}^{\gamma}}{\mathrm{s}^{\gamma}} \cdot \frac{\mathrm{A}^{2\delta} \cdot \mathrm{s}^{2\delta}}{\mathrm{N}^{\delta} \cdot \mathrm{m}^{2\delta}} = \frac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{s}}$$ **0.1**

From this follows:

$$N :\to \delta = -1, \quad \mathrm{A} :\to \beta + 2 \cdot \delta = 0, \quad \mathrm{m} :\to \alpha + \gamma - 2\delta = 1, \quad \mathrm{s} :\to -2 \cdot \alpha + \beta - \gamma + 2\delta = -1$$ **0.2**

Two equations correct **0.1**

And therefore:

$$\to \alpha = 2, \beta = 2, \gamma = -3, \delta = -1$$ **0.1**

---

if dimensions: kg and As [ϵ_0]=A^{2}s^4(kg· m^3)$^{-1}$

$$\frac{\mathrm{m}^{\alpha}}{\mathrm{s}^{2\alpha}} \cdot \mathrm{A}^{\beta} \cdot \mathrm{s}^{\beta} \cdot \frac{\mathrm{m}^{\gamma}}{\mathrm{s}^{\gamma}} \cdot \frac{\mathrm{A}^{2\delta} \cdot \mathrm{s}^{4\delta}}{\mathrm{kg}^{\delta} \cdot \mathrm{m}^{3\delta}} = \frac{\mathrm{kg} \cdot m^2}{s^3}$$ **0.1**

From this follows:

$$\text{kg} :\rightarrow \delta = -1, \quad \text{A} :\rightarrow \beta + 2 \cdot \delta = 0, \quad \text{m} :\rightarrow \alpha + \gamma - 3\delta = 2, \quad \text{s} :\rightarrow -2 \cdot \alpha + \beta - \gamma + 4\delta = -3$$ **0.2**

Two equations correct **0.1**

And therefore:

$$\rightarrow \alpha = 2, \beta = 2, \gamma = -3, \delta = -1$$ **0.1**

Radiated Power:

$$P_{rad} \propto \frac{a^2 \cdot q^2}{c^3 \cdot \epsilon_0}$$ **0.1**

Other solutions with other units are possible and are accepted
No solution but realise that unit of charge must vanish $\beta = 2\delta$ **0.2**

A5 (1.0 pt) Calculate the total radiated power P_{tot} of the LHC for a proton energy of $E = 7.00$ TeV (Note table 1). You may use appropriate approximations.

Solution A5: **[1.0]**

Radiated Power:

$$P_{rad} = \frac{\gamma^4 \cdot a^2 \cdot e^2}{6\pi \cdot c^3 \cdot \epsilon_0}$$ **0.1**

Energy:

$$E = (\gamma - 1)m_p \cdot c^2 \text{ or equally valid } E \simeq \gamma \cdot m_p \cdot c^2$$ **0.2**

Acceleration:

$$a \simeq \frac{c^2}{r} \text{ with } r = \frac{L}{2\pi}$$ **0.2**

Therefore:

$$P_{rad} = (\frac{E}{m_p c^2} + 1)^4 \cdot \frac{e^2 \cdot c}{6\pi\epsilon_0 \cdot r^2} \text{ or } (\frac{E}{m_p c^2})^4 \cdot \frac{e^2 \cdot c}{6\pi\epsilon_0 \cdot r^2}$$ **0.3**

(not required $P_{rad} = 7.94 \cdot 10^{-12}$W)

Total radiated power:

$$P_{tot} = 2 \cdot 2808 \cdot 1.15 \cdot 10^{11} \cdot P_{rad} = 5.13\text{kW}$$ **0.2**

penalty for missing factor 2 (for the two beams): **-0.1** **-0.1**
penalty for wrong numbers 2808 and/or $1.15 \cdot 10^{11}$ (numbers come from table 1): **-0.1** **-0.1**

A6 (1.5 pt) Determine the time T that the protons need to pass through this field.

Solution A6: **[1.5]**

2nd Newton's law

$$F = \frac{dp}{dt} \quad \text{leads to}$$ **0.2**

$$\frac{V \cdot e}{d} = \frac{p_f - p_i}{T} \text{ with } p_i = 0$$ **0.3**

Conservation of energy:

$$E_{tot} = m \cdot c^2 + e \cdot V$$ **0.2**

Since

$$E_{tot}^2 = (m \cdot c^2)^2 + (p_f \cdot c)^2$$ **0.2**

$$\rightarrow p_f = \frac{1}{c} \cdot \sqrt{(m \cdot c^2 + e \cdot V)^2 - (m \cdot c^2)^2} = \sqrt{2e \cdot m \cdot V + \left(\frac{e \cdot V}{c}\right)^2}$$ **0.2**

$$\rightarrow T = \frac{d \cdot p_f}{V \cdot e} = \frac{d}{V \cdot e}\sqrt{2e \cdot m_p \cdot V + \left(\frac{e \cdot V}{c}\right)^2}$$ **0.3**

$$T = 218\text{ns}$$ **0.1**

Alternative solution **[1.5]**

2nd Newton's law

$$F = \frac{dp}{dt} \quad \text{leads to}$$ **0.2**

$$\frac{V \cdot e}{d} = \frac{p_f - p_i}{T} \text{ with } p_i = 0$$ **0.3**

velocity from A1 or from conservation of energy

$$v = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + V \cdot e}\right)^2}$$ **0.2**

and hence for γ

$$\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}} = 1 + \frac{e \cdot V}{m_p \cdot c^2}$$ **0.2**

$$\rightarrow p_f = \gamma \cdot m_p \cdot v = \left(1 + \frac{e \cdot V}{m_p \cdot c^2}\right) \cdot m_p \cdot c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + V \cdot e}\right)^2}$$ **0.2**

$$\rightarrow T = \frac{d \cdot p_f}{V \cdot e} = \frac{d \cdot m_p \cdot c}{V \cdot e} \cdot \sqrt{\left(\frac{m_p \cdot c^2 + e \cdot V}{m_p \cdot c^2}\right)^2 - 1} = \frac{d}{V \cdot e}\sqrt{2e \cdot m_p \cdot V + \left(\frac{e \cdot V}{c}\right)^2}$$ **0.3**

$$T = 218\text{ns}$$ **0.1**

Alternative solution: integrate time **[1.5]**

Energy increases linearly with distance x

$$E(x) = \frac{e \cdot V \cdot x}{d}$$ **0.2**

$$t = \int dt = \int_0^d \frac{dx}{v(x)}$$ **0.2**

$$v(x) = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + \frac{e \cdot V \cdot x}{d}}\right)^2} = c \cdot \frac{\sqrt{\left(m_p \cdot c^2 + \frac{e \cdot V \cdot x}{d}\right)^2 - (m_p \cdot c^2)^2}}{m_p \cdot c^2 + \frac{e \cdot V \cdot x}{d}}$$

$$= c \cdot \frac{\sqrt{\left(1 + \frac{e \cdot V \cdot x}{d \cdot m_p \cdot c^2}\right)^2 - 1}}{1 + \frac{e \cdot V \cdot x}{d \cdot m_p \cdot c^2}}$$ **0.2**

$$\text{Substitution}: \xi = \frac{e \cdot V \cdot x}{d \cdot m_p \cdot c^2} \qquad \frac{d\xi}{dx} = \frac{e \cdot V}{d \cdot m_p \cdot c^2}$$ **0.2**

$$\rightarrow t = \frac{1}{c} \int_0^b \frac{1 + \xi}{\sqrt{(1+\xi)^2 - 1}} \frac{d \cdot m_p \cdot c^2}{e \cdot V} d\xi \qquad b = \frac{e \cdot V}{m_p \cdot c^2}$$ **0.2**

$$1 + \xi := \cosh(s) \qquad \frac{d\xi}{ds} = \sinh(s)$$ **0.1**

$$t = \frac{m_p \cdot c \cdot d}{e \cdot V} \int \frac{\cosh(s) \cdot \sinh(s) ds}{\sqrt{\cosh^2(s) - 1}} = \frac{m_p \cdot c \cdot d}{e \cdot V} [\sinh(s)]_{b_1}^{b_2}$$ **0.2**

$$\text{with} \quad b_1 = \cosh^{-1}(1), \quad b_2 = \cosh^{-1}\left(1 + \frac{e \cdot V}{m_p \cdot c^2}\right)$$ **0.1**

$$T = 218\text{ns}$$ **0.1**

Alternative: differential equation **[1.5]**

$$F = \frac{\mathrm{d}p}{\mathrm{d}t}$$ **0.2**

$$\rightarrow \frac{V \cdot e}{d} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m \cdot v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{m \cdot a \left(1 - \frac{v^2}{c^2}\right) + m \cdot a \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \gamma^3 \cdot m \cdot a$$ **0.4**

$$a = \ddot{s} = \frac{V \cdot e}{d \cdot m}\left(1 - \frac{\dot{s}^2}{c^2}\right)^{\frac{3}{2}}$$ **0.3**

$$\text{Ansatz}: \ s(t) = \sqrt{i^2 \cdot t^2 + k} - l \ \text{ with boundary conditions } s(0) = 0, v(0) = 0$$ **0.1**

$$\rightarrow s(t) = \frac{c}{V \cdot e}\left(\sqrt{e^2 \cdot V^2 \cdot t^2 + c^2 \cdot m^2 \cdot d^2} - c \cdot m \cdot d\right)$$ **0.2**

$$s = d \rightarrow T = \frac{d}{V \cdot e}\sqrt{\left(\frac{V \cdot e}{c}\right)^2 + 2V \cdot e \cdot m}$$ **0.2**

$$T = 218\text{ns}$$ **0.1**

classical solution: **[0.4]**

$$F = \frac{V \cdot e}{d} \rightarrow \text{acceleration } a = \frac{F}{m_p} = \frac{V \cdot e}{m_p \cdot d}$$ **0.1**

$$d = \frac{1}{2} \cdot a \cdot T^2 \rightarrow T = \sqrt{\frac{2d}{a}}$$ **0.1**

And hence for the time

$$T = d \cdot \sqrt{\frac{2 \cdot m_p}{V \cdot e}}$$ **0.1**

$$T = 194\text{ns}$$ **0.1**

Part B. Particle identification (4 points)

B1 (0.8 pt) Express the particle rest mass m in terms of the momentum p, the flight length l and the flight time t assuming that the particles with elementary charge e travel with velocity close to c on straight tracks in the ToF detector and that it travels perpendicular to the two detection planes (see Figure 2).

Solution B1: **[0.8]**

with velocity

$$v = \frac{l}{t}$$ **0.1**

relativistic momentum

$$p = \frac{m \cdot v}{\sqrt{1 - \frac{v^2}{c^2}}}$$ **0.2**

gets

$$p = \frac{m \cdot l}{t \cdot \sqrt{1 - \frac{l^2}{t^2 \cdot c^2}}}$$ **0.2**

→ mass

$$m = \frac{p \cdot t}{l} \cdot \sqrt{1 - \frac{l^2}{t^2 \cdot c^2}} = \frac{p}{l \cdot c} \cdot \sqrt{t^2 \cdot c^2 - l^2}$$ **0.3**

Alternative **[0.8]**

with flight distance: l, flight time t gets:

$$t = \frac{l}{(c \cdot \beta)}$$ **0.1**

relativistic momentum

$$p = \frac{m \cdot \beta \cdot c}{\sqrt{1 - \beta^2}}$$

therefore the velocity:

$$\beta = \frac{p}{\sqrt{m^2 \cdot c^2 + p^2}}$$ **0.2**

insert into the expression for t:

$$t = l \frac{\sqrt{m^2 \cdot c^2 + p^2}}{c \cdot p}$$ **0.2**

→ mass:

$$m = \sqrt{\left(\frac{p \cdot t}{l}\right)^2 - \left(\frac{p}{c}\right)^2} = \frac{p}{l \cdot c} \sqrt{(t \cdot c)^2 - (l)^2}$$ **0.3**

---

non-relativistic solution: **[0.0]**
flight time: $t = l/v$ velocity:

$$v = \frac{p}{m} \rightarrow t = \frac{l \cdot m}{p} \quad \text{and} \quad m = \frac{p \cdot t}{l}$$

this solution gives no points **0.0**

B2 (0.7 pt) Calculate the minimal length of a ToF detector that allows to safely distinguish a charged kaon from a charged pion given both their momenta are measured to be 1.00 GeV/c. For a good separation it is required that the difference in the time-of-flight is larger than three times the time resolution of the detector. The typical resolution of a ToF detector is 150 ps (1 ps = 10^{-12} s).

Solution B2: **[0.7]**

Flight time difference between kaon and pion

$$\Delta t = 450\text{ps} = 450 \cdot 10^{-12}\text{s}$$ **0.1**

Flight time difference between kaon and pion

$$\Delta t = \frac{l}{cp}(\sqrt{m_\pi^2 \cdot c^2 + p^2} - \sqrt{m_K^2 \cdot c^2 + p^2}) = 450\text{ps} = 450 \cdot 10^{-12}\text{s}$$ **0.2**

$$\rightarrow l = \frac{\Delta t \cdot p}{\sqrt{m_K^2 + p^2/c^2} - \sqrt{m_\pi^2 + p^2/c^2}}$$ **0.2**

$$\sqrt{m_K^2 + p^2/c^2} = 1.115 \text{ GeV}/c^2 \text{and} \sqrt{m_\pi^2 + p^2/c^2} = 1.010 \text{ GeV}/c^2$$

$$l = 450 \cdot 10^{-12} \cdot \frac{1}{1.115 - 1.010} \text{s GeV}c^2/(\text{GeV}c)$$ **0.1**

$$l = 4285.7 10^{-12}\text{s} \cdot c = 4285.7 \cdot 10^{-12} \cdot 2.998 \cdot 10^8\text{m} = 1.28\text{m}$$ **0.1**

Penalty for < 2 or > 4 significant digits **-0.1**

Non-relativistic solution: **[0.3]**

Flight time difference between kaon and pion

$$\Delta t = \frac{l}{p}(m_K - m_\pi) = 450ps = 450 \cdot 10^{-12}\text{s}$$ **0.1**

length:

$$l = \frac{\Delta t p}{m_K - m_\Pi} = \frac{450 \cdot 10^{-12}\text{s} \cdot 1\text{GeV}/c}{(0.498 - 0.135)\text{GeV}/c^2}$$ **0.1**

$$l = 450 \cdot 10^{-12}/0.363 \cdot cs = 450 \cdot 10^{-12}/0.363 \cdot 2.998 \cdot 10^8\text{m}$$

$$l = 3716 \cdot 10^{-4}\text{m} = 0.372\text{m}$$ **0.1**

Penalty for < 2 or > 4 significant digits **-0.1**

B3 (1.7 pt) Express the particle mass as a function of the magnetic flux density B, the radius R of the ToF tube, fundamental constants and the measured quantities: radius r of the track and time-of-flight t.

Solution B3: **[1.7]**

Particle is travelling perpendicular to the beam line hence the track length is given by the length of the arc
Lorentz force → transverse momentum, since there is no longitudinal momentum, the momentum is the same as the transverse momentum
Use formula from B1 to calculate the mass
track length: length of arc

$$l = 2 \cdot r \cdot \operatorname{asin}\frac{R}{2 \cdot r}$$ **0.5**

penalty for just taking a straight track ($l = R$) **-0.4**

partial points for intermediate steps, maximum 0.4
Lorentz force

$$\frac{\gamma \cdot m \cdot v_t^2}{r} = e \cdot v_t \cdot B \to p_T = r \cdot e \cdot B$$ **0.4**

partial points for intermediate steps, maximum 0.3
longitudinal momentum=0 $\to p = p_T$ **0.1**

momentum

$$p = e \cdot r \cdot B$$ **0.1**

$$m = \sqrt{\left(\frac{p \cdot t}{l}\right)^2 - \left(\frac{p}{c}\right)^2} = e \cdot r \cdot B \cdot \sqrt{\left(\frac{t}{2r \cdot \operatorname{asin}\frac{R}{2r})}\right)^2 - \left(\frac{1}{c}\right)^2}$$ **0.6**

partial points for intermediate steps, maximum 0.5

Non-relativistic: track length: length of arc **[0.9]**

$$l = 2 \cdot r \cdot \operatorname{asin}\frac{R}{2 \cdot r}$$ **0.5**

penalty for just taking a straight track ($l = R$) **-0.4**

partial points for intermediate steps, maximum 0.4

$$m = \frac{p \cdot t}{l} = \frac{e \cdot r \cdot B \cdot t}{2r \cdot \operatorname{asin}\frac{R}{2r}} = \frac{e \cdot B \cdot t}{2 \cdot \operatorname{asin}\frac{R}{2r}}$$ **0.4**

partial points for intermediate steps, maximum 0.3

B4 (0.8 pt) Identify the four particles by calculating their mass.

Particle	Radius r [m]	Time of flight [ns]
A	5.10	20
B	2.94	14
C	6.06	18
D	2.32	25

Solution B4: **[0.8]**

Particle	arc [m]	p $[\frac{MeV}{c}]$	p $[\frac{mkg}{s}]$ 10^{-19}	pt/l $[\frac{MeVs}{cm}]$ 10^{-6}	pt/l $[\frac{MeV}{c^2}]$	pt/l [kg] 10^{-27}	Mass $[\frac{MeV}{c^2}]$	Mass [kg] 10^{-27}
A	3.786	764.47	4.0855	4.038	1210.6	2.158	938.65	1.673
B	4.002	440.69	2.3552	1.542	462.2	0.824	139.32	0.248
C	3.760	908.37	4.8546	4.349	1303.7	2.324	935.10	1.667
D	4.283	347.76	1.8585	2.030	608.6	1.085	499.44	0.890

Particles A and C are protons, B is a Pion and D a Kaon

correct mass and identification: per particle **0.2**

penalty for correct mass but no or wrong identification **for 1 or 2 particles** **-0.1**

penalty for correct mass but no or wrong identification **for 3 or 4 particles** **-0.2**

wrong mass, correct momentum:**per particle** **0.1**

wrong momentum, correct arc **for 3 or 4 particles** **0.2**

wrong momentum, correct arc **for 1 or 2 particles** **0.1**

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non relativistic solution $m = pt/l$ **Particle identification is not possible** **[0.4]**

Particle	arc [m]	p $[\frac{MeV}{c}]$	p $[\frac{mkg}{s}]$ 10^{-19}	$m = p \cdot t/l$ $[\frac{MeVs}{cm}]$ 10^{-6}	$m = p \cdot t/l$ $[\frac{MeV}{c^2}]$	$m = p \cdot t/l$ [kg] 10^{-27}
A	3.786	764.47	4.0855	4.038	1210.6	2.158
B	4.010	440.69	2.3552	1.542	462.2	0.824
C	3.760	908.37	4.8546	4.349	1303.7	2.324
D	4.283	347.76	1.8585	2.030	608.6	1.085

correct mass or correct momentum: **per particle** **0.1**

wrong momentum, correct arc **for 3 or 4 particles** **0.2**

wrong momentum, correct arc **for 1 or 2 particles** **0.1**