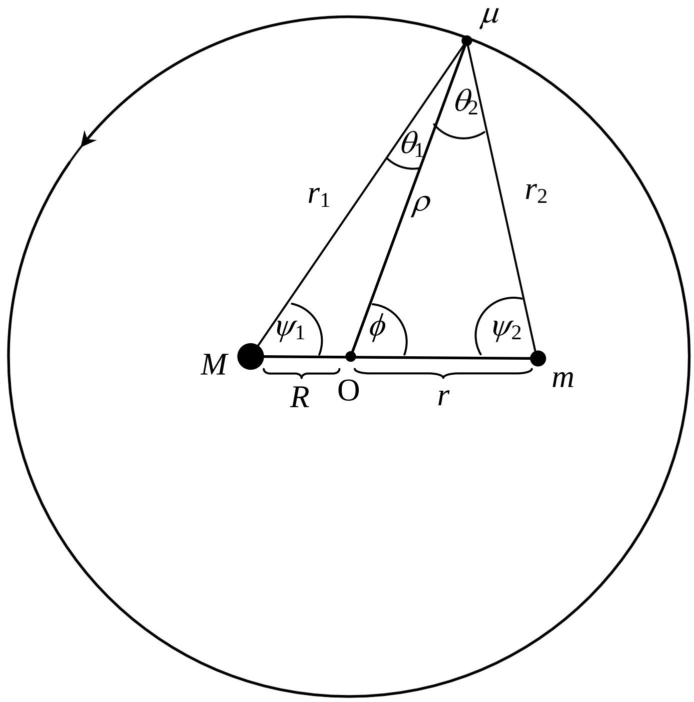
I. Solution



1.1 Let O be their centre of mass. Hence

$$MR - mr = 0 \qquad \text{(1)}$$

$$m\omega_0^2 r = \frac{GMm}{(R+r)^2}$$
$$M\omega_0^2 R = \frac{GMm}{(R+r)^2} \qquad \text{(2)}$$

From Eq. (2), or using reduced mass, $\omega_0^2 = \frac{G(M+m)}{(R+r)^3}$

Hence, $\omega_0^2 = \frac{G(M+m)}{(R+r)^3} = \frac{GM}{r(R+r)^2} = \frac{Gm}{R(R+r)^2}$. $\qquad$ (3)

1.2 Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m. For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1+\frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega_0^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{(4)}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{(5)}$$

Substituting $\frac{GM}{r_1^2}$ from Eq. (5) into Eq. (4), and using the identity

$\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2=\sin(\theta_1+\theta_2)$, we get

$$m\frac{\sin(\theta_1+\theta_2)}{r_2^2} = \frac{(M+m)}{(R+r)^3}\rho\sin\theta_1 \qquad \text{(6)}$$

The distances r_2 and ρ, the angles θ_1 and θ_2 are related by two SineRule equations

$$\frac{\sin\psi_1}{\rho} = \frac{\sin\theta_1}{R}$$
$$\frac{\sin\psi_1}{r_2} = \frac{\sin(\theta_1+\theta_2)}{R+r} \qquad \text{(7)}$$

Substitute (7) into (6)

$$\frac{1}{r_2^3}=\frac{R}{(R+r)^4}\frac{(M+m)}{m} \qquad \text{(10)}$$

Since $\frac{m}{M+m}=\frac{R}{R+r}$,Eq. (10) gives

$$r_2=R+r \qquad \text{(11)}$$

By substituting $\frac{Gm}{r_2^2}$ from Eq. (5) into Eq. (4), and repeat a similar procedure, we get

$$r_1=R+r \qquad \text{(12)}$$

Alternatively, $\frac{r_1}{\sin(180^\circ-\phi)} = \frac{R}{\sin\theta_1}$ and $\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2}$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1}$$

Combining with Eq. (5) gives $r_1 = r_2$

Hence, it is an equilateral triangle with

$$\psi_1 = 60^\circ$$
$$\psi_2 = 60^\circ \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(13)}$$

The distance ρ is calculated from the Cosine Rule.

$$\rho^2 = r^2 + (R+r)^2 - 2r(R+r)\cos 60^\circ$$
$$\rho = \sqrt{r^2 + rR + R^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(14)}$$

Alternative Solution to 1.2

Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m .For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}$$

Note that

$$\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1}$$

$$\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2} \qquad \text{(see figure)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}$$

Equations (5) and (6):

$$r_1 = r_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{m}{M} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}$$

$$\psi_1 = \psi_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(9)}$$

The equation (4) then becomes:

$$M\cos\theta_1 + m\cos\theta_2 = \frac{(M+m)}{(R+r)^3} r_1^2 \rho \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}$$

Equations (8) and (10): $\sin(\theta_1+\theta_2) = \frac{M+m}{M}\frac{r_1^2\rho}{(R+r)^3}\sin\theta_2$ ……………………… (11)

Note that from figure, $\frac{\rho}{\sin\psi_2} = \frac{r}{\sin\theta_2}$ ……………………… (12)

Equations (11) and (12): $\sin(\theta_1+\theta_2) = \frac{M+m}{M}\frac{r_1^2 r}{(R+r)^3}\sin\psi_2$ ……………………… (13)

Also from figure,

$$(R+r)^2 = r_2^2 - 2r_1r_2\cos(\theta_1+\theta_2)+r_1^2 = 2r_1^2\left[1-\cos(\theta_1+\theta_2)\right] \quad \text{………………} \quad (14)$$

Equations (13) and (14): $\sin(\theta_1+\theta_2) = \frac{\sin\psi_2}{2\left[1-\cos(\theta_1+\theta_2)\right]}$ ……………………… (15)

$$\theta_1+\theta_2 = 180^\circ - \psi_1 - \psi_2 = 180^\circ - 2\psi_2 \quad \text{(see figure)}$$

$$\therefore \quad \cos\psi_2 = \frac{1}{2},\ \psi_2 = 60^\circ,\ \psi_1 = 60^\circ$$

Hence M and m from an equilateral triangle of sides $(R+r)$

Distance μ to M is $R+r$

Distance μ to m is $R+r$

Distance μ to O is $\rho = \sqrt{\left(\frac{R+r}{2}-R\right)^2+\left\{(R+r)\frac{\sqrt{3}}{2}\right\}^2} = \sqrt{R^2+Rr+r^2}$

1.3 The energy of the mass μ is given by

$$E = -\frac{GM\mu}{r_1} - \frac{Gm\mu}{r_2} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \rho^2\omega^2\right) \quad \text{…………………………..(15)}$$

Since the perturbation is in the radial direction, angular momentum is conserved ($r_1 = r_2 = \Re$ and $m = M$),

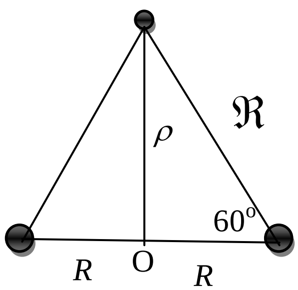
$$E = -\frac{2GM\mu}{\Re} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \frac{\rho_0^4\omega_0^2}{\rho^2}\right)$$

…………………………..(16)

Since the energy is conserved,

$$\frac{dE}{dt} = 0$$

$$\frac{dE}{dt}=\frac{2GM\mu}{\Re^2}\frac{d\Re}{dt}+\mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2}-\mu\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}\frac{d\rho}{dt}=0 \quad \text{...............(17)}$$



$$\frac{d\Re}{dt}=\frac{d\Re}{d\rho}\frac{d\rho}{dt}=\frac{d\rho}{dt}\frac{\rho}{\Re} \quad \text{................(18)}$$

$$\frac{dE}{dt}=\frac{2GM\mu}{\Re^3}\rho\frac{d\rho}{dt}+\mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2}-\mu\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}\frac{d\rho}{dt}=0 \quad \text{................(19)}$$

Since $\frac{d\rho}{dt}\neq 0$, we have

$$\frac{2GM}{\Re^3}\rho+\frac{d^2\rho}{dt^2}-\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}=0 \text{ or}$$

$$\frac{d^2\rho}{dt^2}=-\frac{2GM}{\Re^3}\rho+\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}. \quad \text{..............................(20)}$$

The perturbation from $\Re_0$ and ρ_0 gives $\Re=\Re_0\left(1+\frac{\Delta\Re}{\Re_0}\right)$ and $\rho=\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)$.

Then

$$\frac{d^2\rho}{dt^2}=\frac{d^2}{dt^2}(\rho_0+\Delta\rho)=-\frac{2GM}{\Re_0^3\left(1+\frac{\Delta\Re}{\Re_0}\right)^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)+\frac{\rho_0{}^4\omega_0{}^2}{\rho_0^3\left(1+\frac{\Delta\rho}{\rho_0}\right)^3} \quad \text{..................(21)}$$

Using binomial expansion $(1+\varepsilon)^n\approx 1+n\varepsilon$,

$$\frac{d^2\Delta\rho}{dt^2}=-\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)\left(1-\frac{3\Delta\Re}{\Re_0}\right)+\rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \quad \text{...................(22)}$$

Using $\Delta\rho=\frac{\Re}{\rho}\Delta\Re$,

$$\frac{d^2\Delta\rho}{dt^2}=-\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}-\frac{3\rho_0\Delta\rho}{\Re_0^2}\right)+\rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \quad \text{...................(23)}$$

Since $\omega_0^2=\frac{2GM}{\Re_0^3}$,

$$\frac{d^2\Delta\rho}{dt^2}=-\omega_0^2\rho_0\left(1+\frac{\Delta\rho}{\rho_0}-\frac{3\rho_0\Delta\rho}{\Re_0^2}\right)+\omega_0^2\rho_0\left(1-\frac{3\Delta\rho}{\rho_0}\right) \quad \text{...................(24)}$$

$$\frac{d^2\Delta\rho}{dt^2}=-\omega_0^2\rho_0\left(\frac{4\Delta\rho}{\rho_0}-\frac{3\rho_0\Delta\rho}{\Re_0^2}\right) \quad \text{...................(25)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4 - \frac{3\rho_0^2}{\Re_0^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

From the figure, $\rho_0 = \Re_0 \cos 30°$ or $\frac{{\rho_0}^2}{{\Re_0}^2} = \frac{3}{4}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4 - \frac{9}{4}\right) = -\frac{7}{4}\omega_0^2\Delta\rho\,. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(27)$$

Angular frequency of oscillationis $\frac{\sqrt{7}}{2}\omega_0$.

Alternative solution:

$M = m$ gives $R = r$ and ${\omega_0}^2 = \frac{G(M+M)}{(R+R)^3} = \frac{GM}{4R^3}$. The unperturbed radial distance of μ is $\sqrt{3}R$, so the perturbed radial distance can be represented by $\sqrt{3}R + \zeta$ where $\zeta << \sqrt{3}R$ as shown in the following figure.

Using Newton's 2[nd] law, $-\frac{2GM\mu}{\{R^2 + (\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \mu\frac{d^2}{dt^2}(\sqrt{3}R+\zeta) - \mu\omega^2(\sqrt{3}R+\zeta)$.

(1)

The conservation of angular momentum gives $\mu\omega_0(\sqrt{3}R)^2 = \mu\omega(\sqrt{3}R+\zeta)^2$.

(2)

Manipulate (1) and (2) algebraically, applying $\zeta^2 \approx 0$ and binomial approximation.

$$-\frac{2GM}{\{R^2 + (\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \frac{d^2\zeta}{dt^2} - \frac{{\omega_0}^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{2GM}{\{4R^2 + 2\sqrt{3}\zeta R\}^{3/2}}(\sqrt{3}R+\zeta) \approx \frac{d^2\zeta}{dt^2} - \frac{{\omega_0}^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{GM}{4R^3}\sqrt{3}R\frac{(1+\zeta/\sqrt{3}R)}{(1+\sqrt{3}\zeta/2R)^{3/2}} = \frac{d^2\zeta}{dt^2} - \frac{{\omega_0}^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-{\omega_0}^2\sqrt{3}R\left(1 - \frac{3\sqrt{3}\zeta}{4R}\right)\left(1 + \frac{\zeta}{\sqrt{3}R}\right) \approx \frac{d^2\zeta}{dt^2} - {\omega_0}^2\sqrt{3}R\left(1 - \frac{3\zeta}{\sqrt{3}R}\right)$$

$$\frac{d^2}{dt^2}\zeta = -\left(\frac{7}{4}{\omega_0}^2\right)\zeta$$

1.4 Relative velocity

Let v = speed of each spacecraft as it moves in circle around the centre O.
The relative velocities are denoted by the subscripts A, B and C.
For example, v_{BA} is the velocity of B as observed by A.

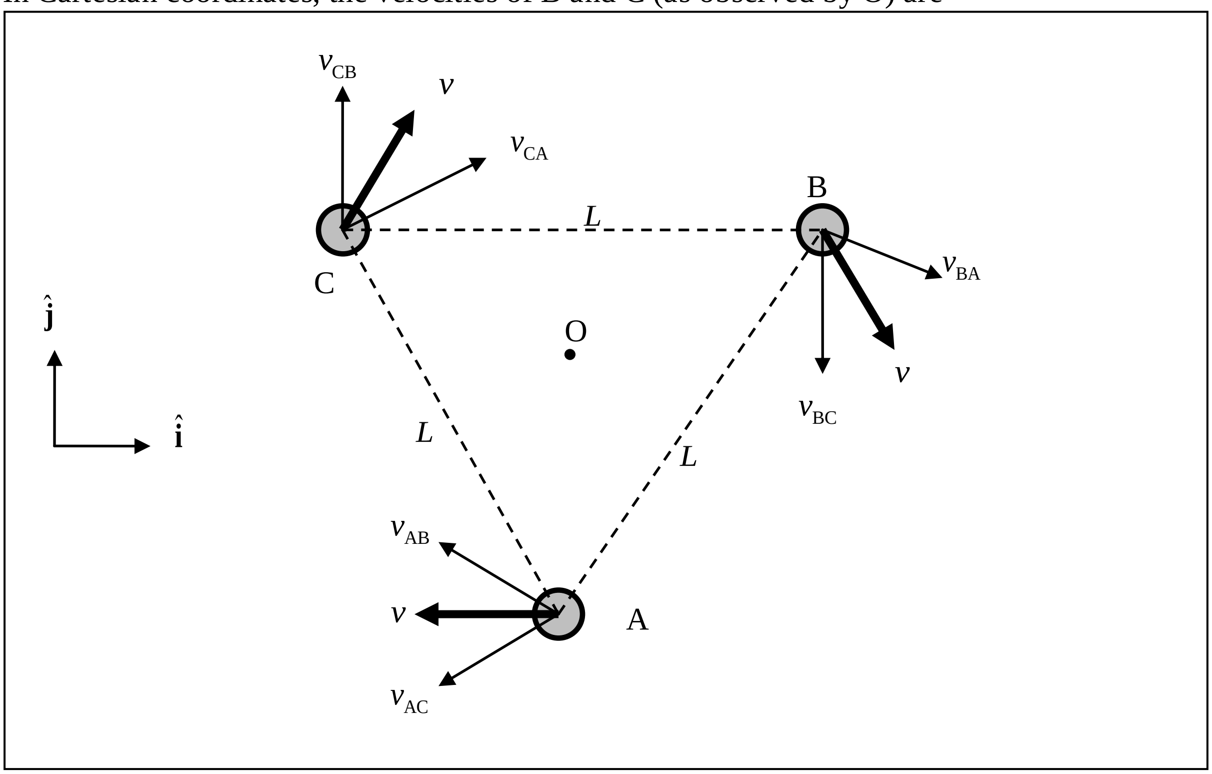
The period of circular motion is 1 year $T = 365 \times 24 \times 60 \times 60$ s. ………… (28)

The angular frequency $\omega = \dfrac{2\pi}{T}$

The speed $v = \omega \dfrac{L}{2\cos 30°} = 575$ m/s ………… (29)

The speed is much less than the speed light → Galilean transformation.

In Cartesian coordinates, the velocities of B and C (as observed by O) are



For B, $\vec{v}_B = v\cos 60°\hat{\mathbf{i}} - v\sin 60°\hat{\mathbf{j}}$

For C, $\vec{v}_C = v\cos 60°\hat{\mathbf{i}} + v\sin 60°\hat{\mathbf{j}}$

Hence $\vec{v}_{\text{BC}} = -2v\sin 60°\hat{\mathbf{j}} = -\sqrt{3}v\hat{\mathbf{j}}$

The speed of B as observed by C is $\sqrt{3}v \approx 996$ m/s ………… (30)

Notice that the relative velocities for each pair are anti-parallel.

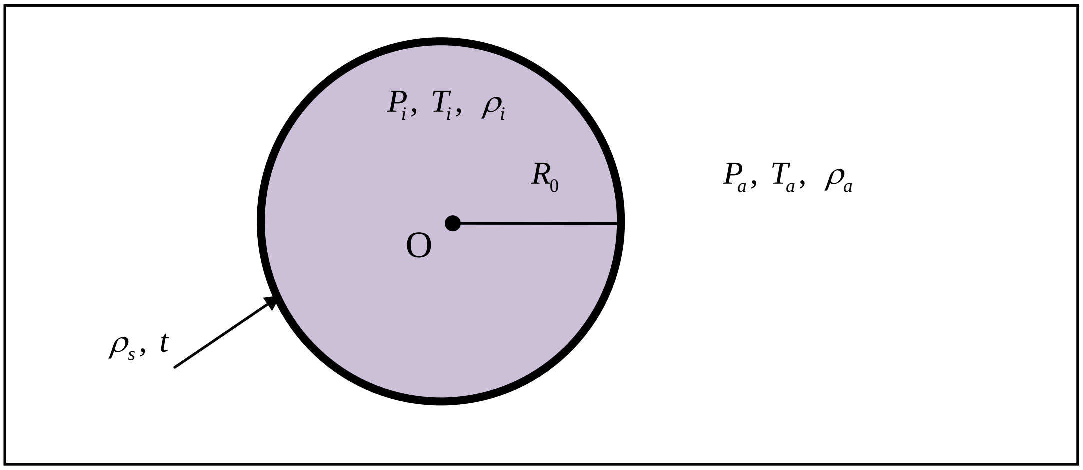
Alternative solution for 1.4

One can obtain v_{BC} by considering the rotation about the axis at one of the spacecrafts.

$$v_{\text{BC}} = \omega L = \frac{2\pi}{365 \times 24 \times 60 \times 60 \text{ s}}(5 \times 10^6 \text{ km}) \approx 996 \text{ m/s}$$

2. SOLUTION

2.1. The bubble is surrounded by air.



Cutting the sphere in half and using the projected area to balance the forces give

$$P_i \pi R_0^2 = P_a \pi R_0^2 + 2(2\pi R_0 \gamma)$$

$$P_i = P_a + \frac{4\gamma}{R_0} \qquad \text{… (1)}$$

The pressure and density are related by the ideal gas law:

$PV = nRT$ or $P = \dfrac{\rho RT}{M}$, where M = the molar mass of air. … (2)

Apply the ideal gas law to the air inside and outside the bubble, we get

$$\rho_i T_i = P_i \frac{M}{R}$$

$$\rho_a T_a = P_a \frac{M}{R},$$

$$\frac{\rho_i T_i}{\rho_a T_a} = \frac{P_i}{P_a} = \left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \text{… (3)}$$

2.2. Using $\gamma = 0.025\,\text{Nm}^{-1}$, $R_0 = 1.0$ cm and $P_a = 1.013\times10^5\,\text{Nm}^{-2}$, the numerical value of the ratio is

$$\frac{\rho_i T_i}{\rho_a T_a} = 1 + \frac{4\gamma}{R_0 P_a} = 1 + 0.0001 \qquad \dots (4)$$

(The effect of the surface tension is very small.)

2.3. Let W = total weight of the bubble, F = buoyant force due to air around the bubble

$$W = (\text{mass of film+mass of air})\,g$$
$$= \left(4\pi R_0^2\,\rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g \qquad \dots (5)$$
$$= 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g$$

The buoyant force due to air around the bubble is

$$B = \frac{4}{3}\pi R_0^3 \rho_a g \qquad \dots (6)$$

If the bubble floats in still air,

$$B \ge W$$
$$\frac{4}{3}\pi R_0^3 \rho_a g \ge 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g \qquad \dots (7)$$

Rearranging to give

$$T_i \ge \frac{R_0 \rho_a T_a}{R_0 \rho_a - 3\rho_s t}\left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \dots (8)$$
$$\ge 307.1\text{ K}$$

The air inside must be about 7.1°C warmer.

2.4. Ignore the radius change → Radius remains $R_0 = 1.0$ cm

(The radius actually decreases by 0.8% when the temperature decreases from 307.1 K to 300 K. The film itself also becomes slightly thicker.)

The drag force from Stokes' Law is $F = 6\pi\eta R_0 u$ … (9)

If the bubble floats in the updraught,

$$F \geq W - B$$

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2 \rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g - \frac{4}{3}\pi R_0^3 \rho_a g \quad \text{… (10)}$$

When the bubble is in thermal equilibrium $T_i = T_a$.

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2 \rho_s t + \frac{4}{3}\pi R_0^3 \rho_a \left[1 + \frac{4\gamma}{R_0 P_a}\right]\right) g - \frac{4}{3}\pi R_0^3 \rho_a g$$

Rearranging to give

$$u \geq \frac{4R_0 \rho_s t g}{6\eta} + \frac{\frac{4}{3} R_0^2 \rho_a g \left(\frac{4\gamma}{R_0 P_a}\right)}{6\eta} \quad \text{… (11)}$$

2.5. The numerical value is $u \geq 0.36$ m/s.

The 2nd term is about 3 orders of magnitude lower than the 1st term.

From now on, ignore the surface tension terms.

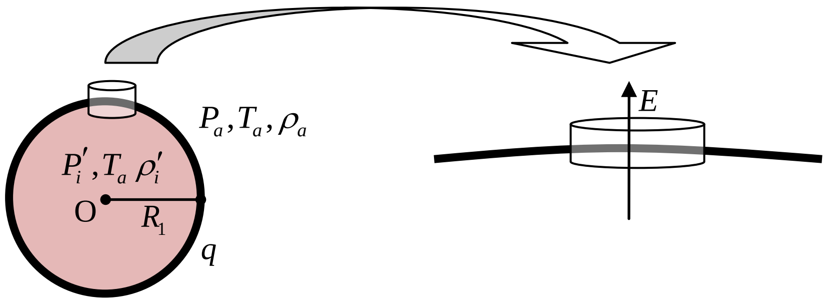
2.6. When the bubble is electrified, the electrical repulsion will cause the bubble to expand in size and thereby raise the buoyant force.

The force/area is (e-field on the surface × charge/area)

There are two alternatives to calculate the electric field ON the surface of *the soap film.*

A. From Gauss's Law

Consider a very thin pill box on the soap surface.



E = electric field on the film surface that results from all other parts of the soap film, excluding the surface inside the pill box itself.

E_q = total field just outside the pill box = $\dfrac{q}{4\pi\varepsilon_0 R_1^2} = \dfrac{\sigma}{\varepsilon_0}$

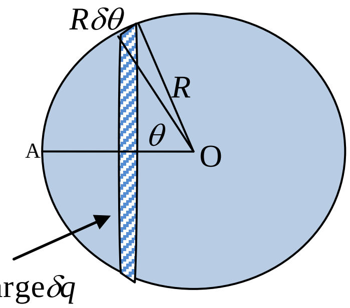
= E + electric field from surface charge σ

= $E + E_\sigma$

Using Gauss's Law on the pill box, we have $E_\sigma = \dfrac{\sigma}{2\varepsilon_0}$ perpendicular to the film as a result of symmetry.

Therefore, $E = E_q - E_\sigma = \dfrac{\sigma}{\varepsilon_0} - \dfrac{\sigma}{2\varepsilon_0} = \dfrac{\sigma}{2\varepsilon_0} = \dfrac{1}{2\varepsilon_0}\dfrac{q}{4\pi R_1^2}$ … (12)

B. From direct integration



$$\delta q = \left(\frac{q}{4\pi R^2}\right) 2\pi R \sin\theta . R\delta\theta$$

To find the magnitude of the electrical repulsion we must first find the electric field intensity E at a point on (not outside) the surface itself.

Field at A in the direction $\overrightarrow{\mathrm{OA}}$ is

$$\delta E_A = \frac{1}{4\pi\varepsilon_0} \frac{\left(q/4\pi R_1^2\right) 2\pi R_1^2 \sin\theta\delta\theta}{\left(2R_1 \sin\frac{\theta}{2}\right)^2} \sin\frac{\theta}{2} = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \cos\frac{\theta}{2} \delta\left(\frac{\theta}{2}\right)$$

$$E_A = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \int_{\theta=0}^{\theta=180^\circ} \cos\frac{\theta}{2} d\left(\frac{\theta}{2}\right) = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \quad \ldots (13)$$

The repulsive force per unit area of the surface of bubble is

$$\left(\frac{q}{4\pi R_1^2}\right) E = \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} \quad \ldots (14)$$

Let P_i' and ρ_i' be the new pressure and density when the bubble is electrified.

This electric repulsive force will augment the gaseous pressure P_i'.

P_i' is related to the original P_i through the gas law.

$$P_i' \frac{4}{3}\pi R_1^3 = P_i \frac{4}{3}\pi R_0^3$$

$$P_i' = \left(\frac{R_0}{R_1}\right)^3 P_i = \left(\frac{R_0}{R_1}\right)^3 P_a \qquad \text{… (15)}$$

In the last equation, the surface tension term has been ignored.

From balancing the forces on the half-sphere projected area, we have (again ignoring the surface tension term)

$$P_i' + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a$$

$$P_a\left(\frac{R_0}{R_1}\right)^3 + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a \qquad \text{… (16)}$$

Rearranging to get

$$\left(\frac{R_1}{R_0}\right)^4 - \left(\frac{R_1}{R_0}\right) - \frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} = 0 \qquad \text{… (17)}$$

Note that (17) yields $\frac{R_1}{R_0} = 1$ when $q = 0$, as expected.

2.7. Approximate solution for R_1 when $\frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} << 1$

Write $R_1 = R_0 + \Delta R,\ \Delta R << R_0$

Therefore, $\frac{R_1}{R_0} = 1 + \frac{\Delta R}{R_0},\ \left(\frac{R_1}{R_0}\right)^4 \approx 1 + 4\frac{\Delta R}{R_0}$ … (18)

Eq. (17) gives:

$$\Delta R \approx \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \qquad \text{… (19)}$$

$$R_1 \approx R_0 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \approx R_0\left(1 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^4 P_a}\right) \qquad \text{… (20)}$$

2.8. The bubble will float if

$$B \geq W$$

$$\frac{4}{3}\pi R_1^3 \rho_a g \geq 4\pi R_0^2\ \rho_s t g + \frac{4}{3}\pi R_0^3 \rho_i g \qquad \ldots (21)$$

Initially, $T_i = T_a \Rightarrow \rho_i = \rho_a$ for $\gamma \to 0$ and $R_1 = R_0\left(1 + \frac{\Delta R}{R_0}\right)$

$$\frac{4}{3}\pi R_0^3 \left(1 + \frac{\Delta R}{R_0}\right)^3 \rho_a g \geq 4\pi R_0^2\ \rho_s t g + \frac{4}{3}\pi R_0^3 \rho_a g$$

$$\frac{4}{3}\pi \left(3\Delta R\right) \rho_a g \geq 4\pi R_0^2\ \rho_s t g$$

$$\frac{4}{3}\pi \frac{3q^2}{96\pi^2 \varepsilon_0 R_0 P_a} \rho_a g \geq 4\pi R_0^2\ \rho_s t g \qquad \ldots (22)$$

$$q^2 \geq \frac{96\pi^2 R_0^3\ \rho_s t \varepsilon_0 P_a}{\rho_a}$$

$$q \approx 256 \times 10^{-9}\ \text{C} \approx 256\,\text{nC}$$

Note that if the surface tension term is retained, we get

$$R_1 \approx \left(1 + \frac{q^2 / 96\pi^2 \varepsilon_0 R_0^4 P_a}{\left[1 + \frac{2}{3}\left(\frac{4\gamma}{R_0 P_a}\right)\right]}\right) R_0$$

QUESTION 3: SOLUTION

1. Using Coulomb's Law, we write the electric field at a distance r is given by

$$E_p = \frac{q}{4\pi\varepsilon_0(r-a)^2} - \frac{q}{4\pi\varepsilon_0(r+a)^2}$$

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{1}{\left(1-\frac{a}{r}\right)^2} - \frac{1}{\left(1+\frac{a}{r}\right)^2}\right) \qquad \text{..................(1)}$$

Using binomial expansion for small a,

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(1+\frac{2a}{r} - 1 + \frac{2a}{r}\right)$$

$$= +\frac{4qa}{4\pi\varepsilon_0 r^3} = +\frac{qa}{\pi\varepsilon_0 r^3} \qquad \text{................(2)}$$

$$= \frac{2p}{4\pi\varepsilon_0 r^3}$$

2. The electric field seen by the atom from the ion is

$$\vec{E}_{ion} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{................ (3)}$$

The induced dipole moment is then simply

$$\vec{p} = \alpha \vec{E}_{ion} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{................ (4)}$$

From eq. (2)

$$\vec{E}_p = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{r}$$

The electric field intensity $\vec{E}_p$ at the position of an ion at that instant is, using eq. (4),

$$\vec{E}_p = \frac{1}{4\pi\varepsilon_0 r^3}\left[-\frac{2\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}\right] = -\frac{\alpha Q}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$$

The force acting on the ion is

$$\vec{f} = Q\vec{E}_p = -\frac{\alpha Q^2}{8\pi^2\varepsilon_0^2 r^5}\hat{r} \qquad \text{................ (5)}$$

The "-'' sign implies that this force is attractive and Q^2 implies that the force is attractive regardless of the sign of Q.

3. The potential energy of the ion-atom is given by $U = \int_r^{\infty} \vec{f}.d\vec{r}$ ……….……………………….(6)

Using this, $U = \int_r^{\infty} \vec{f}.d\vec{r} = -\dfrac{\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4}$ …………………………………………………………….(7)

[Remark: Students might use the term $-\vec{p}\cdot\vec{E}$ which changes only the factor in front.]

4. At the position $r_{\min}$ we have, according to the Principle of Conservation of Angular Momentum,

$$mv_{\max}r_{\min} = mv_0 b$$

$$v_{\max} = v_0\frac{b}{r_{\min}} \qquad \text{…………….. (8)}$$

And according to the Principle of Conservation of Energy:

$$\frac{1}{2}mv_{\max}^2 + \frac{-\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4} = \frac{1}{2}mv_0^2 \qquad \text{…………….. (9)}$$

Eqs.(12) & (13):

$$\left(\frac{b}{r_{\min}}\right)^2 - \frac{\alpha Q^2 \Big/ \frac{1}{2}mv_0^2}{32\pi^2\varepsilon_0^2 b^4}\left(\frac{b}{r_{\min}}\right)^4 = 1$$

$$\left(\frac{r_{\min}}{b}\right)^4 - \left(\frac{r_{\min}}{b}\right)^2 + \frac{\alpha Q^2}{16\pi^2\varepsilon_0^2 m v_0^2 b^4} = 0 \qquad \text{…………….. (10)}$$

The roots of eq. (14) are:

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 \pm \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{…………….. (11)}$$

[Note that the equation (14) implies that $r_{\min}$ cannot be zero, unless b is itself zero.]

Since the expression has to be valid at $Q = 0$, which gives

$$r_{\min} = \frac{b}{\sqrt{2}}[1 \pm 1]^{\frac{1}{2}}$$

We have to choose "+" sign to make $r_{\min} = b$

Hence,

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 + \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{…………………………………....(12)}$$

5. A spiral trajectory occurs when (16) is imaginary (because there is no minimum distance of approach).

$r_{\min}$ is real under the condition:

$$1 \geq \frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2 b^4}$$

$$b \geq b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}} \qquad \text{……………….. (13)}$$

For $b < b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}}$ the ion will collide with the atom.

Hence the atom, as seen by the ion, has a cross-sectional area A,

$$A = \pi b_0^2 = \pi \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{2}} \qquad \text{……………….. (14)}$$

General marking guidelines

1. Minor mistakes in the calculations e.g. copying expressions incorrectlyfrom line to line	Deduct 20% of the final answer
2. Missing units in the final numerical answers (for each part)	Deduct 0.1 point
3. Final answers (for each part) containing too few or too many significant figures (from +2 or -2 positions, say)	Deduct 0.1 point
4. Using wrong physical concepts (despite correct answers)	No points awarded
5. Error propagated from earlier parts: minor errors	Full points (except for the final answer of the same part. No marks are awarded for the final answer.)
6. Error propagated from earlier parts: major errors (such that the solution becomes trivial).	Deduct 20 - 50% for a particular part

Theoretical Question 1: A Three-body Problem and LISA

Questions	Points	Concepts/Details
1.1 (Total1.5)	**1.0**	**1.1a** Use the centripetal acceleration (0.5) and gravitational force (0.5) . (0.5 = 0.2 for concept + 0.3 for correct form)
	0.5	**1.1bAnswer:**Any of the three following answers: $\omega_0^2 = \frac{G(M+m)}{(R+r)^3}$, $\omega_0^2 = \frac{GM}{r(R+r)^2}$, $\omega_0^2 = \frac{Gm}{R(R+r)^2}$
1.2 (Total3.5)	**1.0**	**1.2a** Newton's 2nd law (0.2) for two components of radial forces (0.1 + 0.4 correct expression) and cirucular motion (0.1 + 0.2 correct expression) $\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega_0^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho$
	0.5	**1.2b**Newton's 1st law (0.1) for tangential forces (0.4 correct expression) $\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2$
	1.0	**1.2c** -Using at least two sine rules or sensible geometric relations (0.2)e.g. $\frac{\sin\psi_1}{\rho} = \frac{\sin\theta_1}{R}$ $\frac{\sin\psi_1}{r_2} = \frac{\sin(\theta_1+\theta_2)}{R+r}$ -showing adequate understanding of geometry and/or trigonometry in the problem (0.2) -algebraic manipulation to find a correct expression for r_1 (0.2) -algebraic manipulation to find a correct expression for r_2 (0.2) -realize that $r_1 = r_2$ (0.2)
	0.4	**1.2d**use the cosine rule or algebraic manipulation to find ρ (0.4)
	0.6	**1.2eAnswer**: $r_1 = R + r$ (0.2) , $r_2 = R + r$ (0.2) $\rho = \sqrt{r^2 + rR + R^2}$ (0.2)

Questions	Points	Concepts/Details
1.3 (Total 3.2)	**0.8**	**1.3a** Express energy in terms of potential (0.2) (0.1 for each term) and kinetic energy (0.2) (0.1 for each term) and the conservation of angular momentum (0.4) (correct form of angular momentum 0.2 and correct substitution 0.2) to obtain $E=-\frac{GM\mu}{r_1}-\frac{Gm\mu}{r_2}+\frac{1}{2}\mu(\dot{\rho}^2+\frac{\rho_0{}^4\omega_0{}^2}{\rho^2})$
	0.5	**1.3b** Use the conservation of energy as $\frac{dE}{dt}=0$
	0.4	**1.3c** Use the equilateral triangle: $\Re^2=\rho^2+R^2$, (0.2) $\rho_0=\Re_0\cos 30°$ or equivalent (0.2)
	0.4	**1.3d** Express $\frac{d\Re}{dt}=\frac{d\Re}{d\rho}\frac{d\rho}{dt}=\frac{\rho}{\Re}\frac{d\rho}{dt}$ (0.2)to obtain $\frac{d^2\rho}{dt^2}=-GM(\frac{\rho}{r_1^3}+\frac{\rho}{r_2^3})+2\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}$ or $\frac{d^2\rho}{dt^2}=-\frac{2GM}{\Re^3}\rho+\frac{\rho_0{}^4\omega_0{}^2}{\rho^3}$ (0.2)
	0.3	**1.3e** Express perturbation for radial components $\Re$ and ρ $\rho=\rho_0(1+\frac{\Delta\rho}{\rho_0})$ (0.1), $\Delta\Re=\frac{\rho_0}{\Re_0}\Delta\rho$ (0.2)
	0.5	**1.3f -**Substitute $\Re$ and ρ to obtain expression for a simple harmonic motion $\frac{d^2\Delta\rho}{dt^2}\propto-\Delta\rho$ (0.3) - Express angular frequency of oscillation in terms of ω_0 only (0.2)
	0.3	**1.3g Answer:** frequency of oscillation $\frac{\sqrt{7}}{2}\omega_0$
1.3 (Total 3.2) Alternate Solution Marking Scheme	**0.4**	**1.3.2a** Initial Condition for $\omega_0=\frac{G(M+M)}{(R+R)^3}=\frac{GM}{4R^3}$ (0.2) $\rho_0=\sqrt{3}R$ (0.2)
	0.1	**1.3.2b** Evidence of perturbed radial distance e.g. $\rho=\sqrt{3}R+\zeta$ (0.1)
	0.3	**1.3.2c** Use of Newton's 2nd law
	0.6	**1.3.2d** Gravitational force(0.2)= mass(acceleration due to change in ρ (0.2) +centripetal force acceleration(0.2)). Note: For each 0.2 point 0.1 will be awarded for evidence for using correct concept and 0.1 for the correct expression)
	0.2	**1.3.2e** Correct distance between μ and M $\sqrt{R^2+(\sqrt{3}R+\zeta)^2}$
	0.2	**1.3.2f**Correctly project force into radial direction.

Questions	Points	Concepts/Details
	0.4	**1.3.1g**Using conservation of angular momentum (0.2) to obtain relationship between ω_0 and ω (0.2)
	0.7	**1.3.1h**Applying $\zeta^2 \approx 0$ approximation (0.1)and using binomial expansion(0.1) and algebraic manipulation (0.2) to obtain simple harmonic equation of ζ (0.3)
	0.3	**1.3.1i Answer** $\omega = \frac{\sqrt{7}}{2}\omega_0$
1.4 (Total1.8)	**0.4**	**1.4a** Find the angular velocity $\omega = \frac{2\pi}{T}$ using $T = 365 \times 24 \times 60 \times 60$ s (0.2 for correct relation between and T and ω and 0.2 for knowing numerical value of period = 1yr)
	0.5	**1.4b** Apply the circulation motion (0.1) and find the correct expression for radius (0.2) for each spacecraft to obtain $v = \omega \frac{L}{2\cos 30°}$ (0.2)
	0.6	**1.4c** Correct expression of relative velocity e.g $\vec{v}_{BC} = \vec{v}_B - \vec{v}_C$ (0.1) using drawing or vectors for each $\vec{v}_B$ (0.2) and $\vec{v}_C$ (0.2) $\vec{v}_{BC} = -2v\sin 60°\hat{\mathbf{j}} = -\sqrt{3}v\hat{\mathbf{j}}$ or $v_{BC} = \sqrt{3}v$ (0.1)
	0.3	**1.4d Answer:** $v_{BC} = 996$ m/s $\approx 1.0 \times 10^3$ m/s
1.4 note		Note 1.4a and 1.4b: Total of 0.9 will be awarded for any correct method for finding v from T. Points for the alternate solution using $v_{BC} = \omega L$ (axis of rotation is at one of the spacecrafts) will be given equivalently to the former solution.

Theoretical Question 2: An Electrified Soap Bubble

Questions	Points	Concepts/Details
2.1 (Total 1.7)	**0.3**	**2.1a** Know that the difference between pressure (or force) inside and outside the bubble comes from the surface pressure.
	0.3	**2.1b** Surface tension with **two** surfaces.
	0.5	**2.1c** use the concept of surface tension $dE = \gamma dA$ with correct $dA = d(4\pi r^2)$ (0.2) $dE = Fdr = \Delta PAdr$ (0.3) (other methods are also acceptable e.g. $F = \gamma L \frac{dE}{dx} = \gamma \frac{dA}{dx}$) If the sign of surface tension pressure is wrong, no mark awards.
	0.3	**2.1d** Correct usage of Ideal gas equation (0.1) $P = \frac{\rho RT}{M}$ (0.2 correct expression)
	0.3	**2.1e Answer:** $\frac{\rho_i T_i}{\rho_a T_a} = \left[1 + \frac{4\gamma}{R_0 P_a}\right]$ -If the sign of surface tension pressure is wrong, no mark awards. -No double penalty from part 2.1b - The term t cannot be included in this part since problem specify so
2.2 (Total 0.4)	**0.4**	**2.2a Answer:** $\frac{\rho_i T_i}{\rho_a T_a} - 1 = 0.0001$ For the answer ≥ 1: -0.2 major error 50% For the answer ≥ 0.5: -0.1 major error 25%
2.3 (Total 2.0)	**0.6**	**2.3a** Total weight from the mass of the bubble (0.2) and the inside air pulling downward (0.3), and substitute for ρ_i (0.1): $W = 4\pi R_0^2 \rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g$ - In case that the student doesn't include the surface tension term, deduct 0.3 point if the answer in 2.2a is greater than 1. (a major error) Otherwise, full points.
	0.6	**2.3b** Use $B = \rho_a g V$ (0.3) Use the correct volume term (0.3) $\frac{4}{3}\pi R_0^3$. The term $R_0 + t$ instead of R_0 is acceptable
	0.4	**2.3c** Setting up $B = W$ or $B \geq W$.
	0.4	**2.3d Answer:** $T_i \geq 307.1$ K - The range of answer within [305,309] is acceptable.

Questions	Points	Concepts/Details
2.4 **(Total 1.6)**	**0.5**	**2.4a** Setting the force balance $F \geq W - B$ ("equal sign" also acceptable) (0.5, but only give 0.1 for incorrect sign).
	0.2	**2.4b** Correct expressions for the weight of the bubble (0.1) plus the inside air (0.1). $W = \left(4\pi R_0^2\, \rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g$
	0.5	**2.4c** Thermal equilibrium means $T_i = T_a$ (0.3) and substitute for ρ_i (0.2)
	0.4	**2.4d Answer**: $u \geq \frac{4R_0\, \rho_s t g}{6\eta} + \frac{\frac{4}{3}R_0^2 \rho_a g \left(\frac{4\gamma}{R_0 P_a}\right)}{6\eta}$ - If the term due to surface tension is neglected in 2.3a, the second term above can also be neglected - In 2.3a, if the student uses $R_0 + t$ instead of R_0, there will be an additional third term. That is acceptable.
2.5 **(Total 0.4)**	**0.4**	**2.5a Answer**: $u \geq 0.36$ m/s or $u_{\min} = 0.36 m/s$ -The numerical value in range of [0.35,0.37] is acceptable
2.6 **(Total 2.0)** **Method A**	**0.2**	**2.6a** Gaussian Law leading to the electric field outside the soap bubble: $E_q = \frac{\sigma}{\varepsilon_0}$ *If no factor 1/2 , no mark for the following part b,c
	0.2	**2.6b** Gaussian Law leading to the electric field on the pill box: $E_\sigma = \frac{\sigma}{2\varepsilon_0}$
	0.3	**2.6c** Symmetry lead to the electric field from all other parts of the film excluding the pill box itself: $E = \frac{1}{2\varepsilon_0}\frac{q}{4\pi R_1^2}$
Or **Method B**	**0.2**	**2.6a** Charge on a small stripe of the bubble film: $\delta q = \left(\frac{q}{4\pi R^2}\right) 2\pi R \sin\theta . R\delta\theta$
	0.2	**2.6b** Form the integration with a correct stripe.
	0.3	**2.6c** Do the integration correctly: $E = \frac{1}{2\varepsilon_0}\frac{q}{4\pi R_1^2}$
2.6 cont.	**0.3**	**2.6d** Repulsive force per unit area of the bubble: $\frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0}$

Questions	Points	Concepts/Details
	0.4	**2.6e** Use Boyle's Law to find the new pressure.
	0.3	**2.6f** Balancing the pressurized force pushing inward and outward
	0.3	**2.6g Answer**: $\left(\frac{R_1}{R_0}\right)^4 - \left(\frac{R_1}{R_0}\right) - \frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} = 0$
2.7 (Total 0.7)	**0.3**	**2.7a** Apply the approximation: $R_1 = R_0 + \Delta R,\ \Delta R << R_0$
	0.4	**2.7b Answer**: $R_1 \approx R_0\left(1 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^4 P_a}\right)$
2.8 (Total 1.2)	**0.7**	**2.8a** Newton's Law (0.3). The balance between the weight (0.2) and the buoyancy (0.2). - Check the correct formula for weigh and buoyant force from (21) in the solution. No double penalty for the wrong formula of W from 2.4b. - If the student write down the weigh W in term of the new radius, R_1, and new density, that solution is acceptable too as long as it is correct.
	0.3	**2.8b Answer**: $q^2 \geq \frac{96\pi^2 R_0^3 \rho_s t \varepsilon_0 P_a}{\rho_a}$
	0.2	**2.8c Answer**: $q \approx 256\times10^{-9}$ C ≈ 256 nC -The numerical value in range of [250,260]nC is acceptable.

Theoretical Question 3: To Commemorate the Centenary of Rutherford's Atomic Nucleus: The scattering of an ion by a neutral atom

Questions	Points	Concepts/Details
3.1 (Total 1.2)	**0.3**	**3.1a** Use Coulomb's law - Write down inverse square law (0.2 pt) - Correct constant (0.1 pt)
	0.3	**3.1b** Take electric field from 2 charges - Write down superposition of electric field (0.2 pt) - Correct charge polarity/direction (0.1 pt)
	0.3	**3.1c** Correct distances - If the student didn't use the figure provided (-0.1pt)
	0.3	**3.1d Answer**: $E_p = +\frac{4qa}{4\pi\varepsilon_0 r^3}$ or $+\frac{qa}{\pi\varepsilon_0 r^3}$ or $\frac{2p}{4\pi\varepsilon_0 r^3}$
3.2 (Total 3.0)	**0.3**	**3.2a** Write down that the force is the product of electric field and charge. { $\vec{f} = Q\vec{E}_p$ }
	0.4	**3.2b Answer**: $\vec{f} = +\frac{4qa}{4\pi\varepsilon_0 r^3} Q\,\hat{r}$ or $+\frac{qa}{\pi\varepsilon_0 r^3} Q\,\hat{r}$ or $\frac{2p}{4\pi\varepsilon_0 r^3} Q\,\hat{r}$
	0.5	**3.2c** Use the electric field seen by the atom from the ion
	0.4	**3.2d** Use Coulomb's law to write down $\vec{E}_{ion} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ (magnitude 0.1 pt, sign 0.3 pt)
	0.2	**3.2e** Use the given expression for polarisability and write down $\vec{p} = \alpha\vec{E}_{ion} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}$
	0.5	**3.2f** Use the concept of induced dipole by substituting $\vec{p} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}$ in equation (2) of question (3.1) { $\vec{E}_p = \frac{1}{4\pi\varepsilon_0 r^3}\left[-\frac{2\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}\right]$ }…….(0.3 pt) Get $\vec{E}_p = -\frac{\alpha Q}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$ (magnitude 0.1 pt, sign 0.1 pt)
	0.3	**3.2g Answer**: $\vec{f} = -\frac{2\alpha Q^2}{(4\pi\varepsilon_0)^2 r^5}\hat{r} = -\frac{\alpha Q^2}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$
	0.2	**3.2h** Point out that the negative sign implies attractive force.

	0.2	**3.2i** Point out Q^2 implies that it is regardless of the sign of the ion.
3.3 (Total 0.9)	**0.5**	**3.3a** Use the definition of potential energy to write down $U = \int_r^\infty \vec{f}.d\vec{r}$ (wrong limit -0.2 pt)
	0.4	**3.3b Answer**: $U = -\dfrac{\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4}$ (magnitude 0.2 pt, sign 0.2 pt)
3.4 (Total 2.4)	**0.6**	**3.4a** State conservation of angular momentum (0.3 pt) Write down $mv_{\max}r_{\min} = mv_0 b$ (0.3 pt)
	0.6	**3.4b** State conservation of mechanical energy(0.3 pt) Write down $\dfrac{1}{2}mv_{\max}^2 + \dfrac{-\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4} = \dfrac{1}{2}mv_0^2$ (0.3 pt)
	0.5	**3.4c** Substituting $v_{\max}$ in term of $r_{\min}$ (0.1pt) Arrange in term of quadratic equation (0.2 pt) **Answer:** $r_{\min} = \dfrac{b}{\sqrt{2}}\left[1 \pm \sqrt{1 - \dfrac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}}$ (0.2pt)
	0.2	**3.4d** Choose "+" sign and write down $r_{\min} = \dfrac{b}{\sqrt{2}}\left[1 + \sqrt{1 - \dfrac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4}}\right]^{\frac{1}{2}}$
	0.5	**3.4e** State the reasoning of the sign with $Q = 0$ or $\alpha \le 0$
3.5 (Total 2.5)	**1.4**	**3.5a** Recognize that a spiral trajectory happens when $r_{\min}$ is imaginary because $b < b_0$. (0.7 pt) Recognize that $r_{\min}$ is imaginary when $1 - \dfrac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2 b^4} < 0$ (0.7 pt)
	0.7	**3.5b** Write down $b < b_0 = \left(\dfrac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2}\right)^{\frac{1}{4}}$
	0.4	**3.5c Answer**: $A = \pi\left(\dfrac{\alpha Q^2}{4\pi^2\varepsilon_0^2 m v_0^2}\right)^{\frac{1}{2}}$