PROBLEM 1

Lets examine collision of B1 and B2

$-m_1 \upsilon + m_2 \upsilon = m_1 \upsilon_1 + m_2 \upsilon_2$ (yy′) (1)

$\frac{1}{2} m_1 \upsilon^2 + \frac{1}{2} m_2 \upsilon^2 = \frac{1}{2} m_1 \upsilon_1^2 + \frac{1}{2} m_2 \upsilon_2^2$ (2)

[2 points]

Solving the system

$\upsilon_1=[(3m_2-m_1)/(m_2+m_1)]\,\upsilon$ $\upsilon 1=[(1-3\, m_1/m_2)/(1+m_1/m_2)]\,\upsilon$

but $m_1/m_2=0$ and $\upsilon_1=3\upsilon$

$I/2\, m_1 (3\upsilon)^2 = m_1 g H$

$\frac{1}{2}\; 9\upsilon^2 = gH$

$H = \frac{1}{2}\; 9\upsilon^2/g = 9h$

And the answer d+9h

[3 points]

Lets examine collision of B2 and B3

$-m_3 \upsilon + m_2 3\upsilon = m_3 \upsilon_3 + m_2 \upsilon_2$ (yy′) (1)

$\frac{1}{2} m_3 \upsilon^2 + \frac{1}{2} m_2 9\upsilon^2 = \frac{1}{2} m_3 \upsilon_3^2 + \frac{1}{2} m_2 \upsilon_2^2$ (2)

Solving the system

$\upsilon_3=7\upsilon$

[2 points]

In general

$\upsilon_n = (2^n - 1)\upsilon$

[8 points]

$H = `l + ((2^n - 1)\upsilon)^2 / 2g = l + (2^n - 1)^2 h$

[1 point]

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming l is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

[4 points]

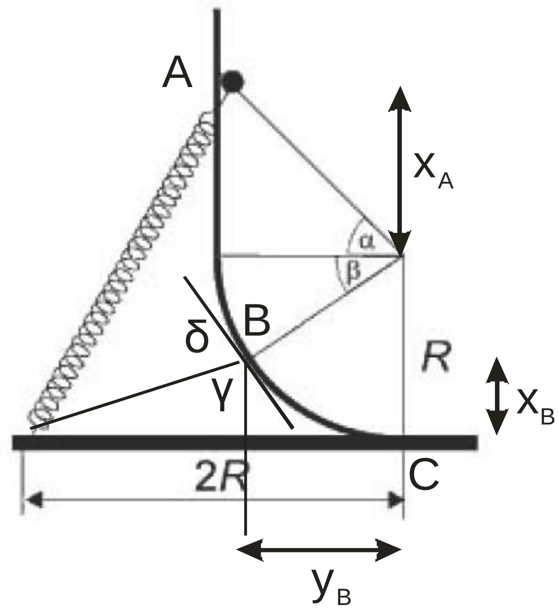
Escape velocity from the earth (which is $\upsilon_{esc} = \sqrt{2gR} \approx 11,200$ m/s) is reached when

$\upsilon_n \geq \upsilon_{esc} \Longrightarrow (2n-1)\,\mathrm{sqrt}(2gh) \geq \mathrm{sqrt}(2gR) \Longrightarrow n \geq \log_2(\mathrm{sqrt}(R/h) + 1)$

With $R = 6.4 \bullet 10^6$ m and h = 1 m, we find n > 12.

[5 points]

PROBLEM 2



1. Calculate the length of the spring when the ball is at the starting position at point A. **[3 points]**

$$L_A^2 = R^2 + (R + x_A)^2$$ **[1 point]**

$$tg\alpha = \frac{x_A}{R}, \quad \alpha = 45^0 \quad \Rightarrow \quad tg\alpha = 1 \quad \Rightarrow \quad x_A = R$$ **[1 point]**

$$\Rightarrow L_A^2 = R^2 + (2R)^2 = 5R^2$$

$$\Rightarrow L_A = R\sqrt{5} = 44.7cm \simeq 45cm.$$ **[0.5 + 0.5 point]**

2. Calculate the elongation of the spring when the ball is at the starting position at point A. **[1 points]**

$$\Delta L_A = L_A - L_0 = 24.7cm \simeq 25cm.$$ **[0.5 + 0.5 point]**

3. Calculate the length of the spring when the ball is at the position of maximal velocity at point B. **[4 points]**

$L_B^2 = (2R - y_B)^2 + x_B^2$ **[1.5 point]**

$x_B = R - R\sin\beta = R(1 - \sin\beta)$ **[0.5 point]**

$y_B = R\cos\beta$ **[0.5 point]**

$\beta = 34^0 \quad \Rightarrow \quad \sin\beta = 0.559, \quad \cos\beta = 0.829$

$\Rightarrow x_B = 8.82cm$

$\Rightarrow y_B = 16.6cm$

$$\Rightarrow L_B^2 = (2R - R\cos\beta)^2 + R^2(1-\sin\beta)^2 =$$
$$= R^2\left[(2-\cos\beta)^2 + (1-\sin\beta)^2\right] =$$ **[1 point]**
$$= R^2(6 - 4\cos\beta - 2\sin\beta) = 1.566R^2$$

$\Rightarrow L_B = R\sqrt{1.566} = 25.02cm \simeq 25cm.$ **[0.5 point]**

4. Calculate the elongation of the spring when the ball is at the of maximal velocity at point B. **[1 points]**

$\Delta L_B = L_B - L_0 = 5.02cm \simeq 5.0cm.$ **[0.5 + 0.5 point]**

5. Calculate the magnitude of the spring force when the ball has maximal velocity at point B. **[2 points]**

$F_k = k\Delta L_B = 5.02N \simeq 5.0N.$ **[1.5 + 0.5 point]**

6. Calculate the acceleration of the ball tangential to the track at point B. **[2 points]**

$v_B = v_{MAX} \quad \Rightarrow \quad a_{Bt} = 0.$

7. Calculate the mass of the ball. **[5 points]**

At point B, using $a_{Bt} = 0$ we can write

$$F_k \cos\delta = mg \cos\beta$$ **[2.5 points]**

$$\Rightarrow m = \frac{F_k \cos\delta}{g \cos\beta}$$ **[1 point]**

$$\delta = 180^0 - (\gamma + \beta)$$ **[0.5 points]**

$$\cos\gamma = \frac{x_B}{L_B} = \frac{8.82}{25.02} = 0.352 \quad \Rightarrow \quad \gamma = 69.3^0$$

$$\Rightarrow \delta = 180^0 - (69.4^0 + 34^0) = 76.7^0$$ **[0.5 points]**

$$\Rightarrow \cos\delta = 0.231^0$$

$$\Rightarrow m = \frac{5.02 \cdot 0.231}{9.81 \cdot 0.829} kg = 0.142 kg \simeq 0.14 kg.$$ **[0.5 points]**

8. Calculate the maximum speed of the ball (at point B). **[4 points]**

We use work-kinetic energy theorem for positions A and B:

$$mgh_A + \frac{1}{2} k \Delta L_A^2 = mgh_B + \frac{1}{2} k \Delta L_B^2 + \frac{1}{2} m v_{MAX}^2$$ **[2 points]**

$$\Rightarrow v_{MAX}^2 = 2g(h_A - h_B) + \frac{k}{m}(\Delta L_A^2 - \Delta L_B^2)$$ **[0.5 points]**

$$h_A = 2R, \quad h_B = x_B = R(1 - \sin\beta)$$ **[0.5 + 0.5 points]**

$$\Rightarrow v_{MAX}^2 = \ldots = 47.01 \frac{m^2}{s^2}$$

$$\Rightarrow v_{MAX} = 6.88 \frac{m}{s} \simeq 6.9 \frac{m}{s}$$ **[0.5 points]**

9. Calculate the speed of the ball at point C. **[3 points]**

We use again work-kinetic energy theorem for positions A and C:

$$mgh_A + \frac{1}{2}k\Delta L_A^2 = mgh_C + \frac{1}{2}k\Delta L_C^2 + \frac{1}{2}mv_C^2$$ **[0.5 points]**

$$\Rightarrow v_C^2 = 2g(h_A - h_C) + \frac{k}{m}(\Delta L_A^2 - \Delta L_C^2)$$ **[0.5 points]**

$h_A = 2R, \quad h_C = 0$ **[0.5 + 0.5 points]**

$\Delta L_C = L_C - L_0 = 2R - L_0 = 20cm$ **[0.5 points]**

$$\Rightarrow v_C^2 = \ldots = 22.54\frac{m^2}{s^2}$$

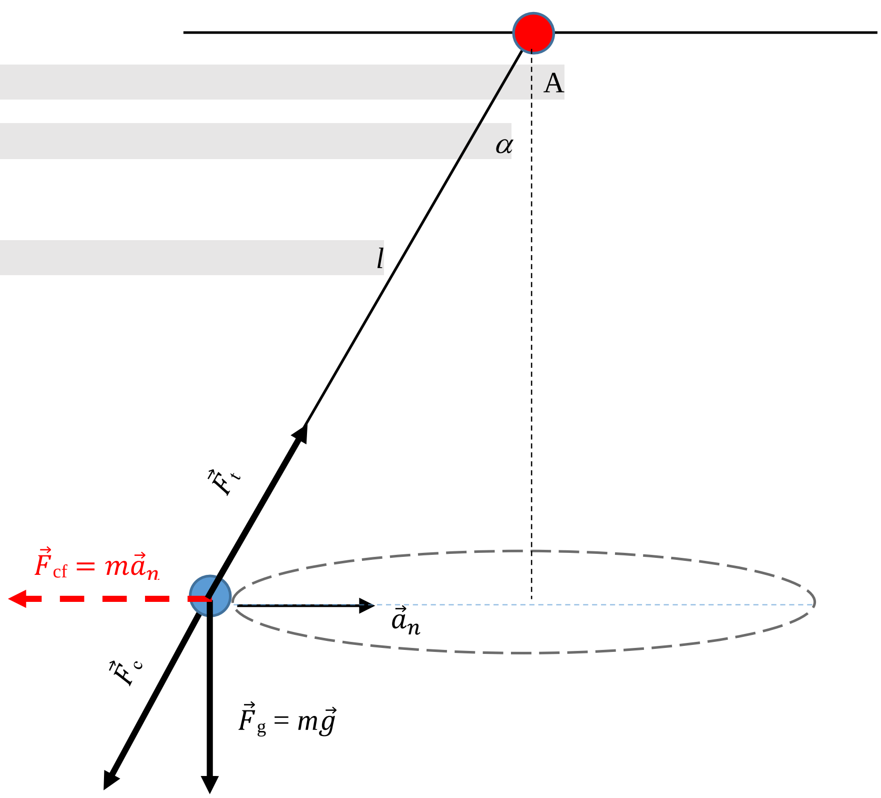
$\Rightarrow v_C = 4.76\frac{m}{s} \simeq 4.8\frac{m}{s}.$ **[0.5 points]**

PROBLEM 3

A ball of mass m = 1 g is charged with an amount of positive charge q = 1 nC. It is suspended by a non – conducting thread of length l = 0.5 m. The mass of the thread is negligible. The thread makes an angle $\alpha = 30^\circ$ with the vertical axis, as shown in figure.
An additional positive charge Q = 2 nC can be placed on the vertical axis in three different positions. In the figure those positions are marked with letters A, B and C. The distance AB is equal to BC. The ball rotates around a vertical axis in a circular path. The friction forces in the system can be neglected. The gravitational acceleration in the system is $g = 9.81\ \mathrm{m/s^2}$. Constant $k = 9\cdot10^9\ \mathrm{Nm^2/C^2}$.

A. When the additional charge is at point A:

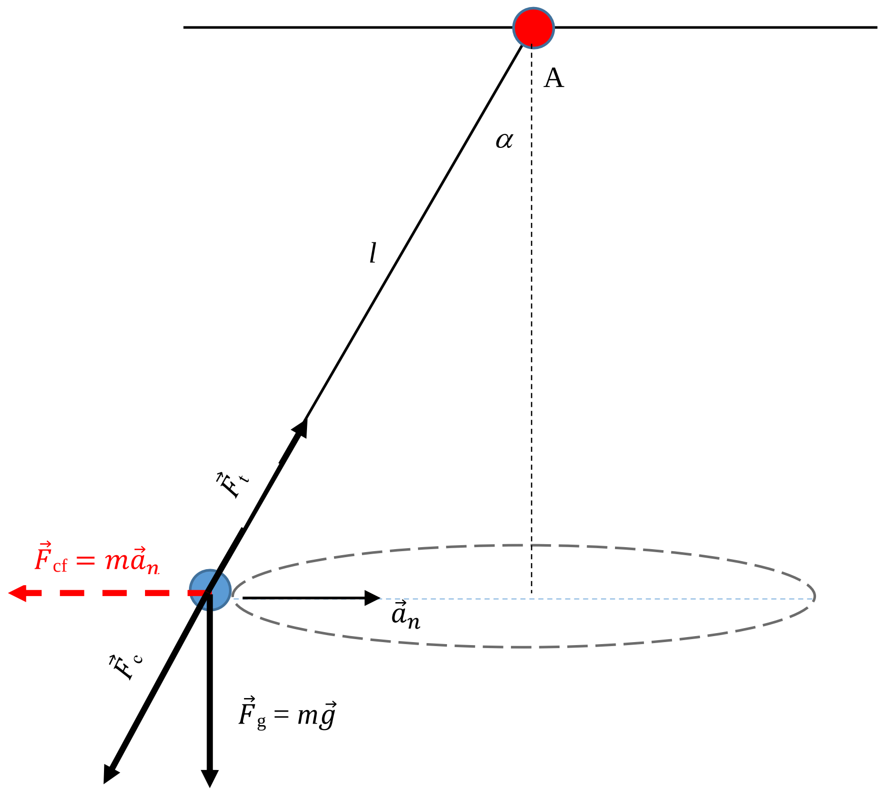
1. Draw the free body diagram for the rotating ball. (**1 point**)



Note:

Since centrifugal force is a fictitious force acting on a ball, it can be omitted from the free body diagram. In that case the marking scheme is the same, i.e. the maximal number of points that can be given without centrifugal force in the diagram is the same.

2. Using the diagram write the equations of motion of the ball. (**2 points**)



The charge q is at rest in its own reference frame. Therefore (**0.5 points**):

$$\vec{F}_g + \vec{F}_{cf} + \vec{F}_c + \vec{F}_t = \vec{0}$$

In this case the centrifugal force must be included. By decomposing the vectors along horizontal and perpendicular axes (**0.5 points**), we get the equations of motion of the ball: (**1 point**)

$$ma_n + k\frac{qQ}{l^2}\sin\alpha = F_t\sin\alpha$$

$$mg + k\frac{qQ}{l^2}\cos\alpha = F_t\cos\alpha$$

3. Determine the tension force in the thread. (**2 points**)

From the equation above, it follows that

$$F_{\mathrm{t}} = \frac{mg}{\cos\alpha} + k\frac{qQ}{l^2} = 11.3\ \mathrm{mN}$$

4. Assuming that the circular motion is uniform, derive the formula for the linear speed of the ball. (**2 points**)

Using the equation (**1 point**)

$$\frac{mv^2}{l\sin\alpha} + k\frac{qQ}{l^2}\sin\alpha = F_{\mathrm{t}}\sin\alpha$$

it can be obtained (**1 point**)

$$v = \sqrt{\frac{gl\sin^2\alpha}{\cos\alpha}}$$

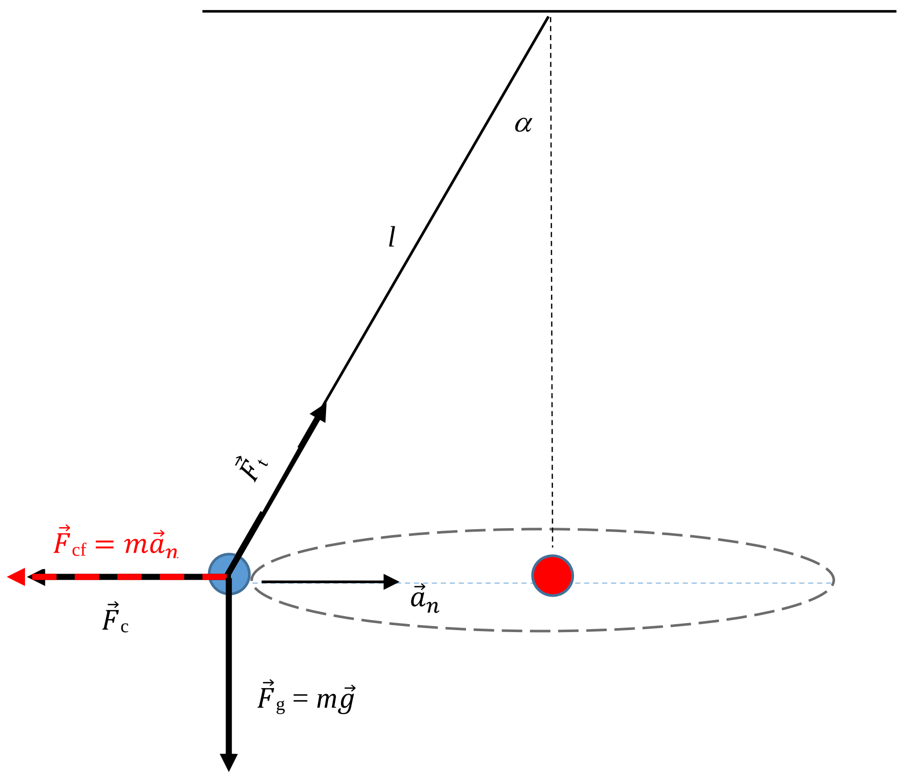
5. Determine the period of rotation of the ball. (**2 points**)

The distance travelled by the ball during one period is equal to the circumference of the circle $s = 2\pi l\sin\alpha$. (**1 point**)
Therefore, the period of rotation is (**1 point**)

$$T = \frac{s}{v} = 2\pi\sqrt{\frac{l\cos\alpha}{g}} = 1.322\ s$$

B. Solve the problems 1 – 5 when the charge Q is at point B.

1. Draw the free body diagram for the rotating ball. (**1 point**)



Note:

Since centrifugal force is a fictitious force acting on a ball, it can be omitted from the free body diagram. In that case the marking scheme is the same, i.e. the maximal number of points that can be given without centrifugal force in the diagram is the same.

2. Using the diagram write the equations of motion of the ball. (**2 points**)

The charge q is at rest in its own reference frame. Therefore (**0.5 points**):

$$\vec{F}_{\mathrm{g}} + \vec{F}_{\mathrm{cf}} + \vec{F}_{\mathrm{c}} + \vec{F}_{\mathrm{t}} = \vec{0}$$

In this case the centrifugal force must be included. By decomposing the vectors along horizontal and perpendicular axes (**0.5 points**), we get the equations of motion of the ball: (**1 point**)

$$ma_n + k\frac{qQ}{l^2\sin^2\alpha} = F_{\mathrm{t}}\sin\alpha$$

$$mg = F_{\mathrm{t}}\cos\alpha$$

3. Determine the tension force in the thread. (**1 point**)

From the equation above, it follows that

$$F_{\mathrm{t}} = \frac{mg}{\cos\alpha} = 11.32\ \mathrm{mN}$$

4. Assuming that the circular motion is uniform, derive the formula for the linear speed of the ball. (**2 points**)

Using the equation (**1 point**)

$$\frac{mv^2}{l\sin\alpha} + k\frac{qQ}{l^2\sin^2\alpha} = F_{\mathrm{t}}\sin\alpha$$

it can be obtained (**1 point**)

$$v = \sqrt{\left(mg\mathrm{tg}\alpha - k\frac{qQ}{l^2\sin^2\alpha}\right)\frac{l\sin\alpha}{m}}$$

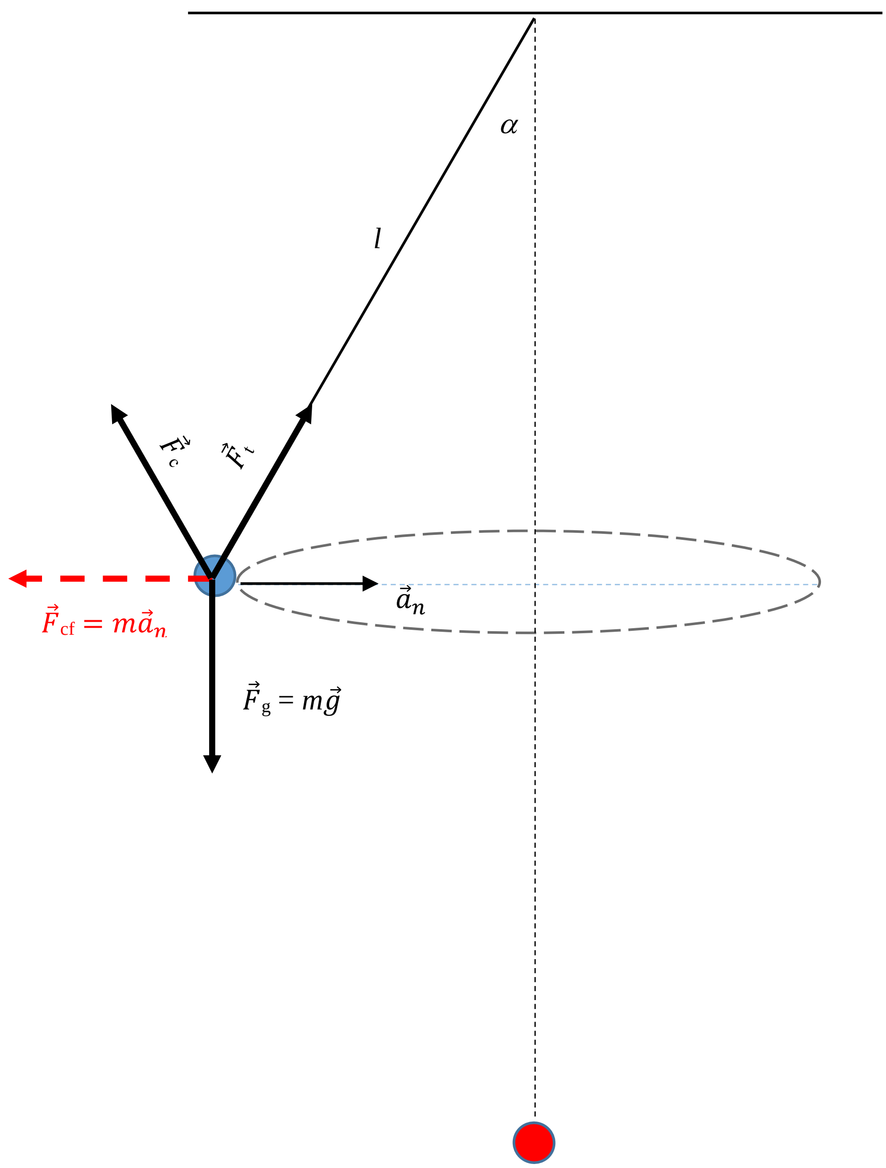
5. Determine the period of rotation of the ball. (**2 points**)

The distance travelled by the ball during one period is equal to the circumference of the circle $s = 2\pi l\sin\alpha$. (**1 point**)
Therefore, the period of rotation is (**1 point**)

$$T = \frac{s}{v} = 2\pi\sqrt{\frac{ml\sin\alpha}{\left(mg\mathrm{tg}\alpha - k\frac{qQ}{l^2\sin^2\alpha}\right)}} = 1.3205\ s$$

C. Solve the problems 1 - 5 when the charge Q is at point C.

1. Draw the free body diagram for the rotating ball. (**1 point**)



Note:

Since centrifugal force is a fictitious force acting on a ball, it can be omitted from the free body diagram. In that case the marking scheme is the same, i.e. the maximal number of points that can be given without centrifugal force in the diagram is the same.

2. Using the diagram write the equations of motion of the ball. (**2 points**)

The charge q is at rest in its own reference frame. Therefore (**0.5 points**):

$$\vec{F}_{\mathrm{g}} + \vec{F}_{\mathrm{cf}} + \vec{F}_{\mathrm{c}} + \vec{F}_{\mathrm{t}} = \vec{0}$$

In this case the centrifugal force must be included. By decomposing the vectors along horizontal and perpendicular axes (**0.5 points**), we get the equations of motion of the ball: (**1 point**)

$$ma_n + k\frac{qQ}{l^2}\sin\alpha = F_{\mathrm{t}}\sin\alpha$$

$$mg = \left(F_{\mathrm{t}} + k\frac{qQ}{l^2}\right)\cos\alpha$$

3. Determine the tension force in the thread. (**1 point**)

From the equation above, it follows that

$$F_{\mathrm{t}} = \frac{mg}{\cos\alpha} - k\frac{qQ}{l^2} = 11.328\ \mathrm{mN}$$

4. Assuming that the circular motion is uniform, derive the formula for the linear speed of the ball. (**2 points**)

Using the equation (**1 point**)

$$\frac{mv^2}{l\sin\alpha} + k\frac{qQ}{l^2}\sin\alpha = F_{\mathrm{t}}\sin\alpha$$

it can be obtained (**1 point**)

$$v = \sqrt{\left(F_{\mathrm{t}} - k\frac{qQ}{l^2}\right)\frac{l}{m}}\ \sin\alpha$$

5. Determine the period of rotation of the ball. (**2 points**)

The distance travelled by the ball during one period is equal to the circumference of the circle $s = 2\pi l \sin\alpha$. (**1 point**)
Therefore, the period of rotation is (**1 point**)

$$T = \frac{s}{v} = 2\pi \sqrt{\frac{ml}{\left(F_{\mathrm{t}} - k\frac{qQ}{l^2}\right)}} = 1.3201\ s$$

PROBLEM 4

A. The simplest choice of variables is:

$$y = \ell/n\,; x = n \tag{1}$$

In this case:

$$y = A + Bx \tag{2}$$

Alternatively, the student may choose to work with:

$$y = \ell/n^2\,; x = 1/n \tag{3}$$

and:

$$y = Ax + B \tag{4}$$

though calculations in this case are more time consuming and the results are less accurate.

Marking scheme for A

Element of solution	Points
Defines a set of variables as in equation (1) or (3)	1.0
Writes explicitly the linear dependence (2) or (4) for the given choice of variables	1.0
Totally on A	**2.0**

B. We will present the solution corresponding to the choice of variables in equations (1) and (2). In this case students must calculate values of *y* only, as shown in Table 1, since *x* coincides with *n*. The final marking scheme for the choice (3)–(4) will follow similar criteria.

Table 1

$n \equiv x$	ℓ (cm)	y (cm)	
5	1.5	0.30	
10	3.9	0.39	
15	7.6	0.51	
20	12.9	0.65	
25	18.3	0.73	
30	25.0	0.83	
35	33.4	0.95	
40	42.8	1.07	
45	53.2	1.18	
50	64.6	1.29	

Figure 1 shows data points on a *x*-*y* graph and the line, approximating the data. If we measure coordinates of two pints of the fitting line, (x_1,y_1) and (x_2,y_2), then:

$$y_1 = A + Bx_1; y_2 = A + Bx_2 \tag{5}$$

and the coefficients A and B could be calculated as:

$$A = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}; B = \frac{y_2 - y_1}{x_2 - x_1} \quad (6)$$

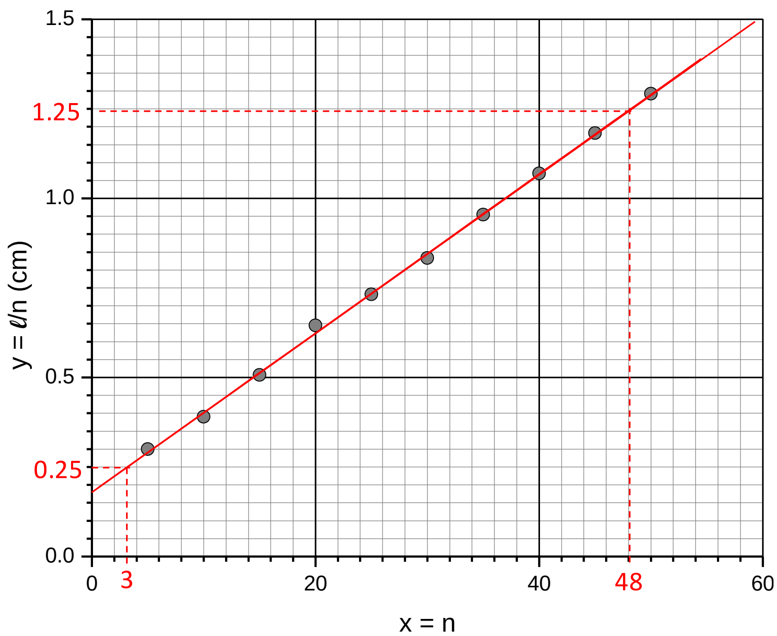
For the choice shown in Fig. 1, $x_1 = 3; y_1 = 0.25$ cm and $x_2 = 48; y_2 = 1.25$ cm. Therefore:

$$A \approx 0.18 \text{ cm}$$

$$B \approx 0.022 \text{ cm}$$

Alternatively A could be determined as the y-coordinate of the intersection between the fitting line and the vertical axis. In this case, however, A is much less accurate and values between 0.15 mm and 0.20 mm could be reported by the students.

Figure 1



Marking scheme for B

Element of solution	Points
Writes calculated y-values (variable choice in eqs. 1–2) or x and y (variable choice in eqs. 3–4) in the empty cells of Table 1. - Values are calculated correctly and are written with as many significant digits as the original data: 0.2 pts./row, maximum 9×0.2 = - Additional table column(s) is (are) annotated with title(s) and unit(s)	1.8 0.2
Axes of the graph are annotated with titles and unit(s) Grid labels on X Grid labels on Y Data points are positioned correctly: 0.2 pts./data point, maximum 9×0.2 =	0.2 0.5 0.5 1.8
Approximating (fitting) line is drown (Deduce 0.5 if number of points below and above the fitting line differ by 2 or more) Coefficients A and B are calculated from measurements performed on two points of the fitting line (eqs. 5-6 or equivalent). Coefficient A could be determined directly from the intersection between fitting line and y-axis, as well.	1.0 1.0
Calculated value of A is within interval [0.16 cm ; 0.20 cm] Additional points if A is within interval [0.17 cm ; 0.19 cm] Additional points if value of A, presented with two significant digits, is 0.18 cm Calculated value of B is within interval [0.024 cm ; 0.020 cm] Additional points if A is within interval [0.023 cm ; 0.021 cm] Additional points if value of A, presented with two significant digits, is 0.022 cm (No points are given for numerical values of A or B, if the units are not specified)	0.5 0.5 0.5 0.5 0.5 0.5
Totally on B	**10.0**

C. We will enumerate the hanging turns from 1 to n starting from the lowest free end toward the upper turn attached to the holder. The mass of one single turn is:

$$m_{\mathrm{t}} = \frac{M}{N} \tag{7}$$

The deformation $\Delta\ell_i$ of i-th turn is due to the total weight $W = (i-1)m_{\mathrm{t}}g$ of all i–1 turns underneath. From the Hook's law:

$$\Delta\ell_i = \frac{W}{k_{\mathrm{t}}} = \frac{(i-1)Mg}{Nk_{\mathrm{t}}} \tag{8}$$

The total deformation of the hanging part is a sum of the deformation of individual turns:

$$\Delta\ell = \Delta\ell_1 + \cdots + \Delta\ell_n = \frac{n(n-1)Mg}{2Nk_{\mathrm{t}}} \tag{9}$$

The measured length is a sum of the undeformed length $n\ell_{\mathrm{t}}$ of the hanging part and the deformation:

$$\ell = n\ell_{\mathrm{t}} + \frac{n(n-1)Mg}{2Nk_{\mathrm{t}}} = \left(\ell_{\mathrm{t}} - \frac{Mg}{2Nk_{\mathrm{t}}}\right)n + \frac{Mg}{2Nk_{\mathrm{t}}}\, n^2 \tag{10}$$

By comparing with equation 1, we identify:

$$A = \ell_\mathrm{t} - \frac{Mg}{2Nk_\mathrm{t}} = \ell_\mathrm{t} - B \tag{11}$$

and

$$B = \frac{Mg}{2Nk_\mathrm{t}} \tag{12}$$

Marking scheme for C

Element of solution	Points
Writes down eq. 7 or equivalent	0.2
Uses the Hook's law to express the deformation of a single turn – eq. 8 or equivalent	0.5
Finds out expression for the total deformation of the hanging part –eq. 9 or equivalent	1.0
Adds up the undeformed length of hanging part to the deformation to obtain the total length – eq. 10 or equivalent	0.5
Writes down an explicit expression for *A* – eq. 11	1.4
Writes down an explicit expression for *A* – eq. 12	1.4
Totally on C	**5.0**

D. From the theoretical expressions for *A* and *B*, and by using their experimental values:

$$k_\mathrm{t} = \frac{Mg}{2NB} = \frac{0.0723\ \mathrm{kg} \cdot 9.81\ \mathrm{m/s}^2}{2 \cdot 108 \cdot 2.2 \times 10^{-4}\ \mathrm{m}} \approx 15\,\mathrm{N/m} \tag{13}$$

and

$$\ell_\mathrm{t} = A + B = 0.18\ \mathrm{cm} + 0.022\ \mathrm{cm} \approx 0.20\ \mathrm{cm}\ (2.0\ \mathrm{mm}) \tag{14}$$

Marking scheme for D

Element of solution	Points
Writes down eq. 13 or equivalent - explicit expression for k_t in terms of *B* - input numeric data are transformed to basic SI units – kg and m - numeric value of k_t is calculated correctly and is written with corresponding SI unit	 0.5 0.5 1.0
Writes down eq. 14 or equivalent - explicit expression for ℓ_t in terms of *A* and *B* - numeric value of ℓ_t is calculated correctly and is written with corresponding SI unit	 0.5 0.5
Totally on D	**3.0**

E. The total length of the spring in undeformed state is:

$$L = N\ell_{\mathrm{t}} = 108 \times 0.20\ \mathrm{cm} \approx 22\ \mathrm{cm} \tag{15}$$

If forces with opposite direction and the same magnitude *F* are applied to both ends of slinky, each single turn will be stretched by:

$$\Delta\ell_{\mathrm{t}} = F/k_{\mathrm{t}} \tag{16}$$

The total deformation of the spring will be:

$$\Delta L = N\Delta\ell_{\mathrm{t}} = NF/k_{\mathrm{t}} \tag{17}$$

Therefore, the stiffness of the whole slinky is:

$$K = \frac{F}{\Delta L} = \frac{k_{\mathrm{t}}}{N} = \frac{15\ \mathrm{N/m}}{108} \approx 0.14\ \mathrm{N/m} \tag{18}$$

Marking scheme for E

Element of solution	Points
Writes down eq. 15 or equivalent - explicit expression for L - numeric value for L with appropriate number of significant digits and SI units Calculated value of L must be consistent with the value of ℓ_{t} obtained by the student in part D and thus may deviate from that in eq. 15.	 1.0 0.5
Realizes that the same force F is stretching all the turns Uses Hook's law to express the deformation of one single turn – eq. 16 Writes down expression for the total deformation of slinky – eq. 17 Derives analytic expression for K in terms of k_{t} and N – eq. 18 Calculates K numerically with the appropriate number of significant digits and SI units. Calculated value of K must be consistent with the value of k_{t} obtained by the student in part D and thus may deviate from that in eq. 18.	0.5 1.0 0.5 1.0 0.5
Totally on E	**5.0**