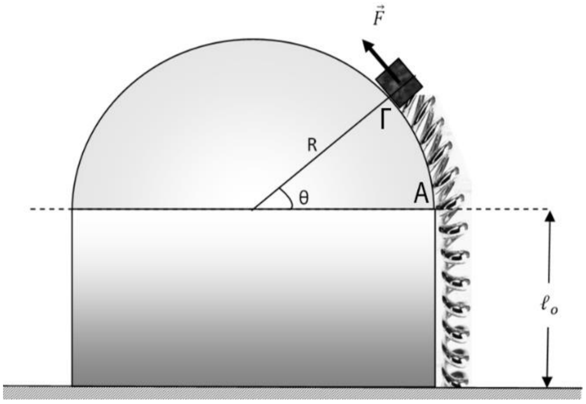
Q1 solution

1. Springs

A. (10 points) A conservative variable force F is slowly pulling a body of weight W as illustrated in the figure, along a smooth semi-sphere with a radius R. The force acts tangential to the semi-sphere. When the body is in the position A the spring has its natural length.

Calculate the work done by F, to move the body from A to Γ.



Solution:

In position A where the spring has its natural length we considered that the gravitational potential energy and the potential energy of the spring is zero.

The body moves very slowly. Consequently its kinetic energy does not change. **(2 p)**

If x is the strain of the spring at position Γ, then, for the spring-body system we have:

$\Delta U = U_2 - U_1 = U_g + U_s - 0 = mg\,R\sin\theta + 1/2\,K\,(R\,\theta)^2$ **(4 p)**

$U_g = mgh = mg\,(R\sin\theta)$ is the gravitational potential energy at position Γ **(2 p)**

$U_s = 1/2\,K\,x^2 = 1/2\,K\,(R\theta)^2$. the potential energy of the spring at position Γ **(2 p)**

B. (15 points) For a particular horizontal spring, the intensity of the elastic force, depends on the deformation x as follows: $F\,(x) = \alpha \cdot x + \beta$, where $\alpha = 50\ \mathrm{N/m}$ and $\beta = 10\ \mathrm{N}$.

B1. (5 points) Calculate the potential energy function $U(x)$ for this spring. Assume that $U(0) = 0$.

Solution:

$$W = -\,(U_2 - U_1)$$ **(1 p)**

By plotting the force F versus position x and calculating the area of the trapezoid formed from $x = 0$ to position x we will find

$$W = -\beta x - 0.5\alpha\, x^2,$$ **(2 p)**

since at $x = 0$, $U_1 = 0$ **(1 p)**

then

$$U(x) = \beta x + 0.5\alpha\, x^2.$$ **(1 p)**

B2. (10 points) An object of mass 2 kg is attached to the end of this spring and it's elongated with 1.5 m on a smooth horizontal surface and then released. Determine the speed of the object when the elongation is 1.0 m for the first time.

Solution:

We apply the principle of conservation of mechanical energy for the two positions $x_1 = 1.5$ m and $x_2 = 1.0$ m of the object and we have

$$K_1 + U_1 = K_2 + U_2 \quad (1)$$ **(4 p)**

$$K_1 = 0 \text{ and } K_2 = 1/2\, m v^2 \quad (2)$$ **(3 p)**

For x=1.5 m

$$U_1 = \beta x + 0.5\alpha\, x^2 = 71{,}25\, J \qquad (3)$$ **(0.5 p)**

For x=1.0 m

$$U_2 = 35\, J \qquad (4)$$ **(0.5 p)**

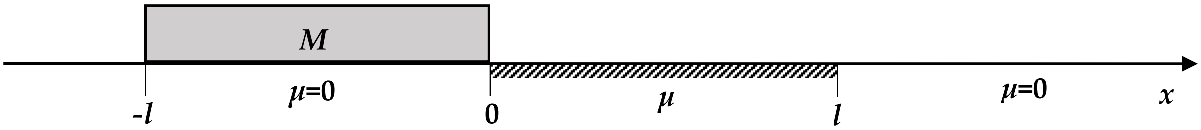
Hence

$$v = 6.0\,\frac{\mathrm{m}}{\mathrm{s}}.$$ **(2 p)**

Question 2: Plank

A plank of mass M and length l lies on a smooth horizontal surface, and it can move across a rough area (*i.e.* along the x axis), characterized by its length l and the sliding friction coefficient μ, the same as the static one. The initial position of the plank is that depicted in the figure below.

A. (6.3 points) The plank is launched with the unknown initial, horizontal speed v_0, towards the rough area. Derive the minimum initial speed of the plank for which:



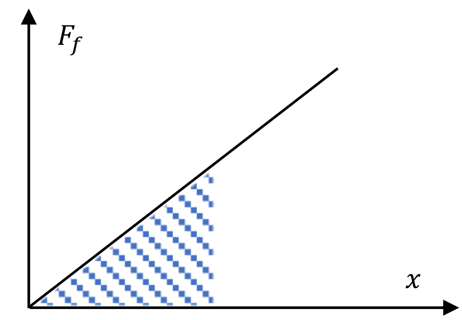
A1. (5 points) it fully enters the rough area.

Solution: (5 p)

When the plank has a portion of length x on the rough area, the friction force acting on it is:

$$F_f(x) = \mu\left(\frac{M}{l}x\right)g. \quad \textbf{(1 p)}$$

Graphing this force **(1 p)**, we get the adjacent representation, the area below which is the work of the friction force.



So, writing the kinetic energy theorem for the plank, it reads:

$$\frac{Mv^2(x)}{2} - \frac{Mv_0^2}{2} = -\frac{\mu Mg}{l}\frac{x^2}{2}, \quad \textbf{(1 p)}$$

from which it follows that

$$v_0 = \sqrt{v^2(x) + \frac{\mu g}{l}x^2}. \quad \textbf{(0.6 p)}$$

This speed is minimal when

$$v(l) = 0, \quad \textbf{(0.7 p)}$$

so

$$v_{0,\min} = \sqrt{\mu g l}. \quad \textbf{(0.7 p)}$$

A2. (1.3 points) it completely surpasses the rough area.

Solution: (1.3 p)

For the plank to completely exit the rough area, the energy lost by friction doubles **(1 p)**, so

$$v_{0,\min} = \sqrt{2\mu g l}. \quad \textbf{(0.3 p)}$$

B. (3.7 points) The plank starts from rest as illustrated in the figure above but is pulled to the right by a constant horizontal force F_0, permanently acting on it. The purpose is to pull the plank on the rough area.

B1. (2.5 points) Determine the minimum value of the force for the plank to completely enter the rough area.

Solution (2.5 p):

In this case, the kinetic energy theorem is

$$\frac{Mv^2(x)}{2} = F_0 x - \frac{\mu Mg}{2l}x^2. \quad \textbf{(1.5 p)}$$

Again, the force has the minimum value when

$$v(l) = 0, \qquad \textbf{(0.5 p)}$$

so

$$F_{0,\min} = \frac{\mu M g}{2}. \qquad \textbf{(0.5 p)}$$

B2. (1.2 points) Derive the maximum value of the plank's speed during its motion analyzed at B1, for the minimum value found for F_0.

Solution (1.2 p):

The plank's speed is a quadratic function of x, having the maximum value when

$$x = \frac{F_0 l}{\mu M g} = \frac{l}{2}. \qquad \textbf{(0.5 p)}$$

For this value, the maximum speed is

$$v_{max} = \frac{1}{2}\sqrt{\mu g l}. \qquad \textbf{(0.7 p)}$$

C. (15 points) The plank starts from rest as illustrated in the figure above but is pulled to the right by a constant horizontal force F_0, permanently acting on it. The purpose is to make the plank surpass the rough area for a minimum value of F_0.

C1. (3.7 points) Make a graph representation of the net force acting on the plank versus the coordinate x of its front end for $x \in [0, 3l]$.

Solution (3.7 p):

For the plank's entrance on the rough area, *i.e.* $x \in [0, l]$, the net force acting on it is

$$F(x) = F_0 - \frac{\mu M g}{l} x. \qquad \textbf{(0.8 p)}$$

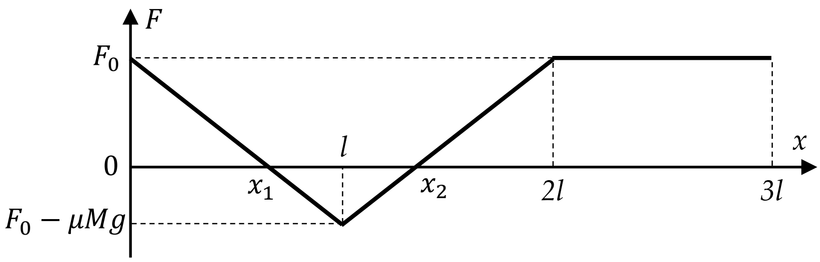
For the plank's exit from the rough area, *i.e.* $x \in [l, 2l]$, the friction force acting on it is

$$F_f(x) = \frac{\mu M g}{l}[l - (x - l)] = \frac{\mu M g}{l}(2l - x), \qquad \textbf{(0.8 p)}$$

so, the net force acting on it is

$$F(x) = F_0 + \frac{\mu M g}{l}(x - 2l). \qquad \textbf{(0.4 p)}$$

The graph is depicted below **(1.2 p)**: (for each value on the graph, the quantities represented



on the axis and each line segment, there will be given 0.1 p).

If F_0 is to be minimized, the lowest point on the graph should be below zero, which means that

$$F_0 < \mu M g. \qquad \textbf{(0.5 p)}$$

C2. (5 points) Determine the minimum value of the force for the plank to completely exit the rough area. It is known that the minimum value of the plank kinetic energy is very small, *i.e.* a very small fraction ε of the maximum value of the kinetic energy the plank had until it reached the minimal value. From the mathematical point of view, the fact that $\varepsilon \ll 1$ means that its algebraic powers higher than one can be neglected. The value of ε is known.

Solution **(5 p):**

The maximum value of the kinetic energy is reached when

$$x = x_1 = \frac{F_0 l}{\mu M g} \qquad \textbf{(0.5 p)}$$

and the minimum value when

$$x = x_2 = 2l - \frac{F_0 l}{\mu M g}. \qquad \textbf{(0.5 p)}$$

The kinetic energy theorem gives:

$$W_{k,\max} = \frac{F_0 x_1}{2} = \frac{F_0^2 l}{2\mu M g} \qquad \textbf{(0.8 p)}$$

and

$$W_{k,\min} = \frac{F_0 x_1}{2} - (\mu M g - F_0)\frac{x_2 - x_1}{2} = \qquad \textbf{(0.7 p)}$$

$$= \frac{F_0^2 l}{2\mu M g}\left[1 - 2\left(\frac{\mu M g}{F_0} - 1\right)^2\right]. \qquad \textbf{(0.6 p)}$$

Since

$$W_{k,\min} = \varepsilon W_{k,\max}, \qquad \textbf{(0.6 p)}$$

from the above two equations it follows that

$$F_0 = \frac{\mu M g}{1 + \sqrt{\frac{1-\varepsilon}{2}}} = \frac{\sqrt{2}\mu M g}{\sqrt{2} + (1-\varepsilon)^{\frac{1}{2}}} \cong \frac{\sqrt{2}\mu M g}{\sqrt{2} + 1 - \frac{1}{2}\varepsilon} = \qquad \textbf{(0.7 p)}$$

$$= \frac{\sqrt{2}}{\sqrt{2}+1}\mu M g\left[1 - \frac{\varepsilon}{2(\sqrt{2}+1)}\right]^{-1} \cong$$

$$\cong (2 - \sqrt{2})\mu M g\left(1 + \frac{\sqrt{2}-1}{2}\varepsilon\right). \qquad \textbf{(0.6 p)}$$

Note: *If $\varepsilon = 0$, then the speed and acceleration are concomitantly zero, so the plank will finish its motion in $x = x_2 < 2l$, which is not what the problem asks.*

C3. (6.3 points) Derive the maximum value of the plank's speed during its motion on the interval $x \in [0, 2l]$. Plot the graph of the plank' speed as a function of x, for $x \in [0, 3l]$.

Note: *If useful, you can use the Bernoulli's approximation $(1+x)^n \cong 1 + nx$, if $|x| \ll 1$.*

Solution (6.3 p):

For $x \in [0, l]$, the maximum speed of the plank is

$$v_{\max,1} = v(x_1) = \frac{F_0}{M}\sqrt{\frac{l}{\mu g}}, \qquad \textbf{(0.7 p)}$$

so

$$v_{\max,1} = (2 - \sqrt{2})\sqrt{\mu g l}\left(1 + \frac{\sqrt{2}-1}{2}\varepsilon\right). \qquad \textbf{(0.5 p)}$$

For $x \in [l, 2l]$, the maximum speed of the plank is $v_{\max,2} = v(2l)$, which is given by the kinetic energy theorem:

$$\frac{M v_{\max,2}^2}{2} - \frac{M v_{\min}^2}{2} = \frac{F_0}{2}(2l - x_2), \qquad \textbf{(0.8 p)}$$

but

$$\frac{F_0}{2}(2l - x_2)\frac{F_0^2 l}{2\mu M g} = \frac{M v_{\max,1}^2}{2}, \qquad \textbf{(0.6 p)}$$

so

$$\frac{M v_{\max,2}^2}{2} = \frac{M v_{\max,1}^2}{2} + \frac{M v_{\min}^2}{2} = (1+\varepsilon)\frac{M v_{\max,1}^2}{2}. \qquad \textbf{(0.6 p)}$$

From here we get

$$v_{\max,2} = \sqrt{1+\varepsilon}\,v_{\max,1} \cong \left(1+\frac{\varepsilon}{2}\right)(2-\sqrt{2})\sqrt{\mu g l}\left(1+\frac{\sqrt{2}-1}{2}\varepsilon\right) \cong$$

$$\cong (2-\sqrt{2})\sqrt{\mu g l}\left(1+\tfrac{\sqrt{2}}{2}\varepsilon\right), \quad \textbf{(0.7 p)}$$

which is the maximum speed reached by the plank on the considered interval.

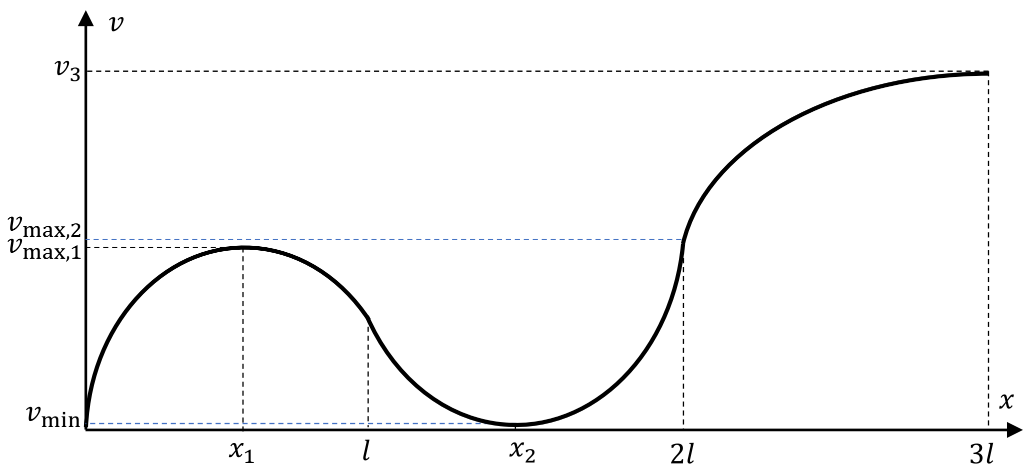
For plotting the requested graph, the value of the plank' speed for $x = 3l$ is needed, which can be obtained again, using the kinetic energy theorem:

$$v_3 = \sqrt{v_{\max,2}^2 + \tfrac{2F_0 l}{M}} \cong \quad \textbf{(0.7 p)}$$

$$\cong \sqrt{2\mu g l(5-3\sqrt{2})\left[1+\frac{3}{14}(3\sqrt{2}-2)\varepsilon\right]} \cong$$

$$\cong \sqrt{2\mu g l(5-3\sqrt{2})}\left[1+\tfrac{3}{28}(3\sqrt{2}-2)\varepsilon\right]. \quad \textbf{(0.7 p)}$$

The graph is represented below **(1 p)**:



3. Tennis ball, Solution

A. Warming up

During warm up, a tennis ball (A) is dropped from rest from Novak Djokovic pocket of height *H*. At the same instant of time grasshopper (B) stands on the ground below Novak's pocket and starts to move vertically towards the ball with initial velocity v_0. When they collide, the ball has twice the speed of the grasshopper. The collision occurs at height *h*.

A1. (1 point) Write down the equation of motion for the ball, y_A as a function of time, in terms of *H*, v_0 and gravitational acceleration *g*.

$$y_{\mathrm{A}}(t) = H - \frac{1}{2}gt^2.$$

A2. (1 point) Write down the equation of motion for the grasshopper, y_B as a function of time, in terms of v_0 and *g*.

$$y_{\mathrm{B}}(t) = v_0 t - \frac{1}{2}gt^2.$$

A3. (3 points) Derive the expression for the velocity of the ball $v_A(t)$ in terms of *H*, g and $y_A(t)$.

$$v_A(t) = -gt \;\Rightarrow\; t = \left|-\frac{v_A(t)}{g}\right| = \frac{v_A(t)}{g},$$

$$\Rightarrow y_{\mathrm{A}}(t) = H - \frac{1}{2}g\left(\frac{v_A(t)}{g}\right)^2,$$

$$\Rightarrow v_A^2(\mathrm{t}) = 2g\big(H - y_{\mathrm{A}}(t)\big) \Rightarrow\; v_B(t) = \sqrt{2g\big(H - y_{\mathrm{A}}(t)\big)}.$$

A4. (1 point) Derive the expression for the velocity of the grasshopper $v_B(t)$ in terms of v_0, g and $y_B(t)$.

$$v_B^2(\mathrm{t}) = v_0^2 - 2gy_B(t) \;\Rightarrow\; v_B(t) = \sqrt{v_0^2 - 2gy_B(t)}.$$

A5. (3 points) Derive the expression for the initial velocity v_0 of the grasshopper in terms of *h*, *H* and *g*.

Using condition $v_A(t_C) = 2v_B(t_C)$ at the moment of collision t_C and $y_{\mathrm{A}}(t_C) = y_{\mathrm{B}}(t_C) = h$ we can write

$$2g(H - h) = 4v_0^2 - 8gh,$$

$$\Rightarrow v_0^2 = \frac{1}{2}g(H + 3h).$$

A6. (3 points) Derive the expression for the moment of the collision t_c in terms of v_0 and *g*.

Using again condition $|v_A(t_C)| = 2v_B(t_C)$ we can write

$$gt_C = 2v_0 - 2gt_C\,,$$

$$\Rightarrow t_C = \frac{2v_0}{3g}.$$

A7. (3 points) Calculate the numerical value of the ratio h/H.

$$H = v_0 t_C = v_0 \frac{2v_0}{3g} = \frac{2}{3g} v_0^2 = \frac{2}{3g}\frac{1}{2} g(H + 3h),$$

$$\Rightarrow \frac{h}{H} = \frac{2}{3}.$$

B. Match

During match, at serve, Novak aims to hit tennis ball horizontally.

B1. (1 point) Write down the expression for the equation of motion of the ball in the horizontal direction $x(t)$.

$$x(t) = v_i t.$$

B2. (1 point) Write down the expression for the equation of motion of the ball in the vertical direction $y(t)$.

$$y(t) = \mathrm{y}_0 - \frac{1}{2} g t^2.$$

B3. (3 points) Calculate the numerical value of the minimal initial velocity v_i required for the ball to pass just above the 0.9 m high net, 15 m in front of Novak, if the ball is launched (horizontally) from a height of 2.5 m.

For $x_1(t_1) = 15\ m$, $\mathrm{y}_1(t_1) = 0.9\ m$ and $\mathrm{y}_0 = 2.5\ m$, where t_1 is instant of time when the ball passed just above the net, we can write

$$x_1(t_1) = v_i t_1 \ \Rightarrow v_i = \frac{x_1(t_1)}{t_1},$$

$$\mathrm{y}_1(t_1) = \mathrm{y}_0 - \frac{1}{2} g t_1^2 \ \Rightarrow \ t_1 = \sqrt{\frac{2}{g}\left(\mathrm{y}_0 - \mathrm{y}_1(t_1)\right)},$$

$$\Rightarrow v_i = x_1(t_1)\sqrt{\frac{g}{2\left(\mathrm{y}_0 - \mathrm{y}_1(t_1)\right)}} = 26.3\ \frac{m}{s}.$$

B4. (4 points) Where will the ball land in the case given under B3?

For $x_2(t_2) = v_i t_2$ and $\mathrm{y}_2(t_2) = \mathrm{y}_0 - \frac{1}{2} g t_2^2 = 0$, where t_2 is instant of time when the ball lands, we can write

$$t_2 = \sqrt{\frac{2}{g}\mathrm{y}_0},$$

$$\Rightarrow x_2(t_2) = v_i \sqrt{\frac{2}{g}\mathrm{y}_0} = x_1(t_1)\sqrt{\frac{g}{2\left(\mathrm{y}_0 - \mathrm{y}_1(t_1)\right)}}\ \sqrt{\frac{2}{g}\mathrm{y}_0},$$

$$\Rightarrow x_2(t_2) = x_1(t_1)\sqrt{\frac{\mathrm{y}_0}{\left(\mathrm{y}_0 - \mathrm{y}_1(t_1)\right)}} = 18.8\,m.$$

B5. (1 point) How long will the ball be in the air before it lands in the case given under B3? Gravitational acceleration is $g = 9.8\ \mathrm{m}/s^2$.

$$t_2 = \sqrt{\frac{2}{g}\mathrm{y}_0} = 0.71\,m.$$

BPU 2023 problem proposal

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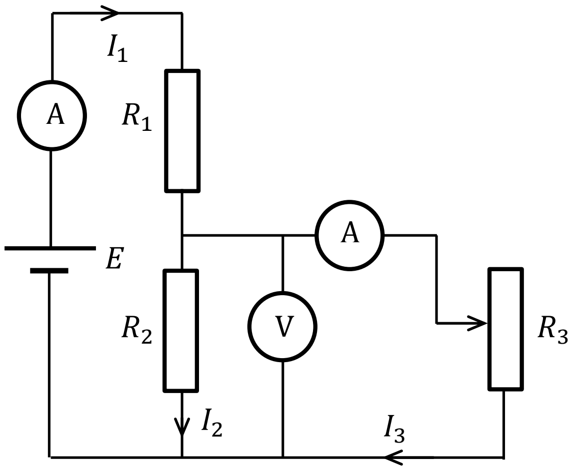
Electric circuit (Solution)

A circuit diagram is given in the figure. It contains a battery with voltage E, two resistors of fixed value R_1 and R_2, one resistor R_3 whose value can be changed, one voltmeter and two ammeters. A series of measurements were made at different values of the resistor R_3, which are given in the table below. The notations of the measured electric currents are given in the figure. From the data presented, calculate (all additionally calculated data must be present in the empty columns of the table):

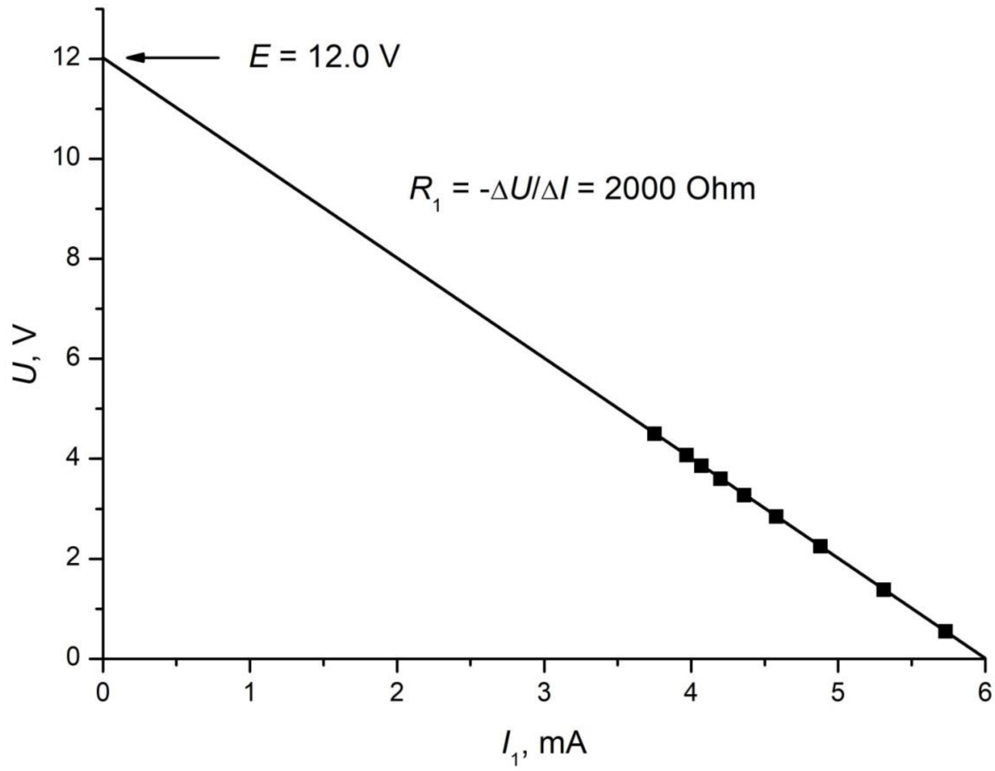


N	I_1, mA	I_3, mA	U, V	R_2, Ω	R_3, Ω	η, %		
1	5.73	5.46	0.545	2019	99.8	4.33		
2	5.31	4.62	1.38	2000	299	10.0		
3	4.88	3.75	2.25	1991	600	14.4		
4	4.58	3.16	2.84	2000	899	16.3		
5	4.36	2.73	3.27	2006	1198	17.1		
6	4.20	2.40	3.60	2000	1500	17.1		
7	4.07	2.14	3.86	2000	1804	16.9		
8	3.97	1.94	4.07	2005	2098	16.6		
9	3.75	1.50	4.50	2000	3000	15.0		

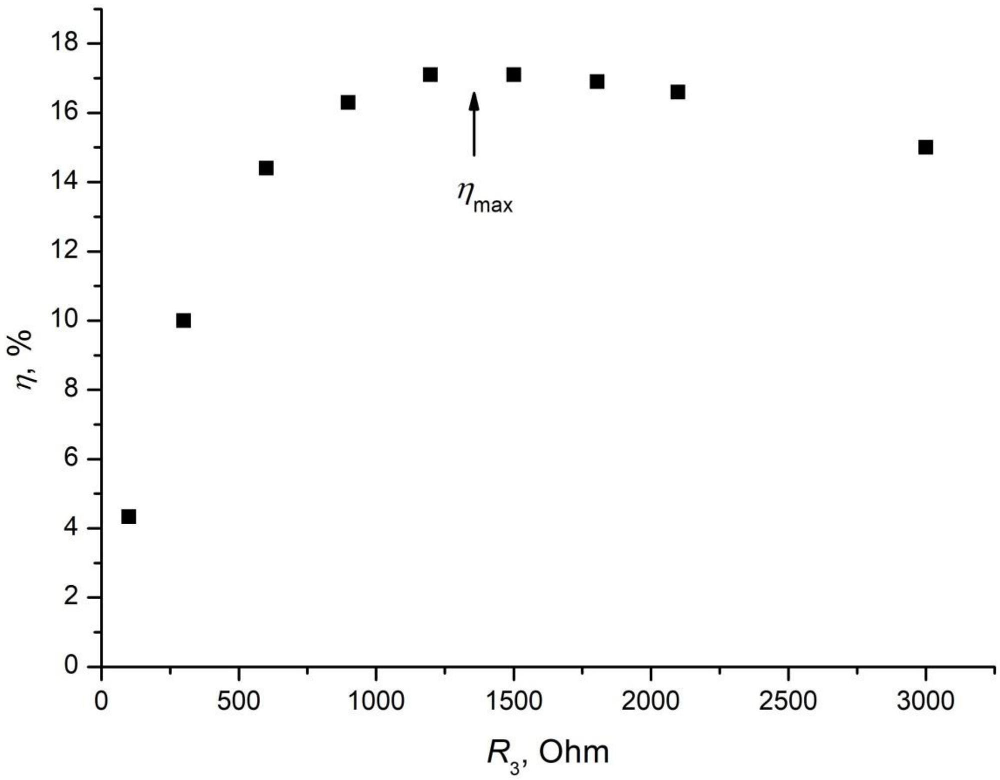
a) (2 points) The value of the resistance of resistor R_2 is calculated as $R_2 = \frac{U}{I_2} = \frac{U}{I_1 - I_3}$. **(0.5 p)** Calculated values are given on the table. **(1.0 p)** The averaged value is $R_2 = 2002\ \Omega \approx 2000\ \Omega$. **(0.5 p)**

b) (4 points) As $U = E - R_1 . I_1$, the dependence $U(I_1)$ is straight line. **(0.5 p)** It crosses the ordinate at E and has a slope $-R_1$. **(0.5 p)** The graph of the dependence is given below. **(3.0 p)**

c) (2 points) From the obtained graph we calculate the values of the voltage $E = 12.0$ V **(1.0 p)** and the resistance $R_1 = 2000\ \Omega$. **(1.0 p)**



d) (5 points) The value of the resistor is $R_3 = \frac{U}{I_3}$. **(0.5 p)** The so defined efficiency of the circuit is $\eta = \frac{U.I_3}{E.I_1}$. **(0.5 p)** The calculated values are listed in the table. **(1.0 p)** The graph of the dependence of the efficiency η on the value of the resistance of the resistor R_3 is given below. **(3.0 p)**



e) (2 points) It is seen that the maximum of η is somewhere in the interval (1200 Ω, 1500 Ω), **(0.5 p)** but as the dependence near the maximum is rather flat, it cannot be determined precisely from the presented data. **(0.5 p)** Good values are $\eta \approx$ 17%, **(0.5 p)** and $R_3 \approx$ 1350 Ω. **(0.5 p)**

f) (5 points) The efficiency of the circuit is $\eta = \frac{U.I_3}{E.I_1} = \frac{{I_3}^2.R_3}{E.I_1}$. **(0.5 p)** In the simpler case $R_1 = R_2 = R$, $I_1 = \frac{E}{R+\frac{R.R_3}{R+R_3}}$. **(0.5 p)** As $R.I_2 = R_3.I_3$, $I_2 = I_3\frac{R_3}{R}$. As $I_1 = I_2 + I_3$, $I_3 = I_1\frac{R}{R+R_3}$. **(0.5 p)** Substituting these results in the equation for the efficiency, $\eta = \frac{(I_1\frac{R}{R+R_3})^2.R_3}{E.I_1} = \frac{I_1.R^2.R_3}{E(R+R_3)^2} = \frac{R^2.R_3}{\left(R+\frac{R.R_3}{R+R_3}\right).(R+R_3)^2}$. **(0.5 p)** After simplifying the expression $= \frac{R.R_3}{2{R_3}^2+3R.R_3+R^2}$. **(0.5 p)** This expression can be written as a quadratic equation with variable R_3, $2\eta{R_3}^2 + (3\eta - 1)R.R_3 + \eta R^2 = 0$. **(0.5 p)** As for the maximal η there is only one solution of R_3, then the discriminant of this quadratic equation must be zero: $D = (3\eta - 1)^2R^2 - 4.2\eta.\eta R^2 = 0$. **(0.5 p)** The new equation for η is quadratic one: $\eta^2 - 6\eta + 1 = 0$. **(0.5 p)** This quadratic equation has roots $\eta = 3 \pm \sqrt{8}$. As $\eta < 1$, **(0.5 p)** only the root $3 - \sqrt{8} \approx 0.172$ is a solution. **(0.5 p)**

g) (4 points) Substituting the obtained value of $\eta_{max} = 3 - \sqrt{8}$ in the quadratic equation for R_3, $2\eta{R_3}^2 + (3\eta - 1)R.R_3 + \eta R^2 = 0$, we obtain $2(3 - \sqrt{8}){R_3}^2 + \left(3(3 - \sqrt{8}) - 1\right)R.R_3 + (3 - \sqrt{8})R^2 = 0$. **(1.0 p)** The discriminant of this quadratic equation is zero. **(1.0 p)** Then $R_3 = \frac{3\sqrt{8}-8}{4(3-\sqrt{8})}R$. **(1.0 p)** After simplification $R_3 = \frac{R}{\sqrt{2}}$. **(1.0 p)**

h) (1 point) The theoretical values of η =17.2% **(0.5 p)** and $R_3 = \frac{2000\,\Omega}{\sqrt{2}} \approx 1414\,\Omega$ **(0.5 p)** are in excellent agreement with the experimental ones.