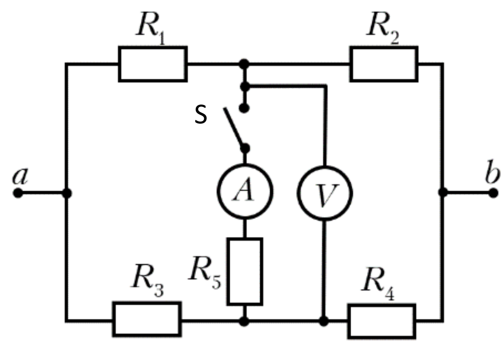
Problem 1: Resistor circuit

A constant voltage U_0 is applied to the *ab* ends of the circuit in figure. The resistors have resistances $R_1=R_2=R_5=R$, $R_3=2R$, $R_4=4R$. The voltmeter and the ammeter are considered ideal.



A. (**9 points**) With the **switch S open** determine:

- **A1.** (**4 points**) The equivalent resistance, R_{ab}, between the *ab* ends of the circuit.
- **A2.** (**3 points**) The values measured by the ammeter and the voltmeter.
- **A3.** (**2 points**) The power dissipated on the resistor R_4.

B. (**13 points**) With the **switch S closed** determine:

- **B1.** (**9 points**) The equivalent resistance, R_{ab}, between the *ab* ends of the circuit.
- **B2.** (**2 points**) The values measured by the ammeter and the voltmeter.
- **B3.** (**2 points**) The power dissipated on the resistor R_4.

C. (**3 points**) The switch S is closed and opened periodically, so that in a time period *T, 2T/3* of the time is closed and *T/3* of the time is opened. Determine the average power dissipated on the resistor R_4.

Assume that all current values are reached instantaneously.

Solution

A1. (4 points)

$$R_s = R_A + R_B \qquad \textbf{(1p)}$$

$$R_p = \frac{R_A R_B}{R_A + R_B} \qquad \textbf{(1p)}$$

$$R_{12} = R_1 + R_2 = 2R \qquad \textbf{(0.5p)}$$

$$R_{34} = R_3 + R_4 = 6R \qquad \textbf{(0.5p)}$$

$$R_{ab} = \frac{R_{12} R_{34}}{R_{12} + R_{34}} = 1,5R \qquad \textbf{(1p)}$$

A2. (3points)

$$I_A = 0 \qquad \textbf{(1p)}$$

$$I_{12} = \frac{U_0}{R_{12}} = \frac{U_0}{2R} \qquad \textbf{(0.5p)}$$

$$I_{34} = \frac{U_0}{R_{34}} = \frac{U_0}{6R} \qquad \textbf{(0.5p)}$$

$$U_V = \left| I_{34}R_3 - I_{12}R_1 \right| = \frac{U_0}{6} \quad \textbf{(1p)}$$

A3. (2 points)

$$P_4 = I_{34}^2 R_4 \qquad \textbf{(1p)}$$

$$P_4 = \frac{U_0^2}{9R} \qquad \textbf{(1p)}$$

B1. (9 points)_

$$I_1R - I_5R - 2I_3R = 0 \qquad \textbf{(1p)}$$

$$I_2R - 4I_4R + I_5R = 0 \qquad \textbf{(1p)}$$

$$I_1 + I_5 = I_2 \qquad \textbf{(1p)}$$

$$I_3 = I_4 + I_5 \qquad \textbf{(1p)}$$

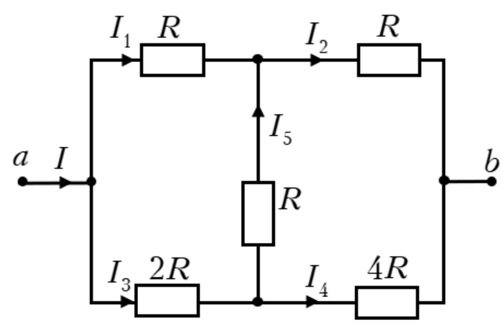
$$I_1R + I_2R = U_0 \qquad \textbf{(1p)}$$

$$I_1 = \frac{16}{34}\frac{U_0}{R} \qquad \textbf{(1p)}$$

$$I_3 = \frac{7}{34}\frac{U_0}{R} \qquad \textbf{(1p)}$$

$$\left(I_1 + I_3\right)R_{ab} = \frac{23}{34}\frac{U_0}{R} \qquad \textbf{(1p)}$$

$$R_{ab} = \frac{34}{23}R \qquad \textbf{(1p)}$$



B2. (2 point)

$$I_5 = \frac{1}{17}\frac{U_0}{R} \qquad \textbf{(1p)}$$

$$U_5 = \frac{1}{17}U_0 \qquad \textbf{(1p)}$$

B3. (2 points)

$$I_4 = \frac{5}{34}\frac{U_0}{R}$$ **(1p)**

$$P_4 = 4I_4^2 R = \frac{25U_0^2}{289R}$$ **(1p)**

C. (3 point)

$$P_{med} = \frac{L}{T}$$ **(1p)**

$$L = P_1\Delta t_1 + P_2\Delta t_2$$ **(1p)**

$$P_{aver} \approx 0,09\frac{U_0^2}{R}$$ **(1p)**

Problem 2: The descent of a skier

A skier descent a steep hill. The angle between the inclined surface of the hill and a horizontal plane is $\alpha = 30{,}0°$. The acceleration of gravity is $g = 9{,}81$ m/s^2. The mass of the skier is $m = 80{,}0$ kg. The coefficient of friction μ of the skies in the snow is unknown. The drag force F_D with which the air acts on the skier, is $F_D = \frac{1}{2}C\rho A v^2$, where C is an unknown number, the so called drag coefficient, depending on the shape of the skier, $\rho = 1{,}28$ kg/m^3 is the density of the air, $A = 0{,}600$ m^2 is the projected frontal area (cross section) of the skier perpendicular to his velocity, and v is his speed relative to the air. The air does not move (there is no wind). The skier's descent is recorded. Analyzing the video, some of his positions x_i (measured on the slope from the starting point of descent) in different moments t_i after the start of his descent are measured. The data are shown on the table below.

i	t_i/s	x_i/m						
1	2,00	7,21						
2	2,50	11,22						
3	3,00	16,08						
4	3,50	21,76						
5	4,00	28,24						
6	4,50	35,49						
7	5,00	43,47						
8	5,50	52,14						
9	6,00	61,49						
10	6,50	71,46						

Using the above data, one can calculate the values of the coefficient of friction μ, the coefficient of the drag force C, the maximum attainable speed v_{max}, of the skier (assuming that he starts his descent from a sufficient high altitude).

A. (7 points) Using suitable new variables, present the dependence of the skier motion on the experimental results in a way that can be studied graphically and the information on the parameters describing the motion to be extracted from the graph, such as the drag coefficient C, the coefficient of the frictional force μ and the maximum attainable speed, v_{max}. Give the expressions for the new variables and for the parameters of the used functional dependence as functions of the given parameters α, g, m, ρ, A.

B. (4 points) Calculate the values of the new variables for each measurement based on the given data and fill the empty columns of the table.

C. (4 points) Draw the dependence in the graph paper provided.

D. (10 points) Using the graph, calculate the values of the coefficient of friction μ, the drag coefficient C and the maximal attainable speed of the skier v_{max} (if the hill is sufficiently high).

Solution

A. (**7 points**)

The friction force F_{fr}, acting on the skier when he slides on the snow, is $F_{fr} = \mu N$, where N is the force of the reaction. As $N = mg\cos\alpha$, then $F_{fr} = \mu mg\cos\alpha$. **(0,5 p)**

The projection of the gravity force on the hill surface is $mg\sin\alpha$. As the drag force is $F_D = \frac{1}{2}C\rho Av^2$, the equation of motion is $mg\sin\alpha - \mu mg\cos\alpha - \frac{1}{2}C\rho Av^2 = ma$, **(0,5 p)** where v and a are the velocity and acceleration of the skier, respectively.

Therefore, $a = g(\sin\alpha - k\cos\alpha) - \frac{C\rho A}{2m}v^2$, i.e. a depends on v^2 linearly, $a = b - dv^2$, where the coefficients of the linear dependence are $b = g(\sin\alpha - k\cos\alpha)$ and $d = \frac{C\rho A}{2m}$. **(1 p)**

As from the experimental data only average velocity $v_{i,i-1} = \frac{x_i - x_{i-1}}{t_i - t_{i-1}}$ can be calculated, the instantaneous velocity $v(t_i) \equiv v_i$ can be evaluated as $v_i = \frac{v_{i,i-1} + v_{i+1,i}}{2}$. **(1 p)**

Therefore, as from the experimental data only average acceleration $a_{i,i-1} = \frac{v_i - v_{i-1}}{t_i - t_{i-1}}$ can be calculated, the instantaneous acceleration $a(t_i) \equiv a_i$ can be evaluated as $a_i = \frac{a_{i,i-1} + a_{i+1,i}}{2}$. **(1 p)**

So, the graph $a_i = f(v_i{}^2)$ is straight line $a_i = b - dv_i{}^2$. The coefficients b and d defined above can be calculated from the graph. After that the unknown values can be calculated: the friction coefficient is $\mu = \frac{\sin\alpha - \frac{b}{g}}{\cos\alpha}$, **(1 p)**

the drag coefficient is $C = \frac{2md}{\rho A}$, **(1 p)**

and maximal attainable speed of the skier is $v_{max} = \sqrt{\frac{b}{d}}$. **(1 p)**

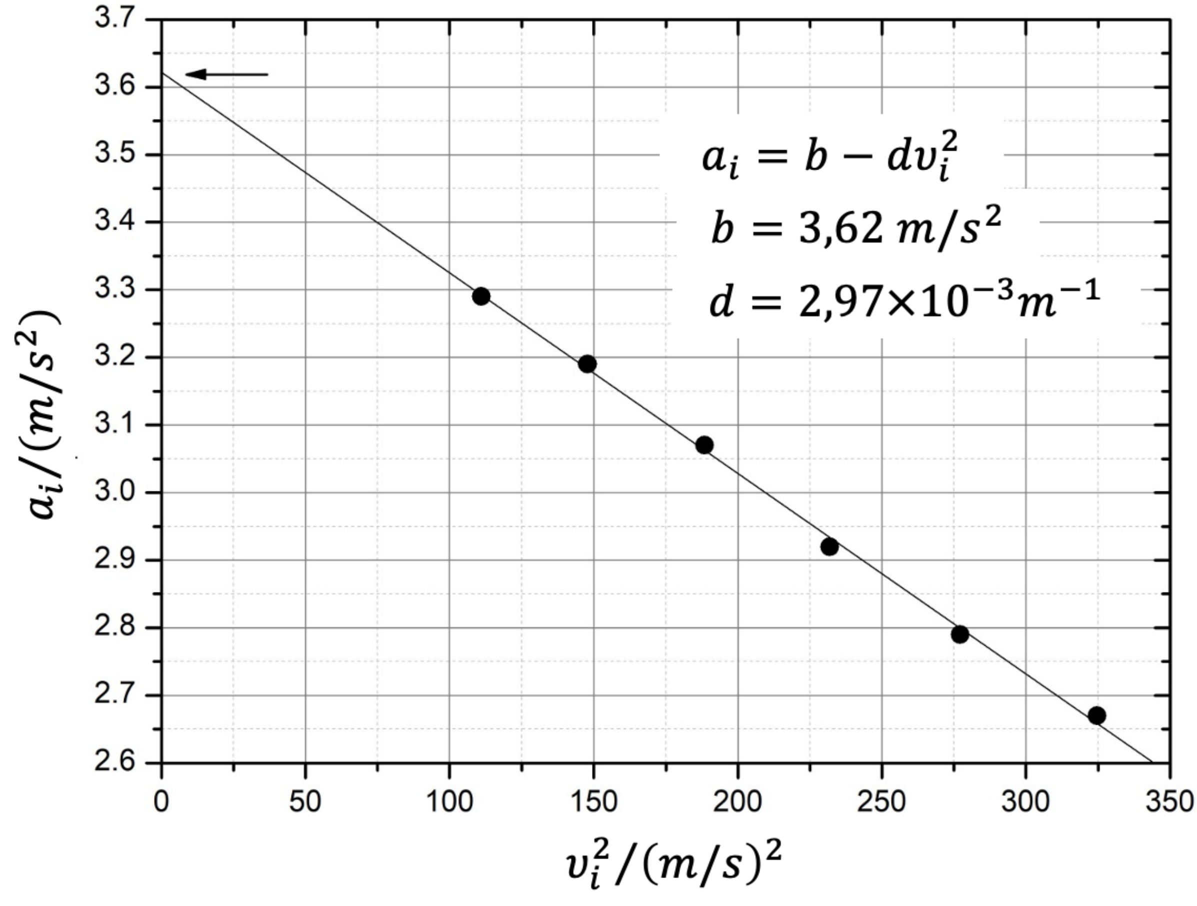
B. (4 points)

The empty columns in the table can be filled with the calculated values of averaged velocity $v_{i,i-1}$, **(0,8 p)** instantaneous velocity v_i, **(0,8 p)** average acceleration $a_{i,i-1}$, **(0,8 p)** instantaneous acceleration a_i, **(0,8 p)** and the square of instantaneous velocity $v_i{}^2$, **(0,8 p)** respectively (see the table below).

i	t_i/s	x_i/m	$v_{i,i-1}$m/s	v_i/m/s	$a_{i,i-1}$m/s²	a_i/m/s²	$v_i{}^2$/(m/s)²
1	2,00	7,21					
2	2,50	11,22	8,02	8,87			
3	3,00	16,08	9,72	10,54	3,34	3,29	111.09
4	3,50	21,76	11,36	12,16	3,24	3,19	147,87
5	4,00	28,24	12,96	13,73	3,14	3,07	188,51
6	4,50	35,49	14,50	15,23	3,00	2,92	231,95
7	5,00	43,47	15,96	16,65	2,84	2,79	277,22
8	5,50	52,14	17,34	18,02	2,74	2,67	324,72
9	6,00	61,49	18,70	19,32	2,60		
10	6,50	71,46	19,94				

C. (4 points)

The dependence is drawn on the graph below.



D. (10 points)

Using the obtained graph, the calculated values are:

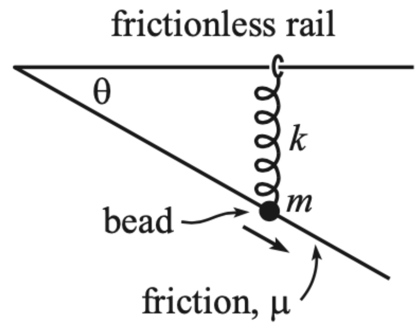
the friction coefficient is $\mu = \frac{\sin\alpha - \frac{b}{g}}{\cos\alpha} \approx 0{,}15$; **(3,5p)**

the drag coefficient is $C = \frac{2md}{\rho A} \approx 0{,}62$; **(3,5 p)**

maximum attainable speed of the skier is $v_{max} = \sqrt{\frac{b}{d}} \approx 34{,}9$ m/s ≈ 126 km/h. **(3 p)**

Problem 3: "The bead on the spring"

Assume there is a setup such as the one shown in the next figure. A bead with mass m is constrained to move along a rail which is connected with another rail at an angle θ. The rail has friction and the coefficient of the kinetic friction with the bead is μ. The bead is connected with a spring of spring constant k. The other end of the spring is constraint to move on a frictionless rail. The spring is always perpendicular to the rail due to the fact the rail is frictionless and the spring does not have a mass. The relaxed length of the spring is zero. Assume also that the system is not affected by gravitational effects.



The bead starts to move at the vertex of the rails and the spring is at its relaxed length of zero. An impulse is given to the system so it starts to move with velocity $\vec{v}_0$ along its rail. The figure shows the configuration of the system at a time $t > 0$.

A. (**6 points**) Draw the free body diagram for the bead at an arbitrary moment and give the mathematical expressions for the forces you draw. If x is the distance the bead has travelled along its rail, determine all the forces and express them in terms of this distance x and the other constants.

B. (**8 points**) Derive the expression for the distance the bead is covering before it comes to a stop for the first time.

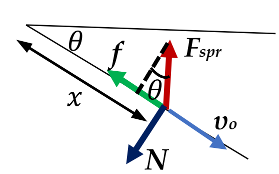
C. (**5 points**) Determine the condition for the bead to start moving again. Specify the direction the bead is going to move. Assume that the coefficient of static friction is also μ.

D. (**6 points**) Assume that the bead really does move towards the vertex of the rails. Determine its speed when it arrives back to the vertex.

Solution:

A. (**6 points**)

The forces acting on the bead are the friction, the force due to spring and the normal force from the rail to the bead. The force of friction points to the opposite direction when the bead moves back towards the vertex of the two rails. If we assume the bead has travelled a distance x then the forces acting on it will be:



(**3p**)

When the bead is at position x then:

Spring Force is $F_{spr} = kxsin\theta$ **(1,25p)**

The component of the spring force perpendicular to the rail is: $F_s cos\theta$ and this force is opposite to the normal force N from the rail to the bead. Its magnitude will be:

$N = (kxsin\theta)cos\theta$ **(1,25p)**

The magnitude of the frictional force will be $f = \mu N \Rightarrow f = \mu kxsin\theta cos\theta$ **(0,5p)**

B. (8 points)

The component of the spring force along the rail, points towards the vertex. Its magnitude is $F_{spr/x} = F_{spr}sin\theta = -kxsin\theta\ sin\theta \Rightarrow F_{spr/x} = -kxsin^2\theta$ **(1p)**

When the bead reaches the point further away from the vertex, it stops momentarily.

Using the work-kinetic energy theorem we obtain: $W_{F_{spr/x}} + W_f = K_f - K_i$ where $W_{F_{spr/x}}$ is the work done by the component of the spring force and W_f is the work of the frictional force. **(1p)**

The forces are of the proportional to the distance travel and their work will becoming either from the integral calculation or considering the graph of force vs displacement and estimating the area of the triangle. The work will be:

$W_{F_{spr/x}} = -\frac{1}{2}kx^2sin^2\theta$ **(2p)**

The work of the friction will be: $W_{F_{spr/x}} = -\frac{1}{2}\mu kx^2sin\theta cos\theta$ **(2p)**

Substitution to the work-kinetic energy formula gives

$$-\frac{1}{2}kx^2sin^2\theta - \frac{1}{2}\mu kx^2sin\theta cos\theta = 0 - \frac{1}{2}mv_0^2 \Rightarrow \frac{1}{2}kx^2(\sin^2\theta + \mu sin\theta cos\theta) = \frac{1}{2}mv_0^2$$

Solving for x, the bead comes to a stop when it traveled a distance x given by: **(2p)**

$$x_{max} = \sqrt{\frac{mv_0^2}{k(\sin^2\theta + \mu sin\theta cos\theta)}}$$

C. (5 points)

The bead will accelerate back towards the vertex if the component of the spring force along the rail is larger than the static frictional force. So the condition to accelerate back is:

$kx_{max}\sin^2\theta > \mu kx_{max}sin\theta cos\theta \Rightarrow tan\theta > \mu$ **(3p)**

This result is independent of x. If the bead accelerates in one place will accelerate in any other place. So, if it starts accelerating back it will make it all the way back to the vertex. **(2p)**

D. (6 points)

Using the work—kinetic energy theorem, as in part **B.**, for the trip back, we will obtain (the work of the spring force is positive and the work of the frictional force is negative):

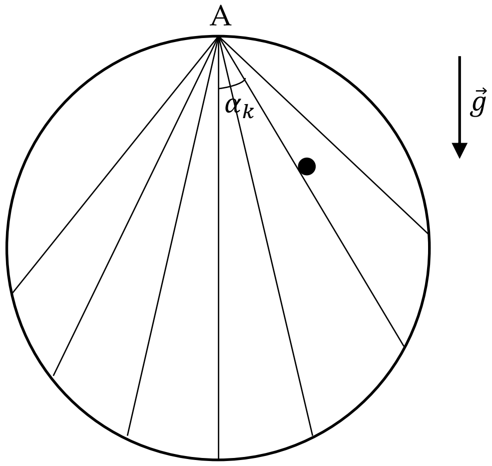
$$W_{F_{spr/x}} + W_f = K_f - K_i \Rightarrow \frac{1}{2}kx_{max}^2 \sin^2\theta - \frac{1}{2}k\mu x_{max}^2 sin\theta cos\theta = \frac{1}{2}mv_f^2$$ **(4p)**

Solving for the speed after substituting the x_{max} gives: **(2p)**

$$v_f^2 = \frac{k}{m}\left(\frac{mv_0^2}{k(\sin^2\theta + \mu sin\theta cos\theta)}\right)(\sin^2\theta - \mu sin\theta cos\theta) \Rightarrow v_f = v_0\sqrt{\frac{sin\theta - \mu cos\theta}{sin\theta + \mu cos\theta}}$$

Problem 4: Galileo Galilei – "The minimum descent time"

This year the world of Physics commemorates 380 years since Galileo Galilei (1564-1642), one of the founders of modern science, passed away. During the final years of his life, Galileo approached an interesting problem dedicated to the "minimum descent time", or, what is today called the "Galilei's problem".



A. (5 points) From a point A, with zero initial speed, a big number of identical material points move in a vertical plane, in a uniform gravitational field (the gravitational acceleration, g, is known), sliding without friction along ramps, inclined at different angles to the vertical (*see* the adjacent figure). If the point A lies at the highest point of a vertical circle of radius R, determine:

A1. (2 points) on which of the ramps the material points reach back to the circle in the shortest time;

Solution: For any given ramp, inclined with an angle α_k with respect to the vertical diameter of the circle, the acceleration of the material point is

$$a_k = gcos\alpha_k,$$ **0.5 p**

which is constant.
The length of that ramp is

$$L_k = 2Rcos\alpha_k.$$ **0.5 p**

On the other hand,

$$L_k = \frac{a_k t_k^2}{2},$$ **0.5 p**

so

$$t_k = \sqrt{\frac{2L_k}{a_k}},$$

or

$$t_k = 2\sqrt{\frac{R}{g}},$$ **0.5 p**

which means that all the material points will reach the circle at the same time.

A2. (3 points) the geometric locus of the points occupied by all the material points at any given time during their motion on the ramps (*i.e.*, the geometrical figure described by the moving material points, at any given moment of time, if a snapshot is taken).

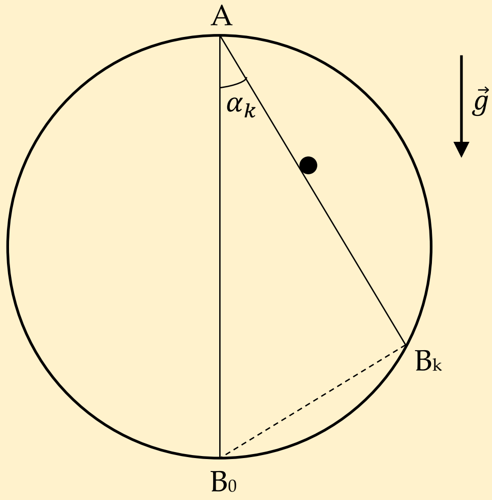
Solution: After a time t, the material point falling vertically will reach the point B_0, the length of the segment AB_0 being

$$AB_0 = \frac{gt^2}{2},$$ **0.5 p**

while the material point sliding along the ramp inclined with an angle α_k with respect to the vertical diameter, will reach the point B_k, the length AB_k being

$$\mathrm{AB}_k = \frac{a_k t^2}{2} = \frac{gt^2}{2} cos\alpha_k.$$ **0.5 p**

drawing:



0.5 p

Hence

$$\mathrm{AB}_k = AB_0 cos\alpha_k,$$ **0.5 p**

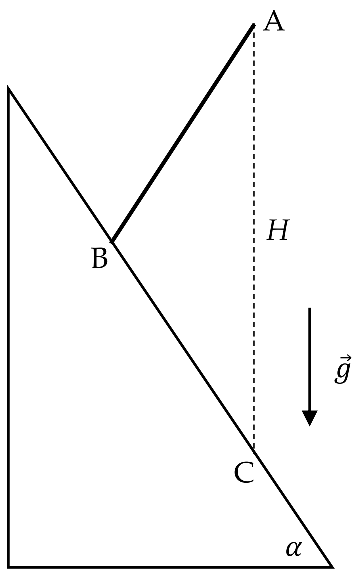
which means that, for any angle α_k,

$$m(\widehat{AB_k B_0}) = 90°.$$ **0.5 p**

This means that, at any instant t, all the material points are on the vertical circle passing through A (the uppermost point of the circle), having the radius

$$r = \frac{\mathrm{AB}_0}{2} = \frac{gt^2}{4}.$$ **0.5 p**

B. **(9 points)** An inclined plane makes the angle $\alpha = 60°$ with the horizontal plane (*see* the figure below). A point A is situated above the inclined plane, at the height $H = \mathrm{AC} = 29{,}43\ \mathrm{m}$, measured along the vertical direction, above the inclined plane.



B1. Find the position of a point B on the inclined plane (*i.e.*, the mathematical expression for the length of the segment BC), such that a material point, leaving A with zero initial speed, and sliding without friction along the rectilinear ramp AB, to arrive in B in the shortest time. The gravitational acceleration is $g = 9{,}81\ \mathrm{m/s^2}$ and has the same value in all the points. Calculate the numerical value for the length of the segment BC.

Solution 1 (geometrical): 4.5p

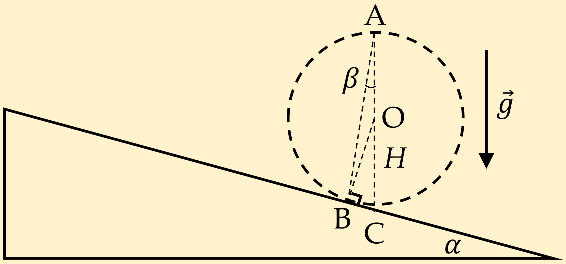
We saw above that, for any inclination of the ramp AB, after a time t, the material point is on the circle having the radius

$$r = \frac{gt^2}{4},$$

which means that the sliding time on the ramp AB is minimum when the circle radius is minimum. **0.25 p**

For the material point to reach the inclined plane, the circle should intersect the plane (the circle can be tangent or secant to the inclined plane). The circle with the minimal value of its radius, satisfying this condition, is the circle tangent to the plane, so B is the tangency point of the circle with the inclined plane (see the figure below). **0.5 p**

Drawing:



0.75 p

The problem is solved if we can derive the expression of the angle β which gives the inclination of the ramp AB with respect to the vertical.

In ΔOBC, the angle $\widehat{BOC}$ has the measure

$$m\left(\widehat{BOC}\right) = 2\beta = \alpha,$$ **2 x 0.25 p**

so

$$\boxed{\beta = \frac{\alpha}{2}}.$$ **0.25 p**

The length of the segment BC is

$$\mathrm{BC} = rtan2\beta = rtan\alpha.$$ **0.25 p**

The radius of the circle follows from

$$cos2\beta = \frac{\mathrm{OB}}{\mathrm{OC}} = \frac{r}{\mathrm{AC} - \mathrm{AO}} = \frac{r}{H - r},$$ **3 x 0.25 p**

hence

$$r = H\frac{cos\alpha}{1 + cos\alpha}.$$ **0.25 p**

So,

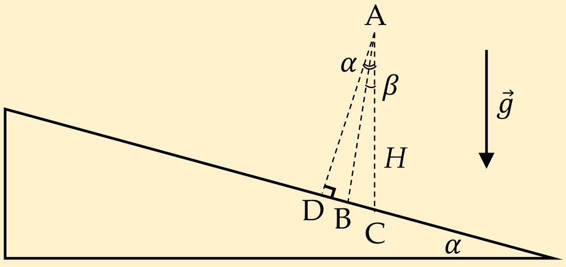
$$\boxed{\mathrm{BC} = H\frac{sin\alpha}{1 + cos\alpha} = Htan\frac{\alpha}{2}}.$$ **0.5 p**

The numerical value for the length of this segment is

$$\boxed{\mathrm{BC} = 9.81\sqrt{3}\ \mathrm{m} = 16.99\ \mathrm{m}}.$$ **0.5 p**

Solution 2 (algebraical): 7.5p

Drawing:



0.75 p

The length of the ramp AB is

$$\mathrm{AB} = \frac{at^2}{2}$$ **0.5 p**

and the acceleration of the material point is

$$a = gcos\beta,$$ **0.5 p**

so

$$t = \sqrt{\frac{2\mathrm{AB}}{g cos\beta}}.$$ **0.5 p**

This means that t is minimum when $\frac{\mathrm{AB}}{cos\beta} = min$. But **0.25 p**

$$\mathrm{AB} = \frac{\mathrm{AD}}{cos(\alpha - \beta)},$$ **0.5 p**

while

$$\mathrm{AD} = \mathrm{AC} cos\alpha = H cos\alpha.$$ **0.5 p**

Hence

$$\frac{\mathrm{AB}}{cos\beta} = \frac{H cos\alpha}{cos\beta cos(\alpha - \beta)}.$$ **0.5 p**

Since $H cos\alpha = const.$, $t = min$ when the function

$$f(\beta) = cos\beta cos(\alpha - \beta) = max.$$ **0.5 p**

This function can be rewritten as

$$f(\beta) = cos^2\beta cos\alpha + sin\beta cos\beta sin\alpha = \frac{1}{2}(cos2\beta + 1)cos\alpha + \frac{1}{2}sin2\beta sin\alpha =$$
$$= \frac{1}{2}cos\alpha + \frac{1}{2}(cos2\beta cos\alpha + sin2\beta sin\alpha) =$$ **0.5 p**
$$= \frac{1}{2}cos\alpha + \frac{1}{2}cos(2\beta - \alpha).$$

This function is maximum when $cos(2\beta - \alpha) = 1$, which means that **0.25 p**

$$\boxed{\beta = \frac{\alpha}{2}},$$ **0.5 p**

meaning that AB is the bisector of the angle $\widehat{DAC}$.

Then

$$\mathrm{BC} = \mathrm{DC} - \mathrm{DB} = H sin\alpha - \mathrm{AD} tan\frac{\alpha}{2} = H sin\alpha - H cos\alpha tan\frac{\alpha}{2} =$$
$$= H\left(tan\alpha - tan\frac{\alpha}{2}\right)cos\alpha.$$ **0.5 p**

Since

$$\tan\alpha = \frac{sin\alpha}{cos\alpha} = \frac{2 sin\frac{\alpha}{2} cos\frac{\alpha}{2}}{cos^2\frac{\alpha}{2} - sin^2\frac{\alpha}{2}} = \frac{2 tan\frac{\alpha}{2}}{1 - tan^2\frac{\alpha}{2}},$$ **0.25 p**

then

$$\mathrm{BC} = H\, tan\frac{\alpha}{2}\frac{1 + tan^2\frac{\alpha}{2}}{1 - tan^2\frac{\alpha}{2}}cos\alpha.$$

Because

$$\frac{1 + tan^2\frac{\alpha}{2}}{1 - tan^2\frac{\alpha}{2}}cos\alpha = \frac{cos^2\frac{\alpha}{2} + sin^2\frac{\alpha}{2}}{cos^2\frac{\alpha}{2} - sin^2\frac{\alpha}{2}}cos\alpha = \frac{1}{cos\alpha}cos\alpha = 1,$$

then

$$\boxed{\mathrm{BC} = H\, tan\frac{\alpha}{2}}.$$ **0.5 p**

The numerical value for the length of this segment is

$$\boxed{\mathrm{BC} = 9.81\sqrt{3}\ \mathrm{m} = 16.99\ \mathrm{m}}.$$ **0.5 p**

B2. Determine the mathematical expression for the minimum descent time on the ramp AB and calculate its numerical value.

(For solution 1): 4.5p

The length of the ramp AB is **0.5 p**

$$\mathrm{AB} = \frac{at^2}{2}$$

and the acceleration of the material point is

$$a = gcos\beta,$$ **0.5 p**

so

$$t = \sqrt{\frac{2\mathrm{AB}}{gcos\beta}},$$ **0.5 p**

where

$$\mathrm{AB} = \frac{\mathrm{AD}}{cos(\alpha - \beta)},$$ **0.5 p**

while

$$\mathrm{AD} = \mathrm{AC}cos\alpha = Hcos\alpha.$$ **0.5 p**

Hence

$$\frac{\mathrm{AB}}{cos\beta} = \frac{Hcos\alpha}{cos\beta cos(\alpha - \beta)},$$ **0.5 p**

so

$$t = \sqrt{\frac{2Hcos\alpha}{gcos\beta cos(\alpha - \beta)}}.$$ **0.5 p**

The minimum descent time is ($\beta = \alpha/2$)

$$t_{min} = \sqrt{\frac{2Hcos\alpha}{gcos^2\frac{\alpha}{2}}},$$

which gives

$$\boxed{t_{min} = 2\sqrt{\frac{H}{g}\frac{cos\alpha}{1 + cos\alpha}}}$$ **0.5 p**

The numerical value of the minimum descent time is

$$\boxed{t_{min} = 2.000\ \mathrm{s}}.$$ **0.5 p**

(For Solution 2): 1.5p

Since

$$t = \sqrt{\frac{2Hcos\alpha}{gcos\beta cos(\alpha - \beta)}}.$$ **0.5 p**

The minimum descent time is ($\beta = \alpha/2$)

$$t_{min} = \sqrt{\frac{2Hcos\alpha}{gcos^2\frac{\alpha}{2}}},$$

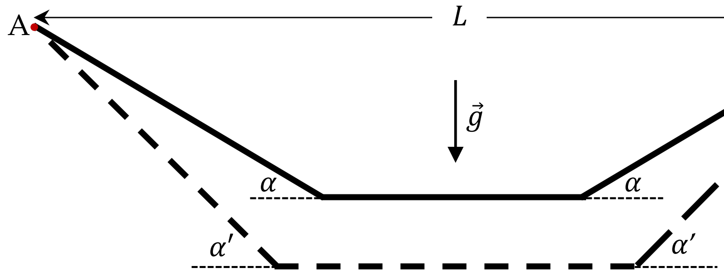
which gives

$$\boxed{t_{min} = 2\sqrt{\frac{H}{g}\frac{cos\alpha}{1 + cos\alpha}}}$$ **0.5 p**

The numerical value of the minimum descent time is

$$\boxed{t_{min} = 2.000\ \mathrm{s}}.$$ **0.5 p**

C. (11 points) In his last book[1], in which motion is treated mathematically for the first time in history, Galileo analyzes the problem of the fastest descent in uniform gravitational field without friction and concludes that the arc of a circle is faster than any number of its chords. Although it was later proved that the circle is not the path of fastest descent, Galilei admits that any possible error will be removed after future advances in Mathematics. The problem of fastest descent (also called the brachistochrone problem) – formulated in 1696 by Johann Bernoulli and correctly solved by his older brother Jakob Bernoulli – actually shows that the curve is different from a circle and is called cycloid, name given by Galilei himself.



Since the brachistochrone problem is complicated, it can be simply modeled by three connected rectilinear ramps, the descent time surpassing the cycloidal one with just a few percent. To further simplify the model, a symmetric arrangement will be considered, as illustrated in the figure above. A material point is released in A, with zero initial speed and slides on the ramps without friction from A to B. The middle ramp is horizontal and the transition from one ramp to the next is smooth. The horizontal distance between the fixed points A and B is $L = 5{,}66$ m.

Note: *By modifying the angle α, while keeping the points A and B fixed, the lengths of all the ramps will change. For an angle $\alpha' > \alpha$, the new configuration is represented with dashed lines on the above figure.*

C1. (8 points) Determine the angle α of the inclined ramps such that the transition time τ from A to B to be minimum.

Solution: Due to the symmetry of the path, the total transition time is

$$\tau = t_1 + t_2 + t_1 = 2t_1 + t_2.$$ **0.5 p**

If the length of one of the inclined ramps is denoted by L_1, the descending time on the first inclined ramp is

$$t_1 = \sqrt{\frac{2L_1}{a}},$$ **0.5 p**

where

$$a = gsin\alpha.$$ **0.5 p**

The transition time on the horizontal ramp is

$$t_2 = \frac{L - 2L_1 cos\alpha}{v},$$ **2 x 0.5 p**

where the speed of the material point on the horizontal path is

2 x 0.5 p

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[1] Galileo Galilei, *Discourses and Mathematical Demonstrations Relating to Two New Sciences*, 1638, Leiden, South Holland.

$$v = \sqrt{2gh} = \sqrt{2gL_1 sin\alpha}.$$

Hence, the total transit time is

$$\tau = 2\sqrt{\frac{2L_1}{gsin\alpha}} + \frac{L - 2L_1 cos\alpha}{\sqrt{2gL_1 sin\alpha}}.$$ **0.25 p**

For optimal values of L_1 and α, the transit time τ will be minimal. Let's consider that α already has that optimal value and that the only variable is L_1. Under these circumstances, we can write

$$\tau = 2\sqrt{\frac{2L_1}{gsin\alpha}} + \frac{L}{\sqrt{2gL_1 sin\alpha}} - \frac{2\sqrt{L_1}cos\alpha}{\sqrt{2gsin\alpha}} =$$

$$= \frac{2(2 - cos\alpha)}{\sqrt{2gsin\alpha}}\sqrt{L_1} + \frac{1}{\sqrt{L_1}}\frac{L}{\sqrt{2gsin\alpha}}.$$ **0.5 p**

This expression has the form

$$\tau = ax + \frac{b}{x},$$

where

$$x = \sqrt{L_1}, a = \frac{2(2 - cos\alpha)}{\sqrt{2gsin\alpha}}, \text{and } b = \frac{L}{\sqrt{2gsin\alpha}}.$$

It can be written as

$$\tau = ax + \frac{b}{x} - 2\cdot\sqrt{ax}\cdot\sqrt{\frac{b}{x}} + 2\cdot\sqrt{ax}\cdot\sqrt{\frac{b}{x}} = \left(\sqrt{ax} - \sqrt{\frac{b}{x}}\right)^2 + 2\cdot\sqrt{ab}.$$

From here we can see that τ is minimal when the binomial is zero, or

$$\sqrt{ax} = \sqrt{\frac{b}{x}},$$

Which gives

$$x = \sqrt{\frac{b}{a}}.$$ **1 p**

In our case, the optimal value for L_1 is

$$L_1 = \frac{L}{2(2 - cos\alpha)}.$$ **0.5 p**

For this result the transit time will be

$$\tau = 2\sqrt{\frac{L(2 - cos\alpha)}{gsin\alpha}}.$$ **0.25 p**

Now, since L_1 is fixed, we can find the optimal value of α which minimizes the value of τ. This will happen when the ratio

$$r = \frac{2 - cos\alpha}{sin\alpha}$$ **0.25 p**

is minimal. From here, by eliminating the denominator and squaring the equation, it follows that

$$r^2 sin^2\alpha = 4 - 4cos\alpha + cos^2\alpha,$$

or

$$r^2(1 - cos^2\alpha) = 4 - 4cos\alpha + cos^2\alpha.$$

Rearranging the terms, we get

$$(1 + r^2)cos^2\alpha - 4cos\alpha + 4 - r^2 = 0.$$ **0.5 p**

The solutions of this quadratic equation are

$$cos\alpha = \frac{2 \pm r\sqrt{r^2 - 3}}{1 + r^2}.$$ **0.25 p**

The minimal value for r to obtain real solutions is

$$r_{min} = \sqrt{3}.$$ **0.25 p**

For this value of r, the optimal value of α is given by

$$cos\alpha = \frac{1}{2},$$ **0.25 p**

which gives the value

$$\boxed{\alpha = 60°}.$$ **0.5 p**

C2. (2 points) Derive the expression for the length of the entire path of the material point, L_{tot}, when the transition time is minimal and calculate its numerical value.

Solution: The optimal value for L_1 is

$$L_1 = \frac{L}{3}.$$ **0.5p**

Since the horizontal path has the length

$$L_2 = L - 2L_1 cos\alpha = \frac{2L}{3},$$ **0.25 p**

then

$$L_{tot} = 2L_1 + L_2,$$ **0.25 p**

having the expression

$$\boxed{L_{tot} = \frac{4L}{3}}$$ **0.5 p**

and the numerical value

$$\boxed{L_{tot} = 7.55 \text{ m}}.$$ **0.5 p**

C3. (1 points) Derive the mathematical expression for the minimum transition time τ_{min} and calculate its numerical value.

Solution: Since

$$\tau_{min} = 2\sqrt{\frac{Lr_{min}}{g}},$$ **0.25 p**

then

$$\boxed{\tau_{min} = 2\sqrt{\frac{L\sqrt{3}}{g}}}.$$ **0.25 p**

The numerical value of this time is

$$\boxed{\tau_{min} = 2.00 \text{ s}}.$$ **0.5 p**