Problem 1: City car

Solution

a) The propulsion force, which accelerates the car, is due to the friction between the leading wheels and the road – explicit statement or a proper drawing. (**1 pt**)

The center of mass of the car is in the middle between the front and the rear wheels. Therefore, the total reaction force on the leading wheels is half of the weight of the car:

(1) $$R = \frac{mg}{2}$$ (**1 pt**)

Correspondingly, the maximum friction force on the leading wheels, and the maximum propulsion force, thereof, is:

(2) $$f_{\max} = \mu R = \frac{\mu mg}{2} = 4000\ \mathrm{N}$$ (**1 pt**)

There is no air drag force on the car, when it is in rest. Therefore, the maximum acceleration at the start is due to the propulsion force only:

(3) $$a_{\max} = \frac{f_{\max}}{m} = \frac{\mu g}{2} = 4\ \mathrm{m/s^2}$$ (**1 pt**)

b) While accelerating from rest to v_{f}, the air drag force is neglected. Therefore, the acceleration of the car is a result of the propulsion force between the leading wheels and the road only. It means also that the work, done by the motor, is transformed completely into kinetic energy of the car. In order that the car can reach v_{f} for a minimum time, the propulsion force should be maintained at the maximum value compliant with the friction between the front wheels and the road (i.e. no tire slipping):

(4) $$f \leq \frac{\mu mg}{2}$$ (**1 pt**)

and the maximum output power of the motor:

(5) $$fv \leq P_{\max}$$ (**2 pt**)

Therefore, the motion of the car can be divided into two stages. During first stage the car starts from rest and maintains a constant propulsion force $f_{\max} = \mu mg/2$, and a constant acceleration $a_{\max}$, until the output power of the motor reaches $P_{\max}$. During the second stage the motor maintains a constant power equal to $P_{\max}$– explicit statement describing the two stages, or implicit understanding of the two stages evident from the subsequent solution. (**1 pt**)

In the first period the velocity of the car increases uniformly in time:

(6) $$v = a_{\max} t$$ (**1 pt**)

The uniformly accelerated motion continues until the velocity reaches specific value v_1 at which:

$$f_{\max} v_1 = P_{\max}$$

i.e.

(7) $$v_1 = \frac{P_{\max}}{f_{\max}} = \frac{8.0 \times 10^4\ \mathrm{W}}{4.0 \times 10^3\ \mathrm{N}} = 20\ \mathrm{m/s}$$ (**2 pt**)

Therefore, the uniformly accelerated stage of the motion lasts for a time:

(8) $$t_1 = \frac{v_1}{a_{\max}} = \frac{20\ \mathrm{m/s}}{4.0\ \mathrm{m/s^2}} = 5\ \mathrm{s}$$ **(1 pt)**

From that moment on, the motor is not able to maintain a constant propulsion force. The car still accelerates, but not uniformly, as the propulsion force decreases with the increase of the velocity according to Equation 5:

$$fv = P_{\max} = \mathrm{const}$$

Let the second stage of motion lasts for a period t_2. The work done by the car engine during that time is:

(9) $$W = P_{\max} t_2$$ **(2 pt)**

According to the work-energy theorem the increase of the kinetic energy of the car in this period is equal to the work done by the motor:

(10) $$\frac{mv_{\mathrm{f}}^2}{2} - \frac{mv_1^2}{2} = W = P_{\max} t_2$$ **(2 pt)**

The time of acceleration from $v_1 = 20\ \mathrm{m/s}$ to $v_{\mathrm{f}} = 30\ \mathrm{m/s}$ is:

(11) $$t_2 = \frac{m(v_{\mathrm{f}}^2 - v_1^2)}{2P_{\max}} = \frac{1000\,.\,500}{2\,.\,80\,000} \approx 3.1\ \mathrm{s}$$ **(1 pt)**

Therefore, the minimum time of acceleration from rest to $v_{\mathrm{f}} = 30\ \mathrm{m/s}$ is:

(12) $$t_{\min} = t_1 + t_2 \approx 8.1\ \mathrm{s}$$ **(1 pt)**

(c) As the velocity surpasses v_{f} the air drag becomes more and more important, and cannot be neglected as far as we consider the maximum velocity achievable by the car. The car reaches a constant terminal velocity when the air drag equilibrates the propulsion force:

(13) $$f = Dv^2$$ **(1 pt)**

During uniform motion, the work done by the motor does not transform into kinetic energy of the car but is dissipated completely into heat due to the work done by the air drag force. Therefore the output power of the motor satisfies the relation:

$$P - F_{\mathrm{d}} v = 0$$

or

(14) $$P = Dv^3$$ **(1 pt)**

Evidently, the terminal velocity is restricted by the friction between the tires and the road:

(15) $$Dv^2 \le \frac{\mu m g}{2}$$ **(1 pt)**

and by the maximum power of the motor:

(16) $$Dv^3 \le P_{\max}$$ **(1 pt)**

It follows from (15) that:

(17) $$v \leq \sqrt{\frac{\mu m g}{2D}} = 89.4\ \mathrm{m/s}$$ **(1 pt)**

and from (16) that:

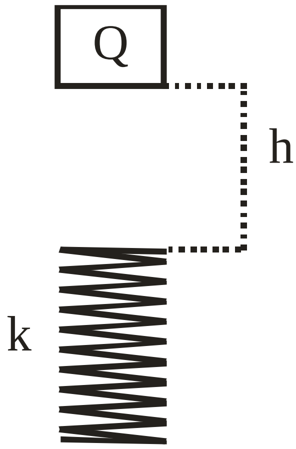
(18) $$v \leq \sqrt[3]{\frac{P_{\max}}{D}} = 54.3\ \mathrm{m/s}$$ **(1 pt)**

Therefore, for the given parameters, the maximum velocity of the car is restricted by the maximum power of the motor:

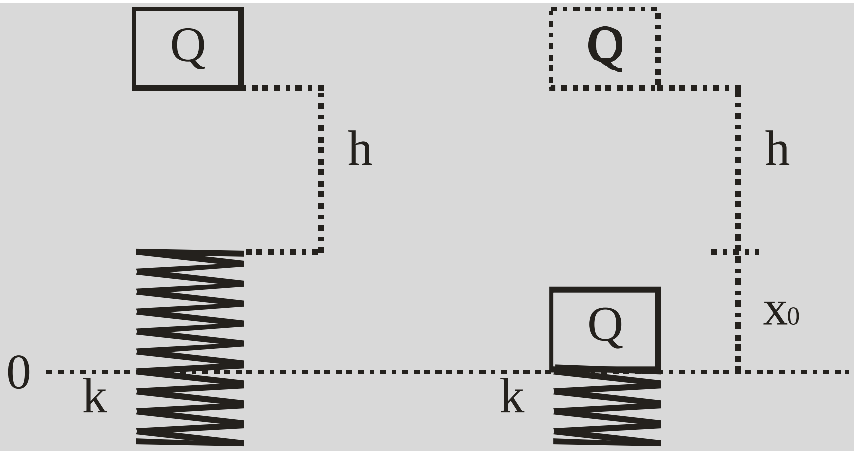
(19) $$v_{\max} = 54.3\ \mathrm{m/s}\ \ (195\ \mathrm{km/h})$$ **(1 pt)**

Problem 2 - solution

Consider a block weighting $Q = 4000N$ is maintained at rest at a height $h = 5m$ above an ideal spring. The spring has a spring constant $k = 8000N/m$ and is initially in its equilibrium state. The block is released and falls towards the spring while subjected to a friction constant force $F_0 = 1000N$ due to air resistance.



a) **(3 points)** Determine the expression for compressing distance x_0 of the spring after the block falls on it, neglecting the friction force, and calculate its numerical value.



For $F_0 = 0$, using the mechanical energy conservation low, we have

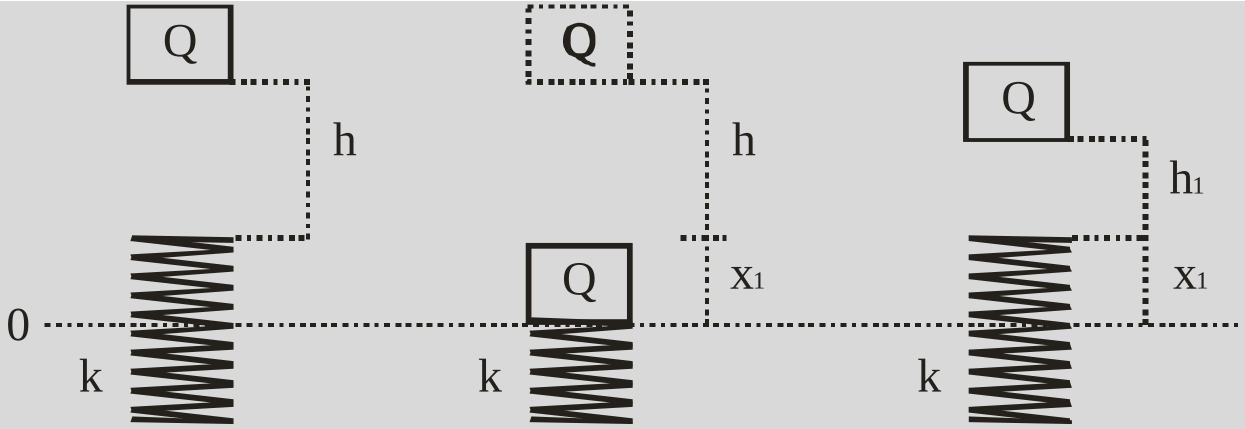
$$Q(h + x_0) = \frac{1}{2}kx_0^2 \text{, } \textbf{(1 point)}$$

$$\Rightarrow x_0 = \frac{Q}{k} \pm \sqrt{\frac{Q^2}{k^2} + \frac{2Q}{k}h} \text{. } \textbf{(1 point)}$$

Taking the + sign the compressed distance x_0 is

$$x_0 = \frac{Q}{k} + \sqrt{\frac{Q^2}{k^2} + \frac{2Q}{k}h} = 2.79m \text{. } \textbf{(1 point)}$$

b) **(4 points)** Taking into account the friction force, determine the expression for the compressing distance x_1 of the spring after the block falls on it and calculate its numerical value.



Taking into account the friction force, using the total energy conservation low, we have

$$Q(h+x_1)=\frac{1}{2}kx_1^2+F_0(h+x_1)\text{, } \textbf{(1 point)}$$

$$\Rightarrow x_1=\frac{(Q-F_0)}{k}\pm\sqrt{\frac{(Q-F_0)^2}{k^2}+\frac{2(Q-F_0)}{k}h}\text{. } \textbf{(2 points)}$$

Taking the + sign the compressed distance x_1 is

$$x_1=\frac{(Q-F_0)}{k}+\sqrt{\frac{(Q-F_0)^2}{k^2}+\frac{2(Q-F_0)}{k}h}=2.34m\text{. } \textbf{(1 point)}$$

c) **(4 points)** Taking into account the friction force, determine the expression for the maximum height h_1 of the block after its first contact with the spring (i.e. after the first „jump“) and calculate its numerical value.

After the first contact between the block and the spring, the block starts to move up. We have

$$\frac{1}{2}kx_1^2=Q(h_1+x_1)+F_0(h_1+x_1)\text{, } \textbf{(1 point)}$$

$$\Rightarrow h_1=\frac{k}{2(Q+F_0)}x_1^2-x_1=2.04m\text{. } \textbf{(2+1 points)}$$

d) **(6 points)** Taking into account the friction force, determine the expression for the maximum height h_2 of the block after its second contact with the spring (i.e. after the second „jump“) and calculate its numerical value.

This time the block free falls from the height h_1 above top of the spring. We have

$$Q(h_1+x_2)=\frac{1}{2}kx_2^2+F_0(h_1+x_2)\text{, } \textbf{(1 point)}$$

$$\Rightarrow x_2=\frac{(Q-F_0)}{k}\pm\sqrt{\frac{(Q-F_0)^2}{k^2}+\frac{2(Q-F_0)}{k}h_1}\text{.}$$

Taking the + sign the compressed distance x_2 is

$$x_2=\frac{(Q-F_0)}{k}+\sqrt{\frac{(Q-F_0)^2}{k^2}+\frac{2(Q-F_0)}{k}h_1}=1.67m\text{. } \textbf{(2 points)}$$

After the second contact between the block and the spring, the block starts to move up again and we have

$$\frac{1}{2}kx_2^2 = Q(h_2 + x_2) + F_0(h_2 + x_2)\text{, } \textbf{(1 point)}$$

$$\Rightarrow h_2 = \frac{k}{2(Q+F_0)}x_2^2 - x_2 = 0.57m\text{. } \textbf{(2 point)}$$

e) **(5 points)** Taking into account the friction force, will the block jump after the third contact between the block and the spring? Explain your answer.

If we try to calculate the maximal height after the third contact we would get negative numerical value

$$h_3 = \frac{k}{2(Q+F_0)}x_3^2 - x_3 = -0.11m\text{, } \textbf{(2 points)}$$

where x_3 is obtained as

$$x_3 = \frac{(Q-F_0)}{k} + \sqrt{\frac{(Q-F_0)^2}{k^2} + \frac{2(Q-F_0)}{k}h_2} = 1.13m\text{. } \textbf{(1 point)}$$

We conclude that the block will not jump off the spring after their third contact. The block stays on top of the spring. **(2 points)**

f) **(3 points)** Taking into account the friction force, calculate the final compressing distance x_f of the spring after the block comes to rest.

The block will come to rest at compressing distance $x_f = (Q-F_0)/k$ while trying to move downwards or at $x_f = (Q+F_0)/k$ while trying to move upwards. We will prove that it happend while trying to move upwards.

The block moves from $x_3 = 1.13m$ upwards until it reaches x_4. Using the theorem of kinetic energy variation, it follows

$$\frac{k}{2}\left(x_4^2 - x_3^4\right) = -\left(Q+F_0\right)\left(x_3 - x_4\right)\text{, } \textbf{(0.5 points)}$$

$$\Rightarrow x_4 = \frac{2}{k}\left(Q+F_0\right) - x_3 = 0.12m\text{. } \textbf{(0.5 points)}$$

From x_4 the block moves downwards until it reaches x_5. Using the same theorem, it follows

$$\frac{k}{2}\left(x_5^2 - x_4^4\right) = \left(Q-F_0\right)\left(x_5 - x_4\right) \Rightarrow x_5 = 0.63m\text{, } \textbf{(0.5 points)}$$

At the end, the block moves upwards from x_5 trying to reach x_6:

$$\frac{k}{2}\left(x_6^2 - x_5^4\right) = -\left(Q+F_0\right)\left(x_5 - x_6\right) \Rightarrow x_6 = 0.62m\text{. } \textbf{(0.5 points)}$$

We see that $x_5 \approx x_6$, which means that the block will come to rest while trying to move upwards (between x_5 and x_6), i.e.

$$\Rightarrow x_f = \frac{Q+F_0}{k} = 0.625m\text{. } \textbf{(1 point)}$$

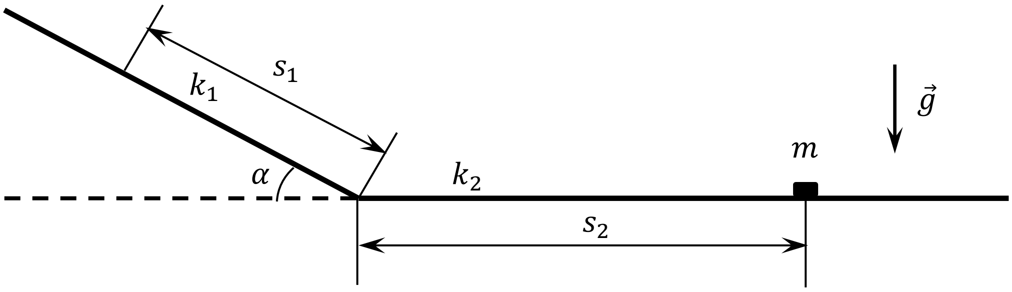
*

<u>Note:</u> If the friction force had acted on the block at rest, then the above position would have been the final equilibrium position of the block on the spring (the resultant force acting on it would have been zero). However, this asumption was not given explicitely in the problem's text.

On the other hand, considering that there is no friction force acting upon the block at rest, but only when it moves, for the above position the net force acting on the block is directed upwards, so the block will go up, in many small steps, until it reaches the final position given by:

$$x_f' = \frac{Q}{k} = 0.5\ \text{m}.$$

Problem 3: Friction Coefficients (Solution)



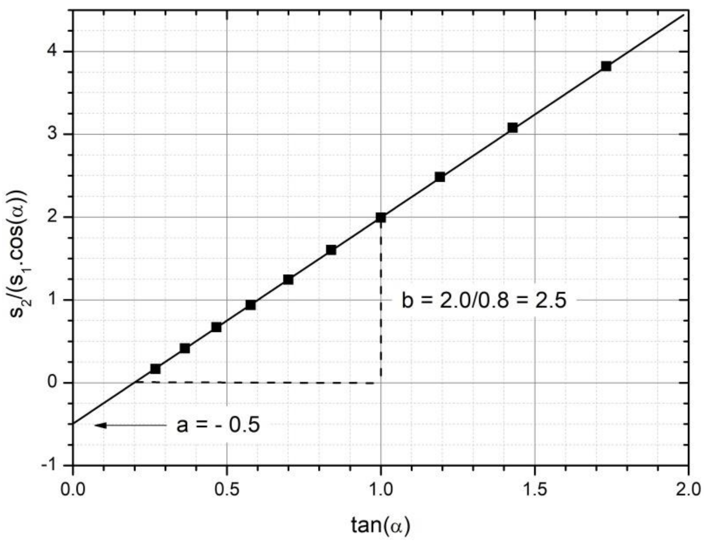
a) The friction force F_{fr1}, acting on the body when it slides on the inclined plane, is $F_{fr1} = k_1 R_1$, where R_1 is the force of the reaction. As $R_1 = mg\cos\alpha$, then $F_{fr1} = k_1 mg\cos\alpha$. **[1 point]** In the same way the friction force F_{fr2}, acting on the body when it slides on the horizontal plane, is $F_{fr2} = k_2 R_2$, where R_2 is the force of the reaction. As $R_2 = mg$, then $F_{fr2} = k_2 mg$. **[1 point]** Using the law of the change of the mechanical energy, $\Delta E_{mech} = A_{friction}$, i.e. $-mgs_1\sin\alpha = -k_1 mg\cos\alpha\, s_1 - k_2 mgs_2$. **[1 point]** After a simplification $s_2 = s_1(\sin\alpha - k_1\cos\alpha)/k_2$. (1) **[1 point]**

b) The formula (1) can be transformed in the form $\frac{s_2}{s_1\cos\alpha} = -\frac{k_1}{k_2} + \frac{1}{k_2}\tan\alpha$. (2) So, if the new variables are $x = \tan\alpha$, **[1.5 points]** and $y = \frac{s_2}{s_1\cos\alpha}$, **[1.5 points]** it can be seen that the equation (2) is a linear dependence $y = a + bx$, where $a = -\frac{k_1}{k_2}$ and $b = \frac{1}{k_2}$. This linear dependence can be easily processed graphically.

c) The calculated values of the new variables for each measurement are filled on the table. **[3 points]** The data are presented in the figure below. **[6 points]** **(2 x 0.5 p for writing the represented quantities on the axes, 4 points for correct representation of the points on the graph and 1 p for drawing the line)**

d) From the obtained graph the parameters of the straight line are $a \approx -0.50$ **[1 point]** and $b \approx 2.50$. **[2 points]** After that the friction coefficients can be calculated, $k_2 = \frac{1}{b} = 0.40$ **[1.5 points]** and $k_1 = -a.k_2 = 0.20$. **[1.5 points]**

α, °	s_2, cm	$\tan\alpha$	$\frac{s_2}{s_1\cos\alpha}$
15	8.0	0.268	0.166
20	19.5	0.364	0.415
25	30.3	0.466	0.669
30	40.6	0.577	0.938
35	51.0	0.700	1.245
40	61.4	0.839	1.603
45	70.5	1.000	1.994
50	79.9	1.192	2.486
55	88.3	1.428	3.079
60	95.5	1.732	3.820



e) From the formula (1) it can be seen that the rest of the body on the inclined plane is possible only if $\tan\alpha \leq k_1$. **[1 point]** So, the value of the maximal angle is $\alpha_{max} = \tan^{-1}(0.2) \approx 11.3°$ **[2 points]**

Problem 4: Resistances measurement - solution

A.

a)	Derive, for each position of the switch (1, respectively 2), the measured value of the resistance (R_m), as a function of R, R_V, and R_A.	**4 p.**

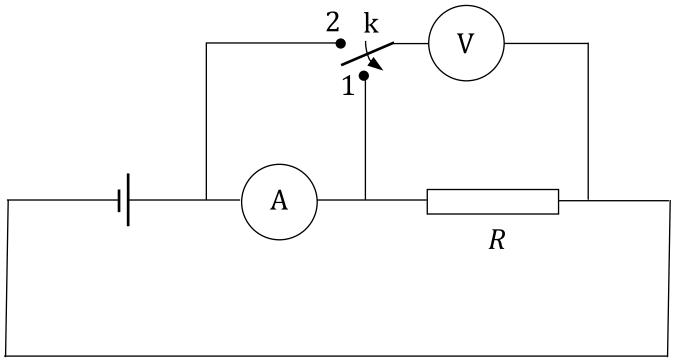


Fig. 1

The measured value of the resistance is given by

$$R_m = \frac{U}{I},$$ **1 p**

where U and I are the indications of the voltmeter and of the amperemeter, respectively.

<u>Switch in position 1:</u>

$$I = \frac{U}{R} + \frac{U}{R_V},$$ **1 p**

such that

$$\boxed{R_m = \frac{U}{I} = \frac{RR_V}{R + R_V}.}$$ **0.5 p**

<u>Switch in position 2:</u>

$$U = I(R + R_A),$$ **1p**

so

$$\boxed{R_m = R + R_A.}$$ **0.5 p**

b)	Derive the expression of the relative variation $\alpha_R = \frac{R_m - R}{R}$ (as a measure of accuracy) for each position of the switch.	**1 p.**

<u>Switch in position 1:</u> **0.5 p**

$$\alpha_{R_1} = \frac{R_m}{R} - 1 = \frac{R_V}{R + R_V} - 1 = -\frac{R}{R + R_V}.$$

For this position of the switch the measured value of the resistance is smaller than the real value.

<u>Switch in position 2:</u>

$$\alpha_{R2} = \frac{R + R_A}{R} - 1 = \frac{R_A}{R}.$$ **0.5 p**

For this position of the switch the measured value of the resistance is greater than the real value.

c)	Based on the results obtained above, derive the range of values for the resistance R for which each position of the switch is preferred. Express the results in terms of R_A and R_V. Using the well-known relationship between R_A and R_V, prove that the (practical) approximate mathematical expression for the limit between the two resistance ranges is $\beta\sqrt{R_A R_V}$ and derive the numerical value for the coefficient β.	**6 p.**

Position 1 of the switch is preferred when

$$|\alpha_{R1}| < \alpha_{R2}.$$ **1 p**

Then

$$\frac{R}{R + R_V} < \frac{R_A}{R},$$ **0.5 p**

or

$$R^2 - R_A R - R_A R_V < 0.$$ **0.5 p**

The solution of this inequality is

$$R \in (R_-, R_+) \cap [0, +\infty).$$ **0.5 p**

Since

$$R_\pm = \frac{1}{2}\left(R_A \pm \sqrt{R_A^2 + 4R_A R_V}\right),$$ **0.5 p**

if follows that $R_- < 0$, so

$$R < R_+ = \frac{1}{2}\left(R_A + \sqrt{R_A^2 + 4R_A R_V}\right).$$ **0.5 p**

<u>Conclusion:</u>

When $R < \frac{1}{2}\left(R_A + \sqrt{R_A^2 + 4R_A R_V}\right)$ the position 1 of the switch is preferred **0.5 p** and

when $R > \frac{1}{2}\left(R_A + \sqrt{R_A^2 + 4R_AR_V}\right)$ the position 2 is preferred.

Since

$$R_A \ll R_V, \qquad \textbf{0.5 p}$$

the limit value of the resistance is

$$R_+ = \frac{1}{2}\left(R_A + \sqrt{R_A^2 + 4R_AR_V}\right) = \frac{1}{2}\left(R_A + R_A\sqrt{1 + 4\frac{R_V}{R_A}}\right) =$$

$$= \frac{R_A}{2}\left(1 + \sqrt{1 + 4\frac{R_V}{R_A}}\right) \cong \frac{R_A}{2}\left(1 + \sqrt{4\frac{R_V}{R_A}}\right) \cong \frac{R_A}{2} \cdot 2\sqrt{\frac{R_V}{R_A}} =$$

$$= \sqrt{R_AR_V}.$$

It follows that **1 p**

$$\beta = 1. \qquad \textbf{0.5 p}$$

B.

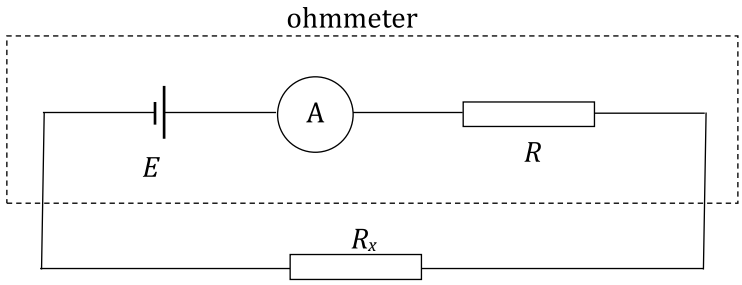


Fig. 2

The student chooses the resistor with the resistance R such that, when their ohmmeter is short-circuited, the needle indicates the maximum value on the scale.

d)	Derive the mathematical formula for R and calculate its numerical value.	**2 p.**

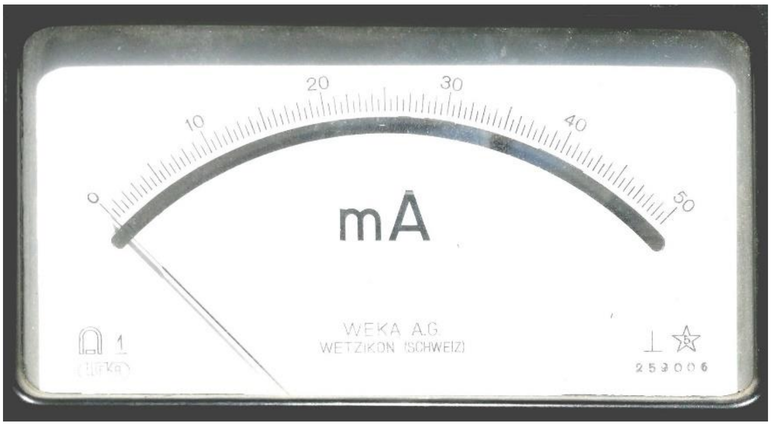


Fig. 3

$$E = I_{max}(R + R_A),$$ **1 p**

so

$$R = \frac{E}{I_{max}} - R_A.$$ **0.5 p**

Its numerical value is:

$$R = 170\ \Omega.$$ **0.5 p**

When the resistor with the unknown resistance is connected to the ohmmeter, the needle deviation is n divisions.

e)	Express R_x as a function of n, N, E, and I_{max}.	**1.5 p.**

The current is

$$I = n\frac{I_{max}}{N}$$ **0.5 p**

so

$$R_x = \frac{E}{I} - R - R_A = \frac{E}{I} - \frac{E}{I_{max}} = \left(\frac{N}{n} - 1\right)\frac{E}{I_{max}}.$$ **1 p**

The student measures the resistances of several resistors and wonders for which of them the precision is the best possible.

f)	Derive the mathematical formula for the relative error of determining R_x. Based on this result, derive the value of n for which this relative error is minimum. *Note: consider that* $\left(\frac{\Delta n}{n}\right)^2 \ll 1$, *where Δn is precision with which the student reads the number n of divisions on the amperemeter scale.*	**4 p.**

$$\varepsilon_{R_x} = \frac{\Delta R_x}{R_x} = \frac{(R_x + \Delta R_x) - R_x}{R_x},$$ **1 p**

where

$$R_x + \Delta R_x = \left(\frac{N}{n - \Delta n} - 1\right)\frac{E}{I_{max}}.$$ **1 p**

Consequently

$$\varepsilon_{R_x} = \frac{\left(\frac{N}{n-\Delta n} - 1\right)\frac{E}{I_{max}} - \left(\frac{N}{n} - 1\right)\frac{E}{I_{max}}}{\left(\frac{N}{n} - 1\right)\frac{E}{I_{max}}} = \frac{\left(\frac{N}{n-\Delta n} - 1\right) - \left(\frac{N}{n} - 1\right)}{\frac{N}{n} - 1} =$$

$$= \frac{\frac{N}{n-\Delta n} - \frac{N}{n}}{\frac{N}{n} - 1} = \frac{N\Delta n}{(n - \Delta n)(N - n)} = \frac{N\Delta n(n + \Delta n)}{[n^2 - (\Delta n)^2](N - n)} =$$

$$= \frac{n^2 N \frac{\Delta n}{n}\left(1 + \frac{\Delta n}{n}\right)}{n^2\left[1 - \left(\frac{\Delta n}{n}\right)^2\right](N - n)},$$

or

$$\boxed{\varepsilon_{R_x} = \frac{N}{(N-n)}\frac{\Delta n}{n}.}$$ **1 p**

This error is minimal when the function

$$f(n) = n(N - n)$$

reaches its maximal value, which happens for

$$\boxed{n = \frac{N}{2}.}$$ **0.5 p**

The numerical value of n is

$$\boxed{n = 50,}$$ **0.5 p**

which means that the most precise measurement is obtained when the needle is pointing exactly at the middle of the scale.

g)	Obtain the expressions for R_x and for the absolute error ΔR_x, using n derived at question f) and calculate their numerical values.	**3.5 p.**

For $n = \frac{N}{2}$, the requested value is

$$R_x = \frac{E}{I_{max}} = 180\ \Omega.$$ **1 p**

The absolute error is

$$\Delta R_x = R_x \varepsilon_{R_x} = 4R_x \frac{\Delta n}{N}.$$ **0.5 p**

Since the precision of the instrument is

$$\Delta n = 1 \text{ division},$$ **0.5 p**

then

$$\Delta R_x = \frac{4R_x}{N} = 7.2\ \Omega \cong 7\ \Omega.$$ **0.5 p**

In conclusion,

$$R_x = 180\ \pm 7\ (\Omega).$$ **1 p**

During the experiments, the student did not place the amperemeter in front of them, but to their right, such that the angle between the reading direction and the perpendicular to the scale is θ. The length of the amperemeter scale is L and the distance between the needle and the scale's plane is h.

h)	Obtain the expression for the systematic error caused by the incorrect reading of the current.	**1.5 p.**

Compared to the real indication, the student sees the tip of the needle deviated to the right with the amount

$$\Delta x = h \times tan\theta,$$ **0.5 p**

so, the error is

$$\Delta I = \frac{I_{max}}{L}\Delta x = I_{max}\frac{h}{L}tan\theta.$$ **1 p**

i)	Calculate the maximum value of the angle θ such that reading error to be less than one division on the amperemeter scale. The length of the amperemeter scale is $L = 10$ cm and the distance between the needle and the scale's plane is $h = 2.0$ mm.	**1.5 p.**

If

$$\Delta I < \frac{I_{max}}{N},$$ **0.5 p**

then

$$tan\theta < \frac{L}{Nh}.$$ **0.5 p**

Numerically,

$$tan\theta < 0.5,$$

which gives

$$\theta_{max} = tan^{-1}0.5 = 27^{\circ}.$$ **0.5 p**