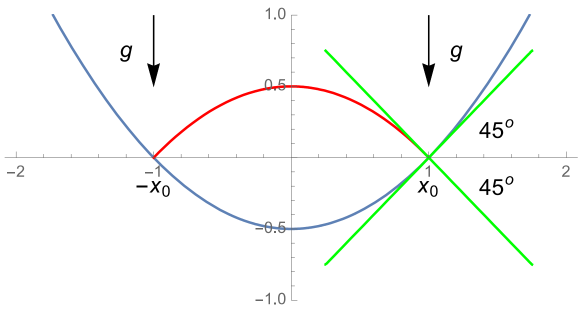
1st Balkan Physics Olympiad – 2019 BPO
July 14-18, Thessaloniki, Greece

Solution to Problem 1

1. In the simplest case the motion is along the vertical axis starting from the origin at the bottom of the parabola with an arbitrary initial velocity v_0. The motion is described by $x = 0$, $y = v_0 t - gt^2/2$, $v_y = v_0 - gt$. The highest point of the trajectory is reached at time $t = v_0/g$. The period of the motion is $T = 2v_0/g$.

2. Another simpler case is a trajectory that is symmetric to the parabola that bounds the motion from below. This is illustrated with the following figure. In order to have periodic motion the trajectory of the particle must reach the boundary at an angle of 90°. The point x_0 is not arbitrary. It is specific for the given bounding parabola. If v_0 is the speed at the instant of collision with the boundary given by $y = ax^2 - b$, $a > 0$, $b > 0$, the trajectory of the particle is described by

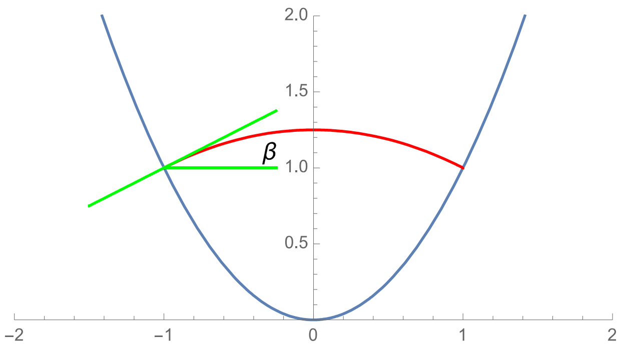
$$x = -x_0 + v_0 \cos\alpha\, t, \quad y = v_0 \sin\alpha\, t - gt^2/2, \quad \alpha = 45^\circ.$$



Elimination of t from the above equations, with the additional requirement that the trajectory has the symmetric form $y = b - ax^2$, and taking into account that $\alpha = 45^\circ$, one finds that the initial speed of the particle at the boundary should be $v_0^2 = 2gx_0$. Therefore the period is $T = 4x_0/(v_0 \cos\alpha) = 4\sqrt{(x_0/g)}$.

3. The boundary may be represented as $y = ax^2$. Let us take $x_0 > 0$ and $(\pm x_0, ax_0^2)$ as the end points of the periodic trajectory. The tangents of the boundary at these end points, at the right and left side, are $y = \pm 2ax_0x - ax_0^2$. The orthogonal direction to the tangent on the left is given by $y = (x + x_0)/(2ax_0) + ax_0^2$. Motion of the point particle is described by

$$x = -x_0 + v_0 \cos\beta\, t, \qquad y = y_0 + v_0 \sin\beta\, t - gt^2/2, \qquad \tan\beta = 1/(2ax_0).$$

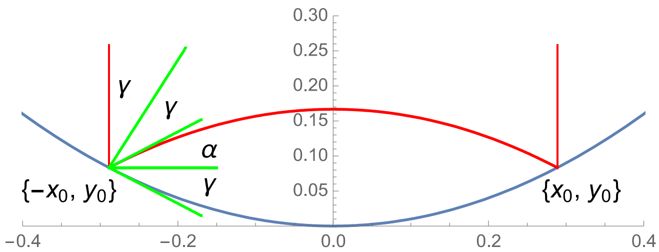


After elimination of t, the symmetry requirement implies the condition $\tan\beta = gx_0/(v_0^2\cos^2\beta)$ and the trajectory is given by $y = y_0 + g\,(x_0^2 - x^2)/(2v_0^2\cos^2\beta)$. The condition of orthogonality at the boundary leads to $2ax_0\,gx_0/(v_0^2\cos^2\beta) = 1$. The speed at the boundary reflection point is

$$v_0^2 = 2agx_0^2/\cos^2\beta = 2agx_0^2\,(1 + \tan^2\beta) = 2agx_0^2\,(1 + 1/(4a^2x_0^2)) = g(1 + 4a^2x_0^2)/(2a)\ .$$

and the period $T = 4x_0/(v_0\cos\beta) = 2\sqrt{(2/ag)}$ is independent of x_0. This is in agreement with the result of section 2, where $\tan\alpha = 2ax_0 = 1$, and $x_0 = 1/2a$, that is $T = 2\sqrt{(2/ag)}$.

4. The trajectory in the following figure has a parabolic section in the middle and two vertical sections at both ends. Indicated angles have to satisfy the relationship $\alpha + 2\gamma = \pi/2$.

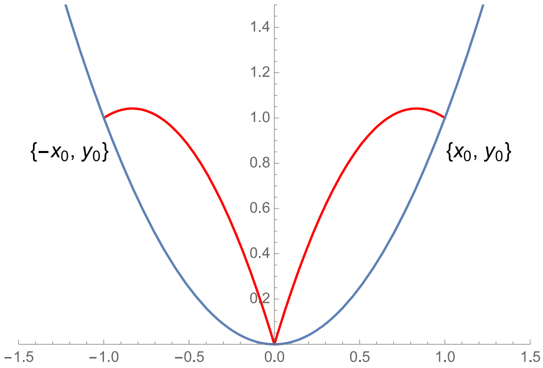


Such trajectory appears when the orthogonal line at the point of collision with the boundary divides in two halves the angle between the vertical line and the tangent at the end of the parabolic section of the trajectory. The period of this motion is given by $T = 4x_0/(v_0\cos\alpha) + 4v_0/g$ and $v_0^2 = gx_0/(\sin\alpha\cos\alpha)$ or

$$T = 4\,[1 + \sqrt{(1 + 1/\tan^2\alpha)}]\,\sqrt{[(x_0/g)\tan\alpha]}\ .$$

5. In this case the orbit, as shown in the figure, touches the boundary at three points. The boundary and the right and left branches of the trajectory are given by $y = ax^2$, $y = bx^2 \pm (a - b)x_0x$, with $\pm x_0$ the abscissa of the collision points with the boundary ($a > 0$, $b < 0$, $y_0 = ax_0^2$). From the orthogonality condition at the boundary $2ax_0^2(a + b) = 1$, it follows $b = 1/(2ax_0^2) - a$.

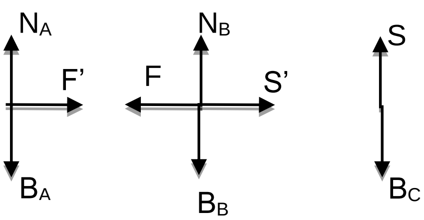
The period is found from $T = 4x_0/(v_0 \cos \alpha)$ where $\tan \alpha = 1/(2ax_0)$ and v_0 is expressed from $y_0 + 1/2a - (gx_0^2/2v_0^2)(1 + 1/4a^2x_0^2) = 0$.



In analogy to the situation in section 4, the trajectory of section 5 can have vertical tails under similar conditions. There are probably two additional periodic trajectories that are not discussed.

Solution to Problem 2

a)



b)

The relevant equations are:

$$2ma_B = mg - kx$$

and

$$ma_A = kx.$$

Let us study the movement of B (and C) with respect to A: multiply the second eq. with 2 and subtract it from the first:

$$2m(a_B - a_A) = mg - 3kx.$$

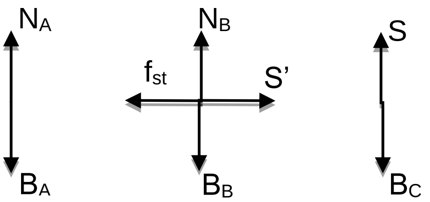
So, the problem is equivalent with suspending a body with the mass $2m$ by a spring with the elastic constant $3k$. The elongation of this spring is

$$x_{max} = \frac{2mg}{3k}.$$

In conclusion, the maximum distance between A and B is

$$L_{max} = L_0 + \frac{2mg}{3k}.$$

(c)

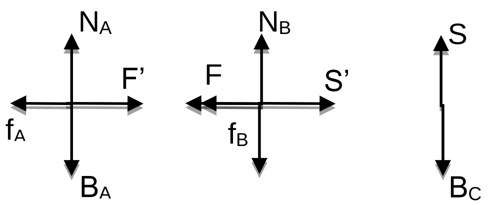


Initially, $d = l_0$. Since the bodies should remain at rest $d = l_0 \Rightarrow F = 0$

$m.g - S = 0$
$S' - \mu.m.g = 0$

$m.g - \mu.m.g = 0 \Rightarrow \mu = 1$

(a)



Constant speed:
$m.g - S = 0$
$S' - F - f_B = 0$
$F' - f_A = 0$
$\Rightarrow m.g - f_A - f_B = 0$
$m.g - 2\mu.m.g = 0$

$\Rightarrow m.g = 2\mu.m.g \Rightarrow \mu = 1/2$

$F - f_A = 0$
$k.x = \mu.m.g$

$$x = \frac{\mu.m.g}{k} = \frac{m.g}{2k} \Rightarrow l = l_0 + \frac{m.g}{2k}$$

Solution to Problem 3

Data:

$$l_1 = 600\, m,\ v_1 = 1\frac{m}{s},$$

$$l_2 = 800\, m,\ v_2 = 2\frac{m}{s}$$

Expression for time t_1 for the motion on the field as a function of x:

$$t_1 = \frac{AD}{v_1} = \frac{\sqrt{x^2 + l_1^2}}{v_1} \qquad (1)$$

Expression for time t_2 for the motion on the road as a function of x:

$$t_2 = \frac{DC}{v_2} = \frac{l_2 - x}{v_2} \qquad (2)$$

Expression for the total time t as a function of x:

$$t = t_1 + t_2 = \frac{1}{v_1}\sqrt{x^2 + l_1^2} + \frac{1}{v_2}(l_2 - x) \qquad (3)$$

Solutions for the total time and position, $t = t_0$ and $x = x_0$:

$$t_0 = \frac{1}{v_1}\sqrt{x_0^{\ 2} + l_1^{\ 2}} + \frac{1}{v_2}(l_2 - x_0) \qquad (4)$$

After some elementary algebraic manipulations in (4) we get

$$v_1^{\ 2}v_2^{\ 2}t_0^{\ 2} + v_1^{\ 2}l_2^{\ 2} + v_1^{\ 2}x_0^{\ 2} - 2v_1^{\ 2}v_2t_0l_2 + 2v_1^{\ 2}v_2t_0x_0 - 2v_1^{\ 2}x_0l_2 - v_2^{\ 2}x_0^{\ 2} - v_2^{\ 2}l_1^{\ 2} = 0 \qquad (5)$$

"Shaking" the solutions with small quantities Δt and Δx:

$$\begin{aligned} &v_1^{\ 2}v_2^{\ 2}(t_0 + \Delta t)^2 + v_1^{\ 2}l_2^{\ 2} + v_1^{\ 2}(x_0 + \Delta x)^2 - 2v_1^{\ 2}v_2(t_0 + \Delta t)l_2 + \\ &+ 2v_1^{\ 2}v_2(t_0 + \Delta t)(x_0 + \Delta x) - 2v_1^{\ 2}(x_0 + \Delta x)l_2 - v_2^{\ 2}(x_0 + \Delta x)^2 - v_2^{\ 2}l_1^{\ 2} = 0 \end{aligned} \qquad (6)$$

$$\begin{aligned} &(6) \quad \Rightarrow \quad v_1^{\ 2}v_2^{\ 2}(t_0^{\ 2} + 2t_0\Delta t + \Delta t^2) + v_1^{\ 2}l_2^{\ 2} + v_1^{\ 2}(x_0^{\ 2} + 2x_0\Delta x + \Delta x^2) - \\ &- 2v_1^{\ 2}v_2(t_0 + \Delta t)l_2 + 2v_1^{\ 2}v_2(t_0 + \Delta t)(x_0 + \Delta x) - 2v_1^{\ 2}(x_0 + \Delta x)l_2 - \\ &- v_2^{\ 2}(x_0^{\ 2} + 2x_0\Delta x + \Delta x^2) - v_2^{\ 2}l_1^{\ 2} = 0 \end{aligned} \qquad (7)$$

$$\begin{aligned} &(5) \to (7) \quad \Rightarrow \quad v_1^{\ 2}v_2^{\ 2}(2t_0\Delta t + \Delta t^2) + v_1^{\ 2}(2x_0\Delta x + \Delta x^2) - \\ &- 2v_1^{\ 2}v_2\Delta t l_2 + 2v_1^{\ 2}v_2(\Delta t x_0 + t_0\Delta x + \Delta t\Delta x) - 2v_1^{\ 2}\Delta x l_2 - \\ &- v_2^{\ 2}(2x_0\Delta x + \Delta x^2) = 0 \end{aligned} \qquad (8)$$

Neglecting all the terms containing very small products, $\Delta t\cdot\Delta t$, $\Delta t\cdot\Delta x$ and $\Delta x\cdot\Delta x$:

$$(8) \Rightarrow v_1^2v_2^2t_0\Delta t+v_1^2x_0\Delta x-v_1^2v_2\Delta t l_2+v_1^2v_2(\Delta t x_0+t_0\Delta x)-v_1^2\Delta x l_2-v_2^2x_0\Delta x=0 \quad (9)$$

Expression for $\Delta t/\Delta x$:

$$(9) \Rightarrow \frac{\Delta t}{\Delta x}=\frac{l_2+(\frac{v_2^2}{v_1^2}-1)x_0-v_2t_0}{v_2^2t_0-v_2l_2-v_2x_0} \quad (10)$$

Equating $\Delta x/\Delta t$ to zero and evaluation of x and t:

$$\frac{\Delta t}{\Delta x}=0 \quad (11)$$

$$(10),(11) \Rightarrow l_2+(\frac{v_2^2}{v_1^2}-1)x_0-v_2t_0=0 \quad (12)$$

$$(4)\rightarrow(12) \Rightarrow x_0=\ldots=\frac{l_1}{\sqrt{\frac{v_2^2}{v_1^2}-1}} \quad (13)$$

$$(13)\rightarrow(4) \Rightarrow t_0=\ldots=\frac{1}{v_2}\left[l_2+l_1\sqrt{\frac{v_2^2}{v_1^2}-1}\right] \quad (14)$$

$$(13) \Rightarrow x_0=\ldots=346.8\,m \quad (15)$$

$$(14) \Rightarrow t_0=\ldots=920\,s \quad (16)$$

Solution to Problem 4

As after the switching of the K1 the resistance of the circuit remains the same, the potentials on the both ends of K1 are equal (voltage on K1 is zero). This situation is only possible when the initial state of K2 is "1". Later, as the $R_{AB\gamma}>R_{AB\delta}$, it can be concluded that the last state of K1 is "1". So, using these conclusions we can fill the state of the switches for all states of the circuit (see the table).

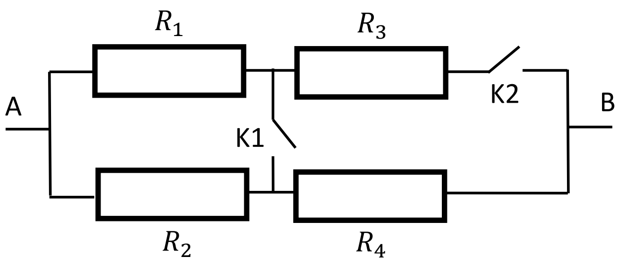
Now we can start calculations of the values of the resistors. As in the beginning both switches are closed and the opening of the K1 does not change the circuit resistance, it follows $U_{R_1}=U_{R_2}$ and $U_{R_3}=U_{R_4}$. The relations of the currents are $I_{R_1}=I_{R_3}$ and $I_{R_2}=I_{R_4}$. From these equations we can obtain that $\frac{R_2}{R_1}=\frac{R_4}{R_3}=x$. To minimize the above expressions, we can note $R_1=R$, $R_2=xR$, let $R_3=yR$, then $R_4=xyR$.

For circuit state "β": $\frac{R(1+y)Rx(1+y)}{R(1+y)+Rx(1+y)}=240\ \Omega$. After simplification $\frac{Rx(1+y)}{1+x}=240\ \Omega$ (1).

For circuit state "γ": $x(1+y)R=400\ \Omega$. (2)

For circuit state "δ": $\frac{xRR}{xR+R}+xyR=280\ \Omega$. (3)

Substituting (2) in (1), $\frac{400\ \Omega}{1+x} = 240\ \Omega$, then $x = \frac{400}{240} - 1 = \frac{2}{3}$.



Subtracting (3) from (2), $x(1+y)R - \frac{xRR}{xR+R} + xyR = 120\ \Omega$. After simplification, $\frac{x^2R}{x+1} = 120\ \Omega$. Substituting x with the obtained value, $R = 450\ \Omega$. Now substituting both x and R with their values in (2), $\frac{2}{3}(1+y)450\ \Omega = 400\ \Omega$, we obtain $y = \frac{1}{3}$. So, the final values of the resistor are, $R_1 = 450\ \Omega$, $R_2 = 300\ \Omega$, $R_3 = 150\ \Omega$, and $R_4 = 100\ \Omega$.

The answer is given in the table below.

State of the circuit	State of the K1	State of the K2	R_{ABi}, Ω		Resistor R_j	Value, Ω
α	1	1	240		R_1	450
β	0	1	240		R_2	300
γ	0	0	400		R_3	150
δ	1	0	280		R_4	100