Theoretical Problem 1: ISS Orbital Decay Dynamics [10.0 points]

Introduction

Figure 1: The International Space Station orbiting above the Earth.

The ISS is currently maintained in a nearly circular orbit with a minimum mean altitude of $370km$ and a maximum of $460km$, in the center of the thermosphere, at an inclination of $\theta = 51.6^{\circ}$ (degrees) to Earth's equator. The trajectory of the spacecraft is similar to a spiral with a slowly changing distance from the station to the Earth's surface, and during one cycle of revolution the change in altitude is inconsiderable.

The ISS mass is $M_S = 4.5 \times 10^5 kg$ and overall length is $L_S = 109m$. Huge solar panels with a width of $W_S = 73m$ provide the ISS with electrical energy [NASA Official Report (2023].

Including all batteries and other parts, the effective cross area (section) of the station is approximately $S \approx 2.5 \times 10^3$ m^2 [European Space Agency, SDC6-23].

The ISS orbital decay is caused by one or more mechanisms which absorb energy from the orbital motion, the essential ones being:

- atmospheric drag at orbital altitude is caused by frequent collisions of gas molecules with the satellite,
- the Ampere force arising from the motion of the conductive apparatus in the Earth's magnetic field,
- the interaction with the atomic oxygen ions.

"... In May 2008, the altitude was 350 kilometers, the ISS lost $4.5km$ and was re-boosted by the Progess-60 supply ship by $5.5km$. Again, the ISS continued to lose altitude by $5.5km$ ..." [https://mod.jsc.nasa.gov]

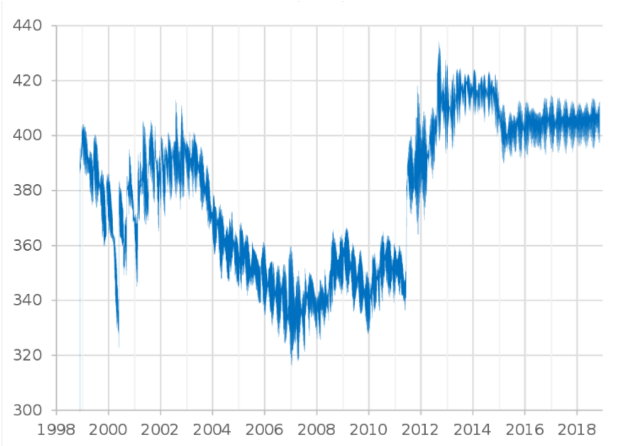


Figure 2: The altitude of ISS (km) over the years.

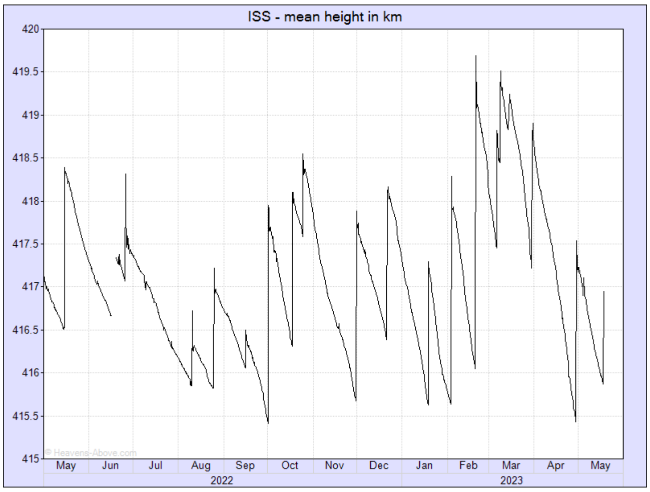


Figure 3: The ISS mean height (km) in 2022-2023.

". . . The ISS loses up to $330ft$ ($100m$) of altitude each day . . . " [NASA Control Data (2021)]. In 2023 the ISS flies at altitudes of 410 km, with an orbital decay about $70m$ every day ($\sim 2km$ per month), and during magnetic storms the daily descent reaches $300m$. The ISS accomplishes the de-orbit maneuvers by using the propulsion capabilities of the ISS and its visiting vehicles [International Space Station Transition Report (2022)].

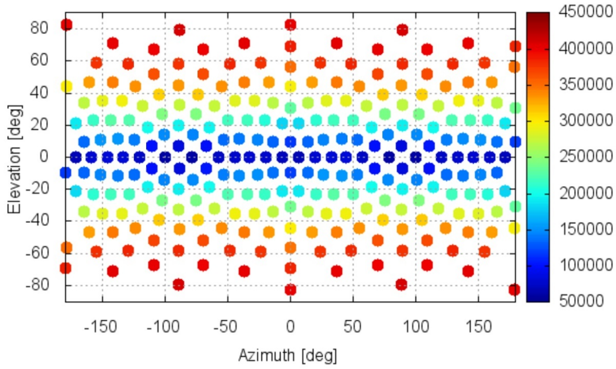


Figure 4: ISS model with the cross sections from different aspect angles (dm^2). The CROC provides $2481m^2$ cross section.

Denotations and Physical constants:

Universal gas constant	R	$=$	$8.31\,J \cdot K^{-1} \cdot mol^{-1}$
Avogadro's number	N_A	$=$	$6.022 \cdot 10^{23}\,mol^{-1}$
The molar mass of gas (for air)	μ	$=$	$0.029\,kg \cdot mol^{-1}$
Mass of the Earth	M_E	$=$	$5.97 \cdot 10^{24}\,kg$
Radius of the Earth	R_E	$=$	$6.38 \cdot 10^{6}\,m$
Gravitational universal constant	G	$=$	$6.67 \cdot 10^{-11}\,m^3 \cdot s^{-2} \cdot kg^{-1}$
Density of air at Earth's surface	ρ_0	$=$	$1.29\,kg/m^3$
Gravitational acceleration at Earth's surface	g_0	$=$	$9.81\,m \cdot s^{-2}$
Average magnitude of Earth's magnetic field	B	$=$	$5.0 \cdot 10^{-5}\,T$
The electron absolute charge	e	$=$	$1.60 \cdot 10^{-19}\,C$

Part A: Modified barometric formula [2.0 points]

The pressure of atmospheric air, composed mainly of neutral O_2 and N_2 molecules, can be found by using the Clapeyron-Mendeleev law (the ideal gas law): $pV = \frac{M}{\mu}RT$. where p, V, T, M and μ are the pressure, volume, temperature, mass and molar mass of a portion of air, R is the ideal gas universal constant.

There are two equations for computing air pressure as a function of height. The first equation is applicable to the standard model of the **troposphere** ($h < 100km$) in which the temperature is assumed to vary with altitude at a lapse rate.

The second equation belongs to the standard model of the **thermosphere** ($h > 250km$) in which the temperature is assumed not to change considerably with altitude and is applicable to ISS.

We may assume that all pressure is hydrostatic and isotropic (i.e., it acts with equal magnitude in all directions).

A.1	Derive the general integral expression for the air pressure p_h at ISS altitude h. This equation is called the general barometric formula. Hint: the temperature and gravitation may depend on h.	0.5pt

Remark 1. The temperature of Earth's thermosphere at altitude $300-600km$ does not change considerably and reaches averagely about $800-900K$ on the solar side [NASA data]. Therefore, one may put $T_h = T = const$ by investigating the ISS orbital flight. Particularly, since the spacecraft spends almost half of its flight time in the shadow side of the Earth, where the temperature drops sharply, we may take the value of $T = 425K$ as the average temperature at these altitudes. This temperature is also in agreement with the air density value $\rho_h \sim 10^{-12} kg/\text{m}^3$ [MSISE-90 Model of Earth's Upper Atmosphere] at $h = 400km$.

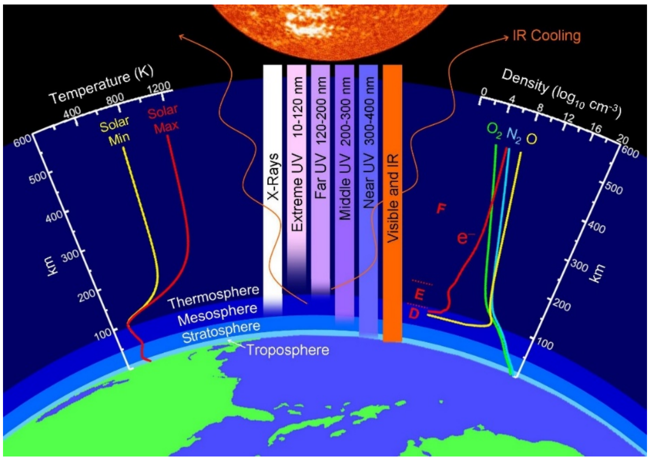


Figure 5: The Earth's thermosphere.

A.2	Write down the air pressure (the standard barometric formula) p_h^{sta}, when the temperature and gravitation g_h do not depend on h. Calculate the parameter $h_0 = \frac{RT}{\mu g_0}$ for $T = 425K$.	0.3pt

A.3	Write down the air pressure (the improved barometric formula) p_h^{imp} when the temperature is constant but the gravitation depends on h. Hint: Use the leading-order correction only, with accuracy $O(z_h^2)$. Hereby, the flight altitude h above the Earth's surface is significantly smaller than the Earth's radius: $z_h \doteq h/R_E \ll 1$.	0.6pt

A.4	Write down the ratio of the 'standard' and 'improved' versions of the barometric formula p_h^{imp}/p_h^{sta}. Estimate it for $h = 4.0 \times 10^5 m$. Further use the 'improved' version.	0.4pt

A.5	Write down the air density ρ_h and the concentration of neutral air molecules n_h at height h, with accuracy $O(z_h^2)$.	0.2pt

Part B: Orbital deceleration and station descent rate [3.0 points]

Let us consider the problem of determining the rate of orbital decay of a satellite with mass M_S that experiences constant friction force $\vec{F}_{drag}$ acting on it. We assume that the decrease in altitude dh is much less than the flight altitude h itself ($dh \ll h$).

B.1	Write down the satellite velocity v_h and revolution period τ_h on a stable orbit of altitude h.	0.5pt

B.2	Write down the total energy E_S of a satellite moving along a circular orbit with radius $R_E + h$.	0.5pt

B.3	The total decelerating force exerted on a satellite of constant mass is given by some external braking force $\vec{F}_{drag}$. As a result, the ISS slows down and its altitude decreases by a height dh for a small time interval, dt. Write down the equation for the total enery balance of the ISS and surrounding system, given a value of F_{drag}.	1.0pt

B.4	Define the rate of descent (de-orbiting) speed u_h of the satellite. Hint: The de-orbiting speed depends on the friction force, and on the altitude of the satellite, and on the mass of the satellite.	0.5pt

B.5	Write down the amount of decent H_h for a revolution around the Earth and the total time T_h for which the satellite will fall from the altitude h to the earth's surface due to the friction. Hint: Take into account relations $h_0 \ll h \ll R_E$.	0.5pt

Part C: Atmospheric drag [1.0 points]

The speed of the satellite v is many times greater than the average velocities (hundreds m/s) of the thermal motion of atmospheric molecules at a height $h \approx 300 - 400 km$, so we can assume that the molecules were at rest before the collision with the ISS. To roughly estimate the drag force, we assume that after the collision the molecules acquire the same speed as the satellite.

C.1	Write down the air drag force F_{air}, the de-orbiting descending velocity u_h^{air} and the descent rate H_h^{air}.	0.5pt

C.2	Define the total time T_h^{air} for which the satellite will fall from the altitude h to the earth's surface due to air drag effect. Hint: Take into account relations $h_0 \ll h \ll R_E$.	0.5pt

Part D: Drag by atomic oxygen ion [1.0 points]

In the thermosphere, under the influence of ultraviolet and X-ray solar radiation and cosmic radiation, air ionization occurs (``polar lights''). Unlike O_2, N_2 does not undergo strong dissociation under the action of solar radiation, therefore, in general, there is much less atomic nitrogen N in the Earth's upper atmosphere than atomic oxygen. At altitudes above $250 km$, atomic oxygen O predominates. Layers containing electrons and ions of oxygen atoms appear on the day side of the atmosphere. In this case, the concentration of atomic oxygen ions reaches $n_{ion} \sim 10^{12}\, m^{-3}$

D.1	Write down the decelerating force F_{ion}, averaged during a 24-hour, associated with the mechanical collisions of these particles. Take into account the strong decrease in ionized layers are negligible during the night. Express the density of ionized oxygen molecules ρ_{ion}.	0.3pt

D.2	Define the speed of fall of the satellite u_h^{ion} due to deceleration by ions of atomic oxygen. Write down the descent rate $H_h^{ion}$for a revolution caused by the ionization effect. Hint: Take into account relations $h_0 \ll h \ll R_E$.	0.7pt

Part E: Drag by the Earth's magnetic field [2.0 points]

We consider the influence on the motion of the satellite of the Earth's magnetic field, the value of which near the Earth's surface is equal to $(3.5-6.5)\cdot 10^{-5}T$ with an average value of $B = 5\cdot 10^{-5}T$.

When a satellite moves at high speed in a magnetic field, an inducted electric current (electromotive force (EMF)) occurs in the current-conducting elements of the satellite's structure. This electromotive force causes a redistribution of electric charges in the current-conducting elements of the satellite structure. An electric field appears around the satellite, which affects the movement electrically charged particles in the environment. Electrons are attracted to those parts of the satellite that have a positive potential (relative to the middle part of the satellite), and positively charged ions are attracted to those parts of the satellite that have a negative potential. Electrons and ions that hit the surface of the satellite structures are combined into neutral oxygen atoms, while the electrons 'travel' in the satellite's conductive structures, creating an electric current. The satellite, moving in space, 'collects' electrons and ions from the surrounding space and collides with them. For a rough estimate of the magnitude of the current that can flow through the conductive structures of the satellite, we will assume that the collection occurs only from an area equal to the cross-sectional area S of the satellite, and all ions and electrons participate in the creation of this current.

E.1	Evaluate approximately the magnitude of the induced electric current I_{ind} .	0.6pt

E.2	Determine an approximate expression for the induced 'braking' Ampere force F_{ind} in the direction opposite to the direction of the satellite's motion. Let ϕ be the angle between the Earth magnetic field $\vec{B}$along the longitude lines. To simplify, you may approximate the length of the satellite L as the square root of the satellite area S. Additionally, instead of computing the average of sin(ϕ), you may approximate it with sin$(\pi/2-\theta)$. You may use a discrete number of sample points to compute an average value.	0.6pt

E.3	Write down the descent speed u_{ind} of the satellite due to Earth's magnetic field. Write down the descent rate H_h^{ind} for a revolution caused by the magnetic drag effect. Hint: Take into account relations $h \ll R_E$.	0.8pt

Part F: Numerical results and conclusion [1.0 points]

F.1	Calculate and fill Table 1 in the Answer Sheet.	0.4pt

F.2	Calculate and fill Table 2 in the Answer Sheet.	0.4pt

F.3	Rank these three orbital slowing processes in order of how strong an impact they have on ISS orbital altitudes higher than $380km$. For the International Space Station, orbiting at an altitude above $380km$, write down the most significant factors contributing to orbital decay.	0.2pt

Theoretical Problem 2: A ball on a turntable [10.0 points]

Preamble

Notations and conventions: The length of a vector $\vec{A}$ is simply denoted as $A \equiv |\vec{A}|$. It's x, y, z components are denoted by A_x, A_y, A_z, respectively. The time derivative of a quantity is denoted by the dot over the quantity: $\dot{\vec{A}} \equiv d\vec{A}/dt$, $\dot{A} \equiv dA/dt$. The unit vector along the direction of vector $\vec{A}$ is denoted as $\hat{A}$. The unit vectors along the Cartesian coordinates are, therefore, $\hat{x}$, $\hat{y}$ and $\hat{z}$. The definitions of scalar and vector products are:

$$(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{A}) = A_xB_x + A_yB_y + A_zB_z = AB\cos\theta$$

$$(\vec{A}\times\vec{B}) = -(\vec{B}\times\vec{A})$$
$$= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_x - A_yB_x)\hat{z}$$

$$|\vec{A}\times\vec{B}| = AB\sin\theta,$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. You may need the following properties of vectors and their multiplications. Triple product rules for vectors:

$$(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A},$$
$$(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A} = (\vec{C}\times\vec{A})\cdot\vec{B}.$$

The vector products are very useful in describing many relations in physics. For example:

$$\vec{v} = \vec{\omega}\times\vec{r},$$
$$\vec{F}_{Lorentz} = Q\vec{v}\times\vec{B},$$

and, often, saves time combining three equations for vector components into a single equation.

The statement

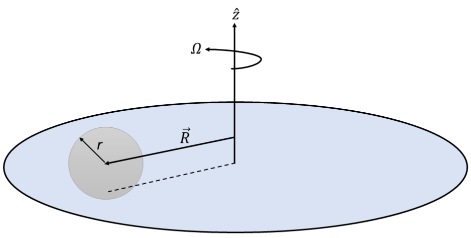


Figure 1. Ball rolling on the turntable without slipping

A ball of mass m and radius r is rolling on a horizontal turntable without slipping (see Figure 1). Its mass density has a spherical symmetry, i.e. only depends on the distance from its center. The moment of inertia of the ball is I . In part B and C, where the turntable can rotate freely, the moment of inertia of the turntable is denoted as I_d. The purpose of the problem is to analyze the motion and trajectory of the ball with respect to the laboratory frame. Throughout the problem, assume the turntable is large enough so that the ball does not fall off. The following notations are used:

Ω - the magnitude of the turntable angular velocity,

$\vec{\omega}$ - the spinning angular velocity of the ball with respect to its spinning axis,

$\vec{R}$ - the horizontal position of the ball center with respect to the rotation axis of the turn table,

$\vec{v}$ - the velocity of the ball at $\vec{R}$ with respect to the laboratory frame.

Assume that the initial position $\vec{R}_0 \equiv \vec{R}(0)$ and velocity $\vec{v}_0 \equiv \vec{v}(0)$ of the ball, the angular velocity of the turn table $\Omega_0 \equiv \Omega(0)$ are known. For the initial vector quantities $\vec{R}_0 \equiv \vec{R}(0)$ and $\vec{v}_0 \equiv \vec{v}(0)$, assume that their directions are known. In addition, whenever you need to express a vector quantity, you may use $\hat{z}$ in your expression. Also, if asked to write your expression in terms of the known quantity you may use any or all of m, r, I and I_d. Unless otherwise stated, keep I as general. The following notations are recommended:

$\alpha = \frac{I}{I+mr^2}$, $\delta = \frac{I_d}{mr^2}$,

You may write the final answers as vector expressions involving cross product (vector product), dot product (scalar product) and unit vectors in axis directions.

Part A: Ball on turntable with constant angular velocity [1.5 points]

First we start with the simplest case wherein the turntable angular velocity with respect to vertical axis $\hat{z}$ is constant, therefore $\Omega = \Omega_0$.

A.1	Express the ball's velocity $\vec{v}$ in terms of Ω, $\vec{\omega}$, r and $\vec{R}$ from a kinematic constraint.	0.1pt

A.2	Using Newton's equation and torque equation with respect to its center, find the acceleration of the ball $\vec{a} \equiv \dot{\vec{v}}$ in terms of Ω, $\vec{v}$, r, m and I.	0.2pt

A.3	Find the velocity $\vec{v}$ in terms of Ω, $\vec{R}$, $\vec{v}_0$, $\vec{R}_0$, r, m and I.	0.2pt

A.4	Write an explicit solution for the trajectory of the ball given the initial conditions $\vec{v}_0$ and $\vec{R}_0$.	0.5pt

A.5	Assume this time that the ball has a uniform mass density, i.e. $I = 2mr^2/5$. Trajectory you have found is a circle and it's radius is R_t. Choose its magnitude to be the same as R_0. How long does it take for the ball to approach the initial spot on the table (the position on the turntable at $t = 0$) with the closest distance?	0.5pt

Part B: Ball on freely rotating turntable [4.0 points]

In this part, the turntable can rotate freely without any friction around z -axis. Therefore its free rotation is hindered only by the ball's friction.

B.1	Find the velocity $\vec{v}$ and acceleration $\dot{\vec{v}}$ of the ball in terms of Ω , $\vec{R}, \Omega_0, \vec{R}_0, \dot{\Omega}, r$, m and I.	0.2pt

B.2	Find the magnitude of the angular acceleration of the turntable $\dot{\Omega}$ in terms of $\Omega, \Omega_0, \vec{R}, \vec{R}_0, \vec{v}_0, r$, m, I and I_d. You may use the constants α and δ defined in the beginning of the problem.	0.6pt

B.3	Find the magnitude of the angular velocity of the turntable Ω as a function of R only. Use this constants in your expression: $\Omega_0, R_0, r, m, I, I_d$.	0.6pt

B.4	From the result of B.3, for a given Ω_0, R_0, find the maximum possible Ω .	0.1pt

B.5	Write down the vertical component the angular momentum $\hat{z}M_z$ of the whole system. Subtract any constant term and rename the remaining part as $\hat{z}L$. In part B.1 you found the velocity of the ball $\vec{v}$, which can be written as the sum of a part that depends on the position of the ball $\vec{R}$ and a constant vector. Let us call this constant vector $\vec{c}$. Choose the direction of x-axis along this vector and y-axis along $\hat{z}\times\vec{c}$. In this frame of reference, find Ω in terms of $L, \vec{R}, \vec{c}, \hat{z}, R^2, r, m, I$ and I_d . Combining this with the result of B.3, write down an equation only containing R^2 and y variables and L, r, m, I, c and I_d. Here c is the magnitude of $\vec{c}$. Substituting $R^2 = x^2 + y^2$, write down an expression containing only x and y variables and describing a curve. From this, list all possible types of trajectories.	2.5pt

Part C: Ball on turntable in magnetic field [4.5 points]

In this part, we consider a density profile so that $I = mr^2/10$. This can be realized, for example, if the ball is filled up to half of its radius with uniform density and the remaining part has a negligible mass. In addition, on its outer surface, the ball has a uniform charge density $Q/(4\pi r^2)$, where Q is the total surface charge. The whole setup is in a uniform magnetic field $\vec{B}$ that is in $\hat{z}$ direction. The turntable rotates with constant Ω like in Part A.

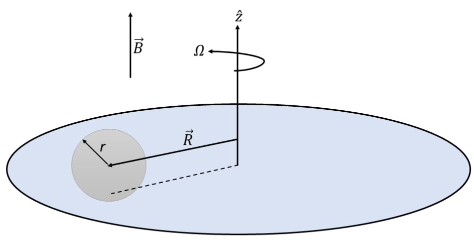


Figure 2. Ball rolling on the turntable in a constant magnetic field $\vec{B}$

C.1	Write down Newton's equation and the torque $\vec{\tau}_s$ equation for the ball. Find expression for the torque due to the spinning of the ball around its axis in terms of $Q, r, \vec{\omega}$ and $\vec{B}$.	0.5pt

C.2	Using the results of C.1, find expression for the linear acceleration of the ball with respect to the laboratory frame in terms of $Q, r, \vec{\omega}$ and $\vec{B}$.	0.5pt

C.3 We assume all quantities of unit lenght are measured by meter, all angular velocities have unit of 1 Hertz, and all quantities of time have the unit of 1 second. The equation for the linear acceleration you found in part C.2 is a second order differential equation for $\vec{R}$ of the following form: **1.0pt**

$\frac{d^2\vec{R}}{dt^2} - \gamma\frac{d\vec{R}}{dt} \times \hat{z} + \beta\vec{R} = 0.$

Write down γ and β constants in terms of Q, r, B, I, m, Ω. Make the following transformation to a polar coordinates for the components of $\vec{R}$:

$x(t) = \rho(t)\cos(\eta(t))$,

$y(t) = \rho(t)\sin(\eta(t))$

so that the new equations do not have the first time derivative term. Here the polar angle $\eta(t)$ is a function of time. Find the form of this function.

Express the coefficient β' of $\rho(t)$ in the new equation in terms of γ and β .

Write down the conditions for different types of behavior of $\rho(t)$ with respect to time: harmonic, exponential etc.

C.4 Consider the following initial conditions for the solution found in part C.3: **0.9pt**

$x(0) = 1\,m,\ \ y = 0\,m,\ \ v_x(0) = \dot{x}|_{t=0} = 1\,m/s,\ \ ,v_y(0) = \dot{y}|_{t=0} = -1\,m/s.$

From these conditions, find β and γ. Using them find the corresponding Ω. Sketch the trajectory. Is the charge of the surface negative or positive? For the negative write $-$ and for the positive write $+$ on your answer sheet.

C.5 Consider the solution you have found in part C.4. If you identified it correctly your solution should have a rotating $\vec{R}(t)$. Find the expressions for the total and per rotation changes in energy for $N \gg 1$ number of rotations. Here you may ignore the terms small compared to N. In this part assume the mass and the radius of the ball are $m = 1\,kg$ and $r = 1\,m$ so that $I = 1/11\,kg \cdot m^2$. **1.6pt**

Theoretical Problem 3: Cavitation [10.0 points]

Introduction

Cavitation is the phenomenon of vapour bubbles or ``cavities'' occurring in a liquid medium due to *drop in pressure*. This is in contrast to boiling, where vapour bubbles are created due to *rise in temperature*. Since the vapour bubbles collapse and generate shock waves as well as supersonic jets when the dropped pressure is restored, cavitation is a constant source of damage and even of catastrophe in hydraulic machines, ships, and more generally in any device involving liquid flow. On the other hand, it has found many positive applications, for example in chemical industry, cleaning, and in treatment of kidney stones.

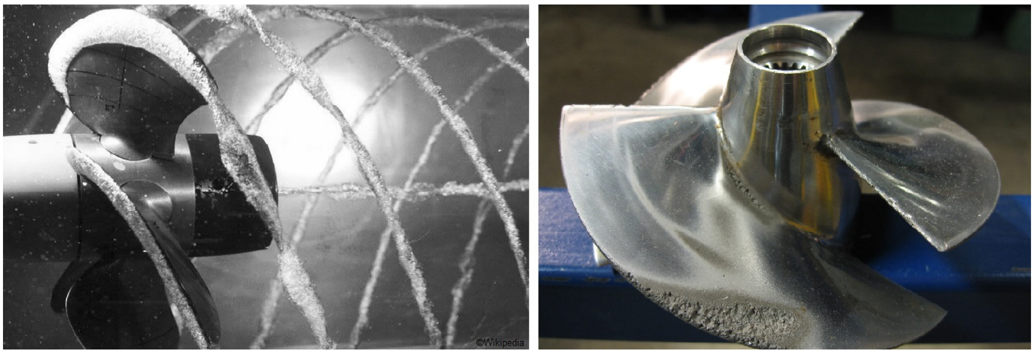


Figure 1. (a) Cavitating propeller (b) Cavitation damage (Source: Wikimedia Commons)

It is understood that cavitation generally grow out of microscopic bubbles, called *nuclei*, that preexisted in the liquid. These micro-bubbles are a few microns in size and contain both vapour and non-condensable gas (the latter is just air when ordinary water is under consideration). If the pressure in the liquid becomes sufficiently low, the nuclei grow into a macroscopic size, initiating cavitation. Liquid purified of such nuclei can even withstand negative pressure without cavitating. One usually compares this with solid under tension, which does not rupture easily if there are no preexisting pockets or cracks in it.

In this problem, we will be concerned with various idealized scenarios related to cavitation. As is often the case, we can glean some nontrivial information from simple dimensional analysis. However, we will need differential equations embodying fundamental laws such as Newton's second law of motion and Fick's law of diffusion, if we want to conduct a more precise study.

One of the first things we want to know is the so called *critical (or threshold) pressure*, that is the minimum value of the water pressure so that the nuclei remain microscopic without growing into macroscopic bubbles. The critical pressure is roughly equal to the vapour pressure at the given temperature, but the exact value is slightly lower due to surface tension and the air content of the nucleus.

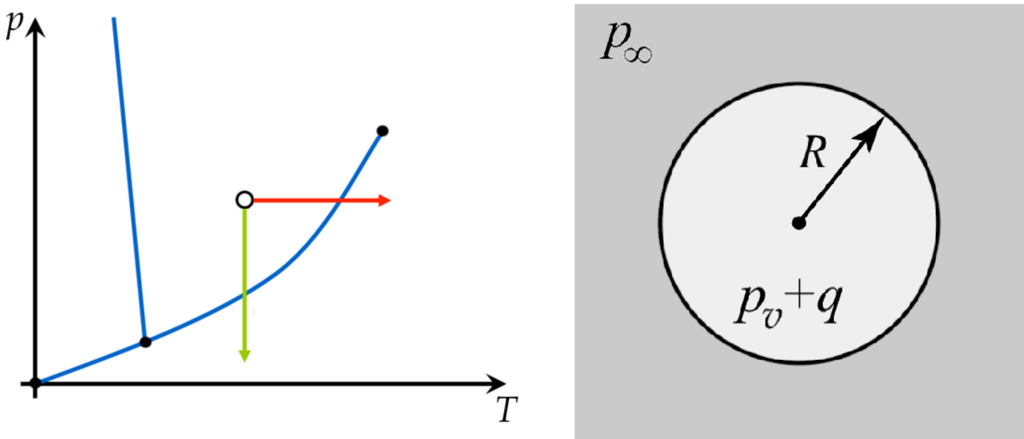


Figure 2. (a) Cavitation (down arrow) and boiling (right arrow) on a phase diagram (b) Typical bubble (see Table 1 for notations)

If the external pressure suddenly drops below the critical pressure of a nucleus, then the nucleus starts expanding and the expansion rate quickly reaches a stable value. In practice, after the bubble becomes macroscopic in size, the pressure is typically restored to its original value, and the bubble starts collapsing. We will model this situation by considering a macroscopic bubble in equilibrium, whose external pressure is then suddenly risen. The collapsing bubble will rebound after reaching a minimum size if the bubble had air in it. On the other hand, a pure vapour bubble would completely dissolve, with the shrinkage rate growing unboundedly as the radius of the bubble reaches zero. In reality, towards the end of the collapse, the bubble would lose its spherical shape, and the compressibility of water would become important. However, unless a particular question explicitly asserts otherwise, we will neglect those effects here.

Another interesting question is what happens when sound wave is transmitted through water containing bubbles. It turns out that not only the bubbles pulsate following the pressure oscillations, but also the sound wave induces translational motions of the bubbles. These effects can be used to manipulate bubbles with the help of acoustic waves. For example, in acoustic cavitation, high intensity ultrasound is employed to generate cavitation or cause collapse of bubbles.

Finally, there is a sort of paradox regarding the existence of nuclei in the first place. The theory predicts that unless water is saturated with dissolved air, diffusion of air from any nucleus into water through the gas-water interface must induce a complete dissolution of the nucleus in a matter of seconds. However, in reality, micron sized nuclei exist in water and it is in fact extremely difficult to get rid of them. We will consider one of a few potential resolutions of this paradox, namely the suggestion that small crevices in solid walls or in solid particles carried by water are responsible for acting as micro-pockets of air and vapour.

Potentially useful information

Vapour pressure

Let us say we have a closed jar containing water and air. If the air is too dry, then its humidity will increase due to evaporation of water. If the air is too wet, then its humidity will decrease due to condensation. It turns out that in equilibrium, the partial pressure $p_v = p_v(T)$ of vapour in air is a function of temperature.

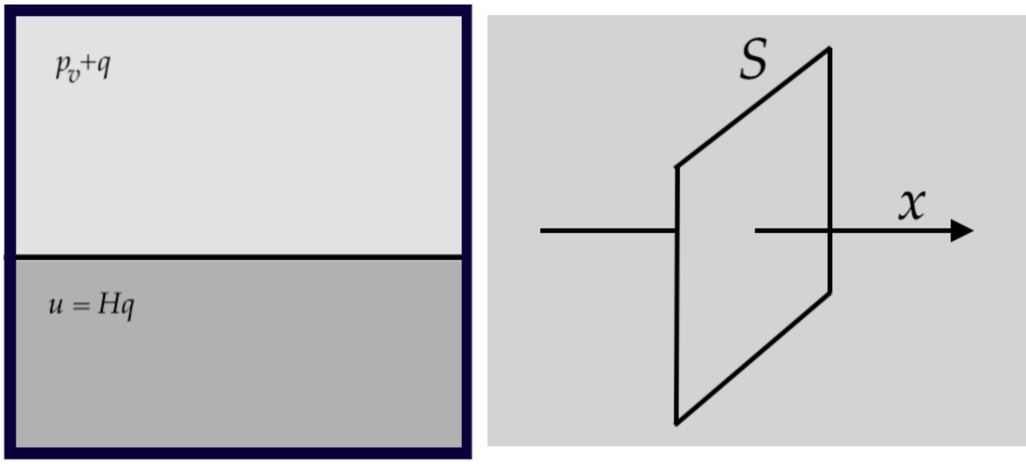


Figure 3. (a) Closed jar containing air and water in equilibrium (b) Diffusion flux through the surface S is proportional to the concentration gradient across S

Now if a bubble changes its volume in an instant, then the humidity inside the bubble will lose its equilibrium with the surrounding water, and a new equilibrium must be reached either by condensation or evaporation. In reality this process is so rapid that we can justifiably assume that equilibrium is maintained at all times. Moreover, the heat lost or gained by the surrounding water during this process is negligible, so that the temperature remains constant. To conclude, we assume that the partial pressure of vapour contained in a bubble remains equal to p_v at all times.

Henry's law

While the concept of vapour pressure gives us a good handle on the vapour content of a bubble, Henry law offers at least a partial handle on the air content. Thinking of a closed jar containing water and air, it says that in equilibrium, the concentration of dissolved air in water is proportional to the partial pressure of air above the water:

$$u = Hq$$

where, u is the concentration of air in water, H is the so called Henry's constant, and q is the partial pressure of air adjacent to water. As before, we will assume that equilibrium of air content in the sense of Henry's law is maintained at least in the immediate vicinity of the bubble at all times, and that this maintenance does not cause any temperature change.

Fick's law

To complement Henry's law, we need to know how dissolved air in water moves from places with high concentration to places with low concentration. This is where Fick's law enters, which states that the diffusion flux across an area element S is proportional to how fast the concentration changes along the direction perpendicular to S, see Figure 3:

$$J = \kappa \frac{\partial u}{\partial x}$$

Here J is the diffusion flux, that is the amount of air moving across the surface per unit area per unit time, κ is the diffusivity coefficient, and we have assumed that the x coordinate axis is perpendicular to S. When u is a function of x and possibly other variables, the notation $\frac{\partial u}{\partial x}$ means that we have taken the derivative of u with respect to the variable x, while holding all other variables constant.

Diffusion equation

If you need to find a function $w = w(x,t)$ in the first quadrant $Q = \{(x,t) : x > 0, t > 0\}$ satisfying $\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}$ in Q, and $\begin{cases} w(x,0) = f(x) & \text{for } x > 0, \\ w(0,t) = 0 & \text{for } t > 0, \end{cases}$ then the solution is given by $w(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left(e^{-(x-y)^2/(4t)} - e^{-(x+y)^2/(4t)}\right) f(y) dy$.

Gaussian type integrals

The following integrals may come in handy. $\int_0^\infty e^{-bx^2} dx = \frac{\sqrt{\pi}}{2\sqrt{b}}$, $\quad \int_0^\infty x^2 e^{-bx^2} dx = \frac{\sqrt{\pi}}{4b\sqrt{b}} \quad (b > 0)$.

Notations and typical values of parameters.

In Table 1, we list the notations used in the statement of the problem, and the typical values of some important constants.

symbol	assigned meaning	typical value
ρ	water density	$997\,\text{kg/m}^3$
p_∞	water pressure far from a bubble	$101\,\text{kPa}$
p_v	vapour pressure	$2340\,\text{Pa}$
σ	surface tension	$72.8 \cdot 10^{-3}\text{N/m}$
R	bubble radius	
R_0	initial radius of a bubble	$10^{-5}\,\text{m}$
δ	density of air	$1.29\,\text{kg/m}^3$
q	partial pressure of air in a bubble	
q_0	initial value of q	
γ	adiabatic exponent of air	1.4
u	concentration of dissolved air in water	
κ	diffusivity coefficient for air in water	$2 \cdot 10^{-9}\,\text{m}^2/\text{s}$
H	Henry's constant for air in water	$0.24 \cdot 10^{-6}\,\text{s}^2/\text{m}^2$
t	time	
f_0	natural/resonant frequency	

Assumptions

Unless otherwise specified, throughout this problem we assume the following.

- Water is incompressible, inviscid, and homogeneous.
- Water fills the entire space.
- Pressure variation due to gravity is negligible.
- No spatial or temporal variation in temperature.
- There is a single bubble.

- The bubble remains spherical and without translatory motion.
- No migration of air between the bubble cavity and the surrounding water.
- Air is an ideal gas.

Part A: Preliminary analysis [1.5 points]

These are warm-up questions to get the initial feel of the phenomenon.

A.1	By performing a simple dimensional analysis, estimate the collapse time T of a pure vapour bubble, in terms of bubble's initial radius R_0, water density ρ, water pressure p_∞, and the vapour pressure p_v. Evaluate the formula with the numerical constant implicit in the formula equal to 1, when $R_0 = 1$ mm and the quantities ρ, p_∞, and p_v take their typical values from previous Notation Table . Assume no surface tension: $\sigma = 0$.	0.5pt

A.2	Suppose that a nucleus consisting of air and vapour, with radius $R_0 = 10^{-5}$ m, is in equilibrium when the external pressure $p_\infty = 101$ kPa. Find the partial pressure q_0 of air in the bubble. Now suppose that the external pressure p_∞ was gradually decreased, and that the air inside the bubble follows an isothermal process. Find the critical pressure p_c, defined by the condition that if $p_\infty < p_c$ the bubble size grows without bound. The quantities p_v and σ take their typical values from the above Notation Table.	1.0pt

Part B: Main dynamics [6.0 points]

Now we will study the detailed dynamics of a spherical bubble consisting of a mixture of air and vapour. Please assume that there is no air migration through the bubble wall, and hence that the whole dynamics is governed by pressure only. Note however that as we have mentioned, there will be evaporation and condensation of water vapour at the bubble wall, that maintains the vapour pressure p_v within the bubble.

B.1	Suppose that a single spherical bubble resides within water that fills space uniformly, and that the bubble may evolve in size without distorting its spherical shape, due to changes, e.g., in the external pressure p_∞ . Derive an equation that relates the bubble radius $R(t)$ and its time derivatives $R'(t)$ and $R''(t)$, surface tension σ , water density ρ , the pressure far from the bubble p_∞ , and the pressure inside the bubble p. Then split the pressure p into two terms, by assuming that the bubble has both vapour (with partial pressure p_v) and air in it, and that the air follows an adiabatic process with exponent γ. To give a reference point, the partial air pressure must be q_0 when the bubble size equals R_0. Assume that evaporation, condensation, or transfer of air between the bubble cavity and the surrounding water has no effect on the water volume.	1.5pt

B.2	A water tank under the external pressure $p_\infty^- = 101$ kPa, containing a nucleus of radius $R_0 = 10^{-5}$ m initially in equilibrium, was exposed to vacuum, so that the system suddenly has $p_\infty = 0$. Estimate the terminal (asymptotic) value of the growth speed R' , as well as the time it reaches this terminal value.	1.0pt

B.3	A water tank under the external pressure $p_{\infty}^{-} = 1.600$ kPa, containing a gas bubble of radius $R_0 = 10^{-5}$ m initially in equilibrium, was suddenly exposed to the atmospheric pressure $p_{\infty} = 101$ kPa . Estimate the minimum radius of the bubble before it rebounds.	1.0pt

B.4	If there is no gas other than water vapour present in a bubble, the bubble completely collapses in finite time. Determine the characteristic exponent α in $R(t) \sim (T-t)^{\alpha}$, where T is the collapse time.	0.5pt

B.5	Based on the equation derived in B3, find the natural frequency of the spherical oscillation of a bubble of radius $R_0 = 0.1$ mm.	1.0pt

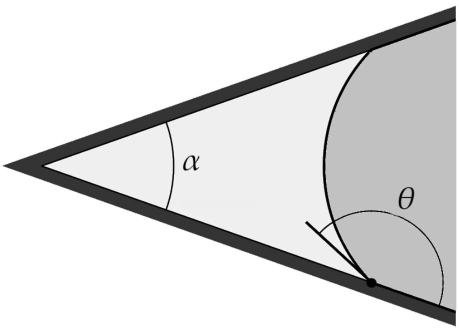
B.6	Suppose that the bubble described in the previous part is subjected to a standing sound wave along the x-axis, whose pressure field is given by $p(x,t) = p_0 + A\sin\left(\frac{2\pi f}{c}(x+a)\right)\sin(2\pi f t)$, where f is the frequency, and c is the speed of sound. The parameters p_0, A and a are constants, whose meanings may readily be deduced from the equation. Find the average force exerted upon the bubble. The bubble is situated at the origin of the xyz coordinate system, and its size is much smaller than the wavelength of the sound.	1.0pt

Part C: Dissolution of nuclei through diffusion [2.5 points]

In this final section, complementary to Part B, we focus on the effect of diffusion across the bubble wall.

C.1	Suppose that a nucleus consisting of air and vapour, with radius $R_0 = 10^{-5}$ m , is placed in water-air solution, in which the dissolved air is in equilibrium with the atmospheric pressure above the water. The partial pressure of air in the bubble is $q = 1.70 \cdot 10^{5}$ Pa, and the vapour pressure can be neglected. Estimate the time required for the bubble to be completely resorbed into water. The quantities $p_{\infty}, \kappa, \delta$ and σ take their typical values from Table 1. Assume that the region surrounding the bubble in which air diffusion takes place immediately gets much larger than the bubble itself.	2.0pt

C.2 Consider a conical crevice in the wall of a water container, with an aperture angle α, see the following Figure. A small amount of air and vapour reside within the cone. Write down the condition of mechanical and diffusive equilibrium. Determine when the pocket of air stays in the crevice without disappearing. The contact angle of water on the surface is θ. **0.5pt**



Conical Crevice