ISS Orbital Decay Dynamics

(Gurjav Ganbold)

The International Space Station (ISS) is the largest modular space station in low Earth orbit. The station serves as a microgravity and space environment research laboratory in which scientific research is conducted in astrobiology, astronomy, meteorology, physics, and other fields. The ISS is suited for testing the spacecraft systems and equipment required for possible future long-duration missions to the Moon and Mars. An international partnership of five space agencies from 15 countries operates ISS.

Figure 1: The International Space Station orbiting above the Earth.

The ISS is currently maintained in a nearly circular orbit with a minimum mean altitude of 370 km and a maximum of 460 km, in the centre of the thermosphere, at an inclination of $\theta = 51.6^0$ (degrees) to Earth's equator. The trajectory of the spacecraft is similar to a spiral with a slowly changing distance from the station to the Earth's surface, and during one cycle of revolution this distance changes inconsiderable.

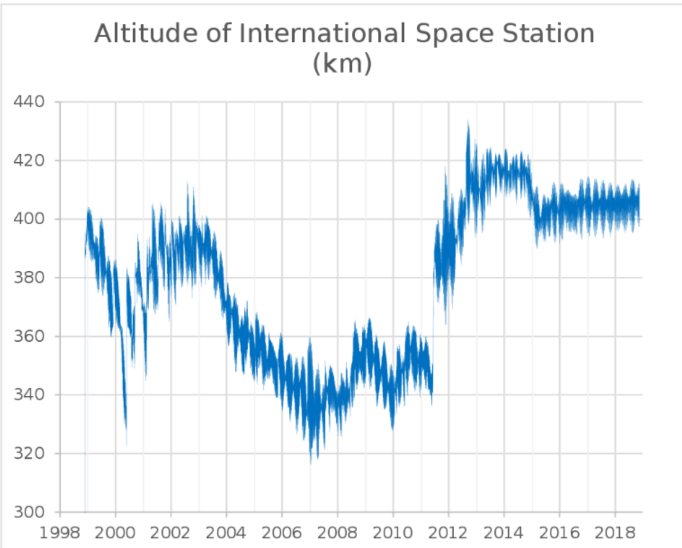


Figure 2: The altitude of ISS (km) over the years.

”In May 2008, the altitude was 350 kilometers, the ISS lost 4.5 km and was re-boosted by the Progess-60 supply ship by 5.5 km. Again, in June, the ISS continued to lose altitude by 5.5 km.” [*https://mod.jsc.nasa.gov/*] .

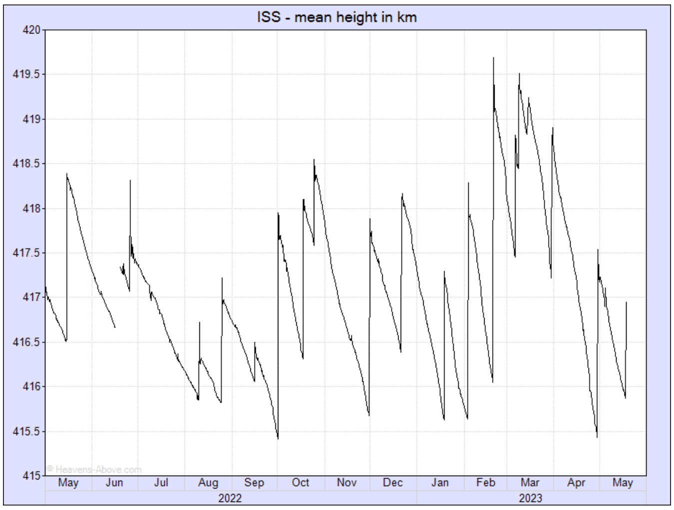


Figure 3: The ISS mean height (km) in 2022-2023.

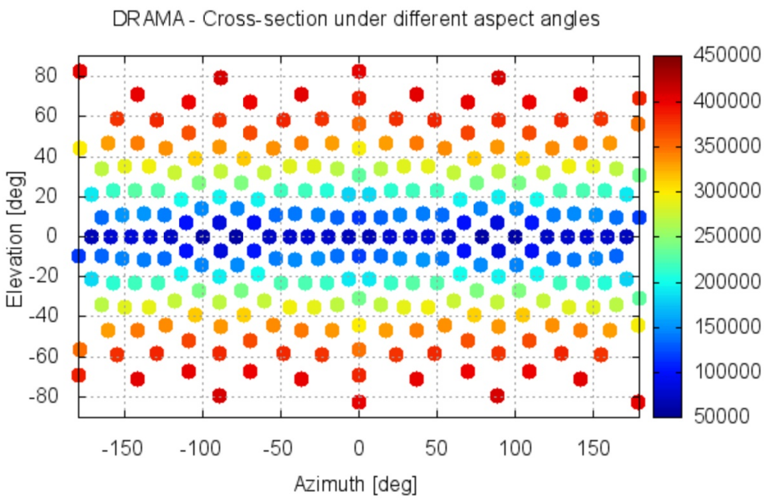


Figure 4: ISS model with the cross sections from different aspect angles ($\mathrm{dm}^2 = 10^{-2}\mathrm{m}^2$). The DRAMA CROC provides $2481\mathrm{m}^2$ cross section.

"The ISS loses up to 330 ft (100 m) of altitude each day." [*NASA Control Data (2021)*].

In 2023 the ISS flies at altitudes of 410 km, with an orbital decay about 70 m every day ($\sim$ 2 km per month), and during magnetic storms the daily descent reaches 300 m. The ISS accomplishes the de-orbit maneuvers by using the propulsion capabilities of the ISS and its visiting vehicles [*International Space Station Transition Report (2022)*].

The ISS mass is $M_S = 4.5 \times 10^5$ kg and overall length is $L_S = 109$ m. Huge solar panels with a width of $W_S = 73$ m provide the ISS with electrical energy [*NASA Official Report (2023)*].

Including all batteries and other parts, the effective cross area (section) of the station is approximately $S \approx 2.5 \times 10^3\,\mathrm{m}^2$ [*European Space Agency, SDC6-23*].

The ISS orbital decay is caused by one or more mechanisms which absorb energy from the orbital motion, the essential ones being:

- atmospheric drag at orbital altitude is caused by frequent collisions of gas molecules with the satellite,

- the Ampere force arising from the motion of the conductive apparatus in the Earth's magnetic field,

- the interaction with the atomic oxygen ions.

Denotations and Physical constants:

R - Universal gas constant ($8.31\,\mathrm{J}\cdot\mathrm{K}^{-1}\cdot\mathrm{mol}^{-1}$)
N_A - Avogadro's number ($6.022\cdot 10^{23}\,\mathrm{mol}^{-1}$)
μ - The molar mass of gas (for air: $0.029\,\mathrm{kg}\cdot\mathrm{mol}^{-1}$, for O_2: $0.032\,\mathrm{kg}\cdot\mathrm{mol}^{-1}$)
M_E - Mass of the Earth ($5.97\cdot 10^{24}\,\mathrm{kg}$)
R_E - Radius of the Earth ($6.38\cdot 10^{6}\,\mathrm{m}$)
G - Gravitational universal constant ($6.67\cdot 10^{-11}\,\mathrm{m}^3\cdot\mathrm{s}^{-2}\cdot\mathrm{kg}^{-1}$)
ρ_0 - Density of air at Earth's surface ($1.29\,\mathrm{kg/m}^3$)
g_0 - Gravitational acceleration at Earth's surface ($9.81\,\mathrm{m}\cdot\mathrm{s}^{-2}$)
B - Average magnitude of Earth's magnetic field ($5.0\cdot 10^{-5}\,\mathrm{T}$)
e - The electron absolute charge ($1.60\cdot 10^{-19}\,\mathrm{Q}$)

A. Modified barometric formula

The pressure of atmospheric air, composed mainly of neutral O_2 and N_2 molecules, can be found by using the Clapeyron-Mendeleev law:

$$pV = \frac{M}{\mu}RT\,. \tag{1}$$

where p, V, T, M and μ are the pressure, volume, temperature, mass and molar mass of a portion of air, R is the ideal gas universal constant.

There are two equations for computing air pressure as a function of height. The first equation is applicable to the standard model of the **troposphere** ($h < 100$ km) in which the temperature is assumed to vary with altitude at a lapse rate.

The second equation belongs to the standard model of the **thermosphere** ($h > 250$ km) in which the temperature is assumed not to change considerably with altitude and is applicable to ISS.

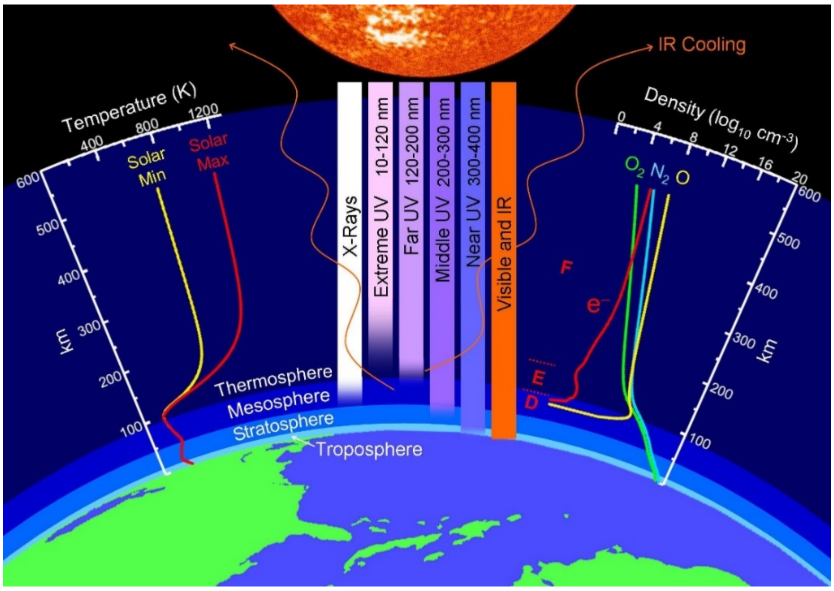


Figure 5: The Earth's thermosphere.

We may assume that all pressure is hydrostatic (i.e., it acts with equal magnitude in all directions).

Then, a perturbation of the air pressure dp_h on a variation of attitude dh may be written:

$$dp_h \doteq p_{h+dh} - p_h = -g_h\,(M/V)\,dh \qquad (2)$$

and dividing the dp_h by the p_h expressed from the Clapeyron-Mendeleev law we obtain

$$\frac{dp_h}{p_h} = -\frac{g_h \mu}{R T_h} dh\,. \qquad (3)$$

Integrating this expression from the surface $h = 0$ to the altitude h we get the air pressure as follows:

$$p_h = p_0 \exp\left(-\frac{\mu}{R}\int_0^h dh\,\frac{g_h}{T_h}\right)\,, \qquad (4)$$

where p_0 is the air pressure at altitude $h = 0$.

Remark 1. The temperature of Earth's thermosphere at altitude 300 - 600 km does not change considerably (see Fig.3) and reaches averagely about 800 - 900 K at solar side [*NASA data*]. Therefore, one may put $T_h = T = const$ by investigating the ISS orbital flight. Particularly, since the spacecraft spends almost half of its flight time in the shadow side of the Earth, where the temperature drops sharply, we may take the value of $\mathbf{T = 425\ K}$ as the average temperature at these altitudes. This temperature is also in agreement with the air density value $\rho_h \sim 10^{-12}\mathrm{m}^{-3}$ [MSISE-90 Model of Earth's Upper Atmosphere] at $h = 400$ km.

Further, by accepting an approximation $g_h = g_0$ one obtains the **standard barometric formula** as follows:

$$p_h^{sta} = p_0 \exp\left(-\frac{h}{h_0}\right)\,, \qquad h_0 \doteq \frac{RT}{\mu\, g_0}\,. \qquad (5)$$

We fix the parameter h_0 as follows:

$$h_0 \doteq \frac{RT}{\mu g_0} = \frac{8.31\ \mathrm{J\ K^{-1}\cdot mol^{-1}425K}}{0.029\ \mathrm{kg\cdot mol^{-1}9.81\ m\cdot s^{-2}}} \approx 12400\ \mathrm{m}\,. \qquad (6)$$

Remark 2. The integral in Eq.(4) may be calculated by taking into account the dependence of g_h on h in the leading-order correction, with accuracy $O(z_h^2)$.

In the leading-order approximation one gets:

$$g_h \simeq g_0\,(1 - 2\,z_h)\,, \qquad \int_0^h dh\, g_h \simeq g_0\, h\,(1 - z_h)\,. \qquad (7)$$

Then, we obtain a **improved barometric formula**

$$p_h^{imp} = p_0 \exp\left(-\frac{h\,(1 - z_h)}{h_0}\right)\,. \qquad (8)$$

Let us estimate the ratio of the 'standard' and 'improved' versions of the barometric formula:

$$\frac{p_h^{imp}}{p_h^{sta}} = \frac{\exp\left(-\frac{h\,(1-z_h)}{h_0}\right)}{\exp\left(-\frac{h}{h_0}\right)} = e^{\frac{h^2}{h_0 R_E}} \approx 7.54 \quad \text{for} \quad h = 4.0\times 10^5\ \mathrm{m}\,. \qquad (9)$$

The gas density rises by almost eight times when the weakening of gravity at ISS altitude is taken into account in the leading order.

Therefore, to avoid significant error in calculation for the ISS, when the air pressure or air density is involved, one should use the improved barometric formula in Eq.(8) instead of Eq.(5).

According to Eq.(8), the air density at height h may be expressed by the formula

$$\rho_h \doteq \frac{M}{V} = \rho_0 \exp(-h\,(1 - z_h)/h_0)\,. \tag{10}$$

The concentration of neutral air molecules at altitude is expressed through a similar law

$$n_h = N_A \frac{\rho_0}{\mu} \exp(-h\,(1 - z_h)/h_0)\,. \tag{11}$$

B. Orbital deceleration and station descent rate

Let us consider the problem of determining the rate of orbital decay of a satellite with mass M_S that experiences friction force $\vec{F}_{drag}$ acting against its velocity $\vec{v}$ during the time dt. We assume that the decrease in altitude dh is much less than the flight altitude h itself ($dh \ll h$).

The satellite's velocity may be found from its equation of motion in orbit (Newton's second law) where the Earth's gravitational force is balanced by the centrifugal force:

$$g_h = \frac{v_h^2}{R_E(1 + z_h)}\,, \qquad g_h \doteq \frac{g_0}{(1 + z_h)^2}\,. \tag{12}$$

The solutions read

$$v_h = \sqrt{\frac{g_0\,R_E}{1 + z_h}}\,, \qquad \tau_h \doteq 2\pi \frac{R_E + h}{v_h} = 2\pi \sqrt{\frac{R_E}{g_0}}(1 + z_h)^{3/2}\,. \tag{13}$$

By the conservation of mechanical energy, the total energy of a satellite moving along an almost circular orbit with radius $R_E + h$ is the sum of kinetic and gravitational potential energies, in an unperturbed two-body orbit:

$$E_S = \frac{M_S \cdot v_h^2}{2} - M_S\,g_h\,R_E\,(1 + z_h) = -\frac{M_S\,g_0\,R_E}{2(1 + z_h)}\,. \tag{14}$$

The total decelerating force exerted on a satellite of constant mass is given by some external braking force $\vec{F}_{drag}$. The rate of loss of orbital energy dE_S is simply the rate at the external force does negative work dA_{drag} on the satellite as the satellite traverses an infinitesimal circular arc-length $dL = vdt$:

$$dA_{drag} = -F_{drag} \cdot v_h \cdot dt\,. \tag{15}$$

The perturbation dE_S of the orbital energy at a change of the radius dh reads:

$$dE_S = +\frac{M_S\,g_0}{2(1 + z_h)^2} dh\,. \tag{16}$$

The total energy conservation $dE_S + dA_{drag} = 0$ leads to the equation

$$\frac{M_S\,g_0}{2(1 + z_h)^2} dh = F_{drag} \cdot v_h \cdot dt\,. \tag{17}$$

Then, we can find the rate of descent speed of the satellite as follows:

$$u_h \doteq \frac{dh}{dt} = \frac{2F_{drag}}{M_S\, g_0}\, v_h\, (1+z_h)^2 = \frac{2F_{drag}}{M_S}\sqrt{\frac{R_E}{g_0}}\,(1+z_h)^{3/2}\,. \tag{18}$$

The de-orbiting speed depends on the friction force, and on the altitude of the satellite, and on the mass of the satellite.

The friction force $\vec{F}_{drag}$ itself, in turn, depends on the flight altitude, on the effective cross section of the satellite S, and on the composition of the space environment at the satellite's flight altitude h.

The descent rate H_h for a revolution around the Earth reads:

$$H_h \doteq u_h\, \tau_h = \frac{4\pi\, R_E}{M_S\, g_0} F_{drag}(h)\cdot(1+z_h)^3\,. \tag{19}$$

The differential equation in Eq.(18) may be integrated out. Then, the total time T_h for which the satellite will fall from the attitude h to the earth's surface due to the friction may be found from the relation:

$$T_h \doteq \int_0^{T_h} dt = \frac{M_S}{2}\sqrt{\frac{g_0}{R_E}}\int_0^h dh \frac{1}{F_{drag}(h)\cdot(1+z_h)^{3/2}}\,. \tag{20}$$

C. Atmospheric drag

The speed of the satellite v is many times greater than the average velocities (hundreds m/s) of the thermal motion of atmospheric molecules at a height $h \approx 300-400$ km, so we can assume that the molecules were at rest before the collision with the ISS. To roughly estimate the drag force, we assume that after the collision the molecules acquire the same speed as the satellite. In this case, the air drag force can be estimated as follows:

$$F_{air} = n_h m_{air}\cdot v_h^2\cdot S = \frac{N_{air} m_{air}}{V}\cdot v_h^2\cdot S = \rho_h\cdot v_h^2\cdot S\,. \tag{21}$$

By substituting this expression into the formula in Eq.(18), we obtain

$$u_h^{air} = \frac{2\rho_0 S\sqrt{g_0\, R_E^3}}{M_S}(1+z_h)^{1/2}\cdot\exp(-h\,(1-z_h)/h_0)\,. \tag{22}$$

The descent rate H_h^{air} for a revolution around the Earth reads:

$$H_h^{air} \doteq u_h^{air}\,\tau_h = \frac{4\pi S\, R_E^2}{M_S}\rho_0\cdot(1+z_h)^2\cdot\exp(-h\,(1-z_h)/h_0)\,. \tag{23}$$

To find the total time T_h^{air} for which the satellite will fall to the earth's surface, we use Eq.(20). We obtain:

$$T_h^{air} \simeq \frac{M_S}{2\rho_0\, S\sqrt{g_0\, R_E^3}}\int_0^h dh\left(1-\frac{h}{2R_E}\right)e^{+h/h_0} \approx \frac{M_S\, h_0}{2\rho_0 S\sqrt{g_0\, R_E^3}}(1-\frac{h}{2R_E})\cdot e^{+h/h_0}\,, \tag{24}$$

where we took into account relations $h_0 \ll h \ll R_E$.

D. Drag by atomic oxygen ions

In the thermosphere, under the influence of ultraviolet and X-ray solar radiation and cosmic radiation, air ionization occurs ("polar lights"). Unlike O_2, N_2 does not undergo strong dissociation under the action of solar radiation, therefore, in general, there is much less atomic nitrogen N in the Earth's upper atmosphere than atomic oxygen. At altitudes above 250 km, atomic oxygen O predominates. Layers containing electrons and ions of oxygen atoms appear on the day side of the atmosphere. In this case, the concentration of atomic oxygen ions reaches $n_{ion} \sim 10^{13}\ m^{-3}$.

The decelerating force associated with the mechanical collisions of these particles on the satellite can be calculated using the formula in Eq.(21) but taking into account the strong decrease in ionization at night. Let the average value of the ion concentration be half the maximum value. Then we have

$$F_{ion} = \frac{1}{2}\rho_{ion} \cdot S \cdot v_h^2\,, \tag{25}$$

where

$$\rho_{ion} = \frac{\mu_{ion}}{N_A} \cdot n_{ion}\,. \tag{26}$$

Therefore, the speed of fall of the satellite due to deceleration by ions of atomic oxygen may be roughly estimated as follows:

$$u_h^{ion} = \rho_{ion} \cdot \frac{S\sqrt{g_0\,R_E^3}}{M_S}(1+z_h)^{1/2}\,. \tag{27}$$

The descent rate H_h^{ion} for a revolution around the Earth reads:

$$H_h^{ion} \doteq u_h^{ion}\,\tau_h = \rho_{ion}\frac{2\pi S\,R_E^2}{M_S} \cdot (1+z_h)^2\,. \tag{28}$$

E. Drag by the Earth's magnetic field

We consider the influence on the motion of the satellite of the Earth's magnetic field, the value of which near the Earth's surface is equal to $(3.5-6.5)\cdot 10^{-5}T$ with an average value of $B = 5\cdot 10^{-5}T$.

When a satellite moves at high speed in a magnetic field, an inducted electric current (electro-motive force, EMF) occurs in the current-conducting elements of the satellite's structure. This electromotive force causes a redistribution of electric charges in the current-conducting elements of the satellite structure. An electric field appears around the satellite, which affects on the movement of electrically charged particles in the environment. Electrons are attracted to those parts of the satellite that have a positive potential (relative to the middle part of the satellite), and positively charged ions are attracted to those parts of the satellite that have a negative potential. Electrons and ions that hit the surface of the satellite structures are combined into neutral oxygen atoms, while the electrons 'travel' in the satellite's conductive structures, creating an electric current. The satellite, moving in space, 'collects' electrons and ions from the surrounding space and collides with them. For a rough estimate of the magnitude of the current that can flow through the conductive structures of the satellite, we will assume that the collection occurs only from an area equal to the cross-sectional area S of the ISS, and all ions and electrons participate in the creation of this current.

The number of electrons hitting the structure of the ISS body during the short time interval dt is

$$dN = n_{ion} \cdot v_h \cdot S \cdot dt\,. \tag{29}$$

Therefore, the magnitude of the current is of the order

$$I_{ind} \approx e\frac{dN}{dt} = e \cdot S \cdot n_{ion} \cdot \sqrt{\frac{g_0\, R_E}{1+z_h}}\,. \tag{30}$$

The orbital 'braking' Ampere's force is proportional to $[\vec{v}_h \times \vec{B}] = v_h B|\sin(\phi)|$, where ϕ is the angle between the Earth's magnetic field $\vec{B}$ and the velocity of the ISS $\vec{v}_h$. Hereby, $\theta = 51.6^0$ (degrees) is the inclination angle of the ISS orbit to Earth's equator.

Let us consider a revolution starting from the 'north' sample point in the ISS orbit with the highest latitude ($\phi = \pi/2 - \theta$). After a half revolution the ISS arrives at the 'south' point with the lowest latitude ($\phi = \pi/2 + \theta$). The second part of the revolution cycle ends at the 'north point'.

The averaging of the value $|\sin(\phi)|$ during a revolution period may be performed as follows:

$$\langle|\sin(\phi)|\rangle = \frac{1}{2\theta}\int\limits_{\pi/2-\theta}^{\pi/2+\theta} d\phi\,|\sin(\phi)| = 0.93 \approx 1\,. \tag{31}$$

An approximate result may be obtained by using four equidistant sample positions in the ISS orbit as follows:

$$\langle|\sin(\phi)|\rangle = \{\sin(\pi/2-\theta) + \sin(\pi/2) + \sin(\pi/2+\theta) + \sin(\pi/2)\}\,/4 = 0.89 \approx 1\,. \tag{32}$$

Further, we will use an approximation $\langle|\sin(\phi)|\rangle \approx 1$.

When the induced current flows through the conductive parts of the satellite, they are affected by the 'braking' Ampere force directed opposite to the direction of the satellite's speed:

$$F_{ind} = B \cdot I_{ind} \cdot \langle|\sin(\phi)|\rangle \cdot L \approx B \cdot I_{ind} \cdot \sqrt{S} = e \cdot B \cdot S^{3/2} \cdot n_{ion} \cdot \sqrt{\frac{g_0\, R_E}{1+z_h}}\,, \tag{33}$$

where for the external linear size of the station, we can use the approximation $L \sim S^{1/2}$.

Then for the rate of descent of the satellite we obtain

$$u_h^{ind} \approx 2n_{ion}\frac{eBS^{3/2}R_E}{M_S} \cdot (1+z_h)\,. \tag{34}$$

The descent rate H_h^{ind} for a revolution around the Earth reads:

$$H_h^{ind} \doteq u_h^{ind}\,\tau_h = \frac{4\pi\, e\, B\,(S\, R_E)^{3/2}}{M_S\sqrt{g_0}} \cdot (1+z_h)^{5/2}\,. \tag{35}$$

F. Numerical results and conclusion

Table 1: Various deorbit velocities on the height h above the Earth surface, compared to the ISS-NASA data estimated for $n_{ion} = 10^{13} m^{-3}$. For $n_{ion} = 10^{12} m^{-3}$ the results for u_{ion} and u_{ind} will decrease in 10 times.

h [km]	T_h^{air} [day]	$u_{air} [m/day]$	$u_{ion} [m/day]$	$u_{ind} [m/day]$	$\sum [m/day]$	$w_{ISS} [m/day]$
350	316	184	14	28	226	~ 170 [in 2008]
375	2360	30.9	14	29	73	-
400	17700	5.3	14	29	47	≤ 100 [in 2021]
410	39500	2.6	14	29	45	≤ 70) [in 2022]

Table 2: The descent rates for a revolution of the ISS around the Earth for $n_{ion} = 10^{13} m^{-3}$. For $n_{ion} = 10^{12} m^{-3}$ the results for H_h^{ion} and H_H^{ind} will decrease in 10 times.

h (km)	$H_h^{air} [m]$	$H_h^{ion} [m]$	$H_h^{ind} [m]$
350	11.7	0.9	1.8
375	2.0	0.9	1.8
400	0.3	0.9	1.8
410	0.2	0.9	1.8

For the International Space Station, orbiting at an altitude above 380 km, the most significant factors ensuring orbital decay are ranked as follows:

1) the Ampere force arising from the motion of the conductive apparatus in the Earth's magnetic field.

2) Collisions of the station with ionized atoms of oxygen.

3) The atmospheric drag caused by frequent collisions of neutral O_2 molecules.

Theoretical Problem 1: ISS Orbital Decay Dynamics

Q1 - Marking scheme

Question part	Total marks	Partial marks	Explanation for partial marks and special cases
A.1	**0.5**	**0.1** **0.1** **0.3**	Perturbation of air pressure correct Differential formula is correct Integral formula is correct (final answer)
A.2	**0.3**	**0.1** **0.2**	correct standard barometric formula correct calculation of h_0
A.3	**0.6**	**0.1** **0.2** **0.3**	Correct dependence of g_{h+} correct integration correct improved barometric formula
A.4	**0.4**	**0.2** **0.2**	correct analytic formula correct numerical values
A.5	**0.2**	**0.1** **0.1**	Correct air density Correct air concentration
B.1	**0.5**	**0.1** **0.1** **0.3**	Gravitation g_h Satellite velocity v_h Correct revolution period
B.2	**0.5**	**0.2** **0.3**	Kinetic and potential energy Total energy
B.3	**1.0**	**0.3** **0.2** **0.5**	Negative work by air drag force Perturbation of total energy Correct conservation formula
B.4	**0.5**	**0.1** **0.4**	Formula of the speed as dh/dt Correct formula for descent speed
B.5	**0.5**	**0.1** **0.4**	Descent rate per a revolution Total falling time written as an integral
C.1	**0.5**	**0.3** **0.1** **0.1**	Air drag force Air drag descending velocity Air drag descent rate per a revolution
C.2	**0.5**	**0.1** **0.1** **0.3**	Correct integral formula for the air descent rate Use of approximations taking into account $h_0 << h << R_E$ Correct total falling time in final integral form
D.1	**0.3**	**0.2** **0.1**	Drag force by oxygen ions Density of ionized oxygen molecules
D.2	**0.7**	**0.3** **0.4**	Descent velocity due to atomic oxygen Descent rate H_h^{ion}
E.1	**0.6**	**0.3** **0.3**	Number of electrons collected by the IS for time dt Magnitude of the inducted electric current
E.2	**0.6**	**0.2** **0.2** **0.2**	Averaged sine of the angle between magnetic field and ISS velocity Formula of Ampere's force Correct final answer
E.3	**0.8**	**0.3** **0.5**	Descent velocity due to Earth's magnetic field Descent rate H_h^{ind}

F.1	**0.4**	**0.02** each filled box	
F.2	**0.4**	**0.1** **0.1** **0.1** **0.1**	Value H_h^{air} at h=350km Value H_h^{air} at h=3750km Value H_h^{air} at h=400km Value H_h^{air} at h=410km
F.3	**0.2**	**0.2**	If answered: 1) Ampere 2) Oxygen ions 3) Air drag

A ball on a turntable

Enkhbat Tsedenbaljir

1 Problem 1: 10 points

1.1 Preamble

Notations and conventions: The length of a vector $\vec{A}$ is simply denoted as $A \equiv |\vec{A}|$. The time derivative of a quantity is denoted by the dot over the quantity: $\dot{\vec{A}} \equiv d\vec{A}/dt$, $\dot{A} \equiv dA/dt$. The unit vector along the direction of vector $\vec{A}$ is denoted as $\hat{A}$. The unit vectors along the Cartesian coordinates are, therefore, $\hat{x}$, $\hat{y}$ and $\hat{z}$. The definitions of scalar and vector products are:

$$
\begin{aligned}
(\vec{A}\cdot\vec{B}) &= (\vec{B}\cdot\vec{A}) = A_xB_x + A_yB_y + A_zB_z = AB\cos\theta, & (1)\\
(\vec{A}\times\vec{B}) &= -(\vec{B}\times\vec{A}) & (2)\\
&= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_x - A_yB_x)\hat{z}, & (3)\\
|\vec{A}\times\vec{B}| &= AB\sin\theta, & (4)
\end{aligned}
$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. You may need the following properties of vectors and their multiplications: Scalar products of vectors vectors:

$$(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{A})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}, \quad (5)$$

$$(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A} = (\vec{C}\times\vec{A})\cdot\vec{B}. \quad (6)$$

Triple product rules for vectors:

$$(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}, \quad (7)$$

$$(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{B}\times\vec{C})\cdot\vec{A} = (\vec{C}\times\vec{A})\cdot\vec{B}. \quad (8)$$

The vector products are very useful in describing many relations in physics. For example:

$$\vec{v} = \vec{\omega}\times\vec{r}, \quad (9)$$

$$\vec{F}_{Lorentz} = Q\vec{B}\times\vec{v}, \quad (10)$$

and, often, saves time combining three equations for vector components into a single equation.

1.2 The statement

A ball of mass m and radius r is rolling on a horizontal turntable without slipping. Its mass density has a spherical symmetry, i.e. only depends on the distance from its center. In part B and C, where the turntable can rotate freely, the moment of inertia of the turntable is denoted as I_d. The purpose of the problem is to analyze the motion and trajectory of the ball with respect to an observer at rest. Throughout the problem, assume the turntable is large enough so that the ball does not fall off. The following notations are used:

Ω – the magnitude of the turntable angular velocity,

$\vec{\omega}$ – the spinning angular velocity of the ball with respect to its spinning axis,

$\vec{R}$ – the horizontal position of the ball center with respect to the rotation axis of the turn table,

$\vec{v}$ – the velocity of the ball at $\vec{R}$.

Assume that the initial position $\vec{R}_0 \equiv \vec{R}(0)$ and velocity $\vec{v}_0 \equiv \vec{v}(0)$ of the ball, the angular velocity of the turn table $\Omega_0 \equiv \Omega(0)$ are known. For the initial vector quantities $\vec{R}_0 \equiv \vec{R}(0)$ and $\vec{v}_0 \equiv \vec{v}(0)$, assume that their directions are known. In addition, whenever you need to express a vector quantity, you may use $\hat{z}$ in your expression. Also, if asked to write your expression in terms of the known quantity you may use any or all of m, r, I and I_d. Unless otherwise stated, keep I as general. The following notations are recommended:

$$\alpha = \frac{I}{I + mr^2}, \quad \delta = \frac{I_d}{mr^2}, \tag{11}$$

1.3 Part A: 2 points

First we start with the simplest case wherein the turntable angular velocity with respect to vertical axis $\hat{z}$ is constant, therefore $\Omega = \Omega_0$.

A1. 0.1 point

Express the ball's velocity $\vec{v}$ in terms of Ω, $\vec{\omega}$, r, m, I and $\vec{R}$ from a kinematic constraint.

A.2 0.2 point

Using Newton's equation and torque equation with respect to its center, find the acceleration of the ball $\vec{a} \equiv \dot{\vec{v}}$ in terms of Ω, $\vec{v}$, r, m and I.

A.3 0.2 points

Find the velocity $\vec{v}$ in terms ofΩ, $\vec{R}$, $\vec{v}_0$, $\vec{R}_0$, r, m and I.

A.4 0.5 points

Find the trajectory of the ball. It means, for the given initial conditions $\vec{v}_0$ and $\vec{R}_0$, completely specify the trajectory.

A.5 1 point

Assume this time that the ball has a uniform mass density, i.e. $I = 2mr^2/5$. Trajectory you have found has a single defining parameter R_t for its size. Choose its magnitude to be the same as R_0. How long does it take for the ball to approach the initial spot on the table (the position on the turntable at $t = 0$) with the closest distance?

1.4 Part B

In this part, the turntable can rotate freely, without any friction, around z–axis. Therefore its free rotation is hindered only by the ball's friction.

B.1 0.2 points

Find the velocity $\vec{v}$ and acceleration $\dot{\vec{v}}$ of the ball in terms of Ω, $\vec{R}$, Ω_0, $\vec{R}_0$, $\dot{\Omega}$, r, m and I.

B.2 0.2 points

Find the magnitude of the angular acceleration of the turntable $\dot{\Omega}$ in terms of Ω, Ω_0, $\vec{R}$, $\vec{R}_0$, $\vec{v}_0$, r, m, I and I_d . You may use the constants α and δ defined in the beginning of the problem.

B.3 0.4 points

Find the magnitude of the angular velocity of the turntable Ω as a function of R only, namely, in terms of Ω_0, R, R_0, r, m, I and I_d.

B.4 0.1 points

From the result of B.3, for a given Ω_0, R_0, find the maximum possible Ω.

B.5 3.1 points

Write down the vertical component the angular momentum $\hat{z}M_z$ of the whole system. Subtract any constant term and rename the remaining part as $\hat{z}L$.

In part B.1 you found the velocity of the ball $\vec{v}$, which can be written as the sum of a part that depends on the position of the ball $\vec{R}$ and a constant vector. Let us call this constant vector $\vec{c}$. Choose the direction of x-axis along this vector and y-axis along $\hat{z} \times \vec{c}$. In this frame of reference, find Ω in terms of L, $\vec{R}$, $\vec{c}$, $\hat{z}$, R^2, r, m, I and I_d. Combining this with the result of B.3, write down an equation only containing R^2 and y variables and L, r, m, I, c and I_d. Here c is the magnitude of $\vec{c}$. Substituting $R^2 = x^2 + y^2$, write down an expression containing only x and y variables and describing a curve. From this, list all possible types of trajectories.

1.5 Part C: 4 points

In this part, we consider a density profile so that $I = mr^2/10$. This can be realized, for example, if the ball is filled up to its half radius with uniform density and the remaining part has a negligible mass. In addition, on its outer surface, the ball has a uniform charge density $Q/(4\pi r^2)$, where Q is the total surface charge. The whole setup is in a uniform magnetic field $\vec{B}$ that is in $\hat{z}$ direction. The turntable rotates with constant Ω like in Part A.

It is often useful to analyze the equations governing the evolution of a system in a unitless form so that the general behavior can be studied without worrying about a specific values or units. For this purpose, we divide the $\vec{R}$ and Ω by 1 meter and 1 Hertz respectively. Also we divide the time variable by 1 second.

C.1 0.3 points

Write down Newton's equation and the torque equation for the ball. Find expression for the torque $\vec{\tau}_s$ due to the spinning of the ball around its axis in terms of Q, r, $\vec{\omega}$ and $\vec{B}$.

C.2 0.2 points

Using the results of C.1, find expression for the linear acceleration of the ball with respect to the laboratory frame in terms of Q, r, $\vec{\omega}$ and $\vec{B}$.

C.3 0.3 points

The equation for the linear acceleration you found in part C.2 is a second order differential equation for $\vec{R}$ of the following form:

$$\frac{d^2\vec{R}}{dt^2} - \gamma \frac{d\vec{R}}{dt} \times \hat{z} + \beta \vec{R} = 0. \tag{12}$$

Write down γ and β constants. From now on we assume we have made the transformation to the unitless forms. This in turn, has an effect on the γ and β as factors of $1/s$ =Hz and $1/s^2$ respectively, rendering them unitless as well. Make the following transformation to a polar coordinates for the components of $\vec{R}$:

$$x(t) = \rho(t)\cos(\eta(t)), \tag{13}$$

$$y(t) = \rho(t)\sin(\eta(t)), \tag{14}$$

so that the new equations do not have the first time derivative term. Here the polar angle $\eta(t)$ is a function of time. Find the form of the form of this function. Express the coefficient β' of $\rho(t)$ in the new equation in terms of γ and β. Write down the conditions for different types of trajectories: harmonic, exponential etc.

C.4 1.5 points

Consider the following initial conditions for the solution found in part C.3:

$$x(0) = 1, \;\; y = 0, \;\; v_x(0) = \dot{x}|_{t=0} = 1, \;\; , v_y(0) = \dot{y}|_{t=0} = -1. \tag{15}$$

find γ and β. Using them find the corresponding Ω. Sketch the trajectory. Is the charge of the surface negative or positive? For the negative write − and for the positive write + on your answer sheet.

C.5 1.5 points

Consider the solution you have found in part C.4. If you identified it correctly your solution should have a rotating $\vec{R}(t)$. Find the expressions for the total and per rotation changes in energy for $N \gg 1$ number of rotations. Here you may ignore the terms small compared to N. In this part assume the mass and the radius of the ball are $m = 1$ and $r = 1$ so that $I = 1/11$ (in our unitless scheme we divide masses by 1 kg).

2 Solution

2.1 Part A

A.1

The velocity of the ball $\vec{v}_b$ with respect to the turntable from the non–slipping condition is given by:

$$\vec{v}_b = \vec{\omega} \times (r\hat{z}). \tag{16}$$

The ball velocity with respect to the Lab frame is then

$$\vec{v} \;\; = \;\; \Omega\hat{z} \times \vec{R} + \vec{v}_b, \vec{v} = \Omega\hat{z} \times \vec{R} + \vec{\omega} \times \hat{z}r. \tag{17}$$

A.2

The force $\vec{F}$ and torque $\vec{\tau}$ due to friction are:

$$\vec{F} = m\dot{\vec{v}}, \tag{18}$$

$$\vec{\tau} = (-r\hat{z}) \times \vec{F} = I\frac{d\vec{\omega}}{dt}. \tag{19}$$

The time derivative of equation (16) gives

$$\dot{\vec{v}} = \Omega\hat{z} \times \vec{v} + \frac{d\vec{\omega}}{dt} \times (r\hat{z}) \tag{20}$$

$$\tag{21}$$

and substituting Eq. (18) and (19) in results in:

$$\dot{\vec{v}} = \Omega\hat{z} \times \vec{v} - \frac{mr^2}{I}\left(\hat{z} \times \dot{\vec{v}}\right) \times \hat{z}. \tag{22}$$

Using the triple vector product rule in the last term of the above equation and keeping in mind that both $\vec{v}$ and $d\vec{v}/dt$ are orthogonal to $\hat{z}$ yields

$$\dot{\vec{v}} = \Omega\hat{z} \times \vec{v} - \frac{mr^2}{I}\dot{\vec{v}} \rightarrow \tag{23}$$

$$\dot{\vec{v}} = \frac{\Omega}{1 + mr^2/I}\hat{z} \times \vec{v}. \tag{24}$$

A.3

The last equation unequivocally shows that the motion of the ball is circular and the corresponding angular velocity of its center is $\frac{\Omega}{1+ma^2/I}$. Now we integrate this equation to find the radius and its center:

$$\vec{v} - \vec{v}_0 = \frac{\Omega}{1 + mr^2/I}\hat{z} \times (\vec{R} - \vec{R}_0), \tag{25}$$

$$\vec{v} = \frac{\Omega}{1 + mr^2/I}\hat{z} \times (\vec{R} - \vec{R}_0 - \frac{1 + mr^2/I}{\Omega}\hat{z} \times \vec{v}_0) \rightarrow \tag{26}$$

$$\vec{v} = \frac{\Omega}{1 + mr^2/I}\hat{z} \times (\vec{R} - \vec{R}_0) + \vec{v}_0 \tag{27}$$

A.4

From this we see that the circle trajectory has radius R_t and its center is located at

$$\vec{R}_c = \vec{R}_0 + \frac{1 + mr^2/I}{\Omega}\hat{z} \times \vec{v}_0. \tag{28}$$

$$R_t = |\vec{R}_0 - \vec{R}_c| = \frac{1 + mr^2/I}{\Omega}|\hat{z} \times \vec{v}| = \frac{1 + mr^2/I}{\Omega}v_0 \tag{29}$$

A.5

In the case of a solid ball of uniform density, the moment of inertia is

$$I = \frac{2mr^2}{5}, \tag{30}$$

and therefore the angular velocity of the ball's center is

$$\omega_c = \frac{2}{7}\Omega. \tag{31}$$

The time to return the initial point on the turntable is then

$$t = \frac{14\pi}{\Omega}. \tag{32}$$

This solution is true for most cases. But there are special cases where this time is shorter. Trajectory is a circle and its size is defined by its radius R_t and , as stated, we solve for $R_t = R_0$. It could happen that the red spot happens to cross path with the ball at a moment before the turntable could make a full circle. In this case we can find the distance between the starting and the crossing positions:

$$2R_0 \sin\left(\frac{\omega_c t}{2}\right) = 2R_t \sin\left(\frac{2\pi - \Omega t}{2}\right), \tag{33}$$

$$t = \frac{2\pi}{\omega_c + \Omega} = \frac{14\pi}{9\Omega}. \tag{34}$$

2.2 Part B

Now we examine the case wherein the turntable rotates freely, i.e. without friction, around vertical axis. In this case the total kinetic energy and the angular momentum are conserved.

B.1

Integrating the torque equation for the ball one gets:

$$\vec{\omega} \times \hat{z} = \vec{\omega}_0 \times \hat{z} - \frac{mr}{I}(\vec{v} - \vec{v}_0). \tag{35}$$

Substituting this into the non slipping condition we get

$$\vec{v} = \Omega\hat{z} \times \vec{R} + \vec{\omega}_0 \times \hat{z}r - \frac{mr}{I}(\vec{v} - \vec{v}_0), \tag{36}$$

$$\vec{v}_0 = \Omega_0\hat{z} \times \vec{R}_0 + \vec{\omega}(0) \times \hat{z}r, \tag{37}$$

which gives

$$\vec{v} = \frac{I}{I + mr^2}\hat{z} \times \left(\Omega\vec{R} - \Omega_0\vec{R}_0\right) + \vec{v}_0, \tag{38}$$

$$\dot{\vec{v}} = \frac{I}{I + mr^2}\hat{z} \times \left(\dot{\Omega}(t)\vec{R} + \Omega\vec{v}\right). \tag{39}$$

B.2

The torque equation for the turntable is:

$$I_d\dot{\Omega}\hat{z} = -m\vec{R} \times \dot{\vec{v}}. \tag{40}$$

If we substitute the velocity and accelaration in the above equation and use the triple vector product rule we get

$$I_d\dot{\Omega}\hat{z} = -m\vec{R} \times \left(\frac{I}{I + mr^2}\hat{z} \times \left(\dot{\Omega}\vec{R} + \Omega\vec{v}\right)\right)$$

$$I_d\dot{\Omega} = -\frac{mI}{I + mr^2}\left(\dot{\Omega}R^2 + \Omega\left(\vec{v} \cdot \vec{R}\right)\right) \tag{41}$$

$\vec{v} \cdot R$ can be obtained using equation 44 as:

$$\vec{v} \cdot \vec{R} = \left(\vec{v}_0 + \frac{I}{I + mr^2}\hat{z} \times \left(\Omega\vec{R} - \Omega_0\vec{R}_0\right)\right) \cdot \vec{R}, \tag{42}$$

$$= \left(\vec{v}_0 - \frac{I}{I + mr^2}\Omega_0\hat{z} \times \vec{R}_0\right) \cdot \vec{R}. \tag{43}$$

Applying this to the turntable torque equation (41), we obtain:

$$\left(I_d + \frac{mI}{I+mr^2}R^2\right)\dot{\Omega} \quad = \quad -\frac{mI}{I+mr^2}\Omega\left(\vec{v}_0 - \frac{I}{I+mr^2}\Omega_0\hat{z}\times\vec{R}_0\right)\cdot\vec{R}. \quad (44)$$

We may rewrite the equation into a simpler form as:

$$\dot{\Omega} = -\frac{\alpha\Omega/r^2\left(\vec{C}\cdot\vec{R}\right)}{\delta + \alpha R^2/r^2}, \quad (45)$$

where

$$\alpha \equiv \frac{I}{I+mr^2}, \quad (46)$$

$$\delta \equiv \frac{I_d}{mr^2}, \quad (47)$$

$$\vec{c} \equiv \vec{v}_0 - \alpha\Omega_0\hat{z}\times\vec{R}_0. \quad (48)$$

B.3

Observe that the velocity of the ball can be written as

$$\vec{v} \quad = \quad \alpha\Omega\hat{z}\times\vec{R} + \vec{c}, \quad (49)$$

and, therefore, using equation (42) we see that:

$$\vec{v}\cdot\vec{R} \quad = \quad \frac{1}{2}\frac{d(\vec{R}\cdot\vec{R})}{dt} = \frac{1}{2}\dot{R}^2 = \vec{R}\cdot\vec{c}. \quad (50)$$

Substituting this in equation (45) we get:

$$\frac{1}{\Omega}\frac{d\Omega}{dt} = -\frac{1}{2}\frac{1}{\delta + \alpha R^2/r^2}\frac{d\left(\alpha R^2/r^2\right)}{dt}. \quad (51)$$

The integration of this leads to:

$$\ln\left(\frac{\Omega}{\Omega_0}\right)^2 \quad = \quad \ln\left(\frac{\delta + \alpha R_0^2/r^2}{\delta + \alpha R^2/r^2}\right), \quad (52)$$

$$\Omega^2 \quad = \quad \Omega_0^2\frac{\delta + \alpha R_0^2/r^2}{\delta + \alpha R^2/r^2} \quad (53)$$

B.4 From this result we see that the maximum possible Ω is achieved when R^2, i.e when the ball crosses the center of the turntable:

$$\Omega_{max} \quad = \quad \Omega_0\sqrt{1 + \frac{\alpha R_0^2}{\delta r^2}} \quad (54)$$

B.5

Now we determine the trajectory of the ball. The total angular momentum along $\hat{z}$ is:

$$M_z\hat{z} = I_d\Omega\hat{z} + m\vec{R}\times\vec{v} + I\omega_z\hat{z}. \tag{55}$$

Since there is no torque along $\hat{z}$ acting on the ball ω_z is constant. So we define the following conserved quantity:

$$L\hat{z} = I_d\Omega\hat{z} + m\vec{R}\times\vec{v} = I_d\Omega_0\hat{z} + m\vec{R}_0\times\vec{v}_0. \tag{56}$$

The velocity of the ball $\vec{v}$ was written as the sum of a part that depends on the position of the ball $\vec{R}$ and a constant vector $\vec{c}$. Then, we have:

$$\vec{R}\times\vec{v} = \vec{R}\times\left(\alpha\Omega\hat{z}\times\vec{R}+\vec{c}\right) \tag{57}$$

$$= -\alpha\Omega R^2\hat{z} + \vec{R}\times\vec{c}. \tag{58}$$

Substituting this in equation (59) one gets:

$$L\hat{z} = I_d\Omega\hat{z} + \alpha\Omega m R^2\hat{z} + m\vec{R}\times\vec{c}, \tag{59}$$

$$\Omega = \frac{L - m\hat{z}\cdot\left(\vec{R}\times\vec{c}\right)}{I_d + \alpha m R^2} \tag{60}$$

Choosing the direction of x-axis along $\hat{c}$ and y-axis along $\hat{z}\times\hat{c}$,

$$\Omega = \frac{L/mr^2 + cy/r^2}{\delta + \alpha R^2/r^2}, \tag{61}$$

Combining this with equation (52) we have:

$$\Omega_0^2\frac{\delta + \alpha R_0^2/r^2}{\delta + \alpha R^2/r^2} = \left(\frac{L/mr^2 + cy/r^2}{\delta + \alpha R^2/r^2}\right)^2, \tag{62}$$

$$\Omega_0^2\left(\delta + \alpha R_0^2/r^2\right)\left(\delta + \alpha R^2/r^2\right) = \left(L/mr^2 + cy/r^2\right)^2. \tag{63}$$

Observe that this is the equation for conic section. Let us elaborate on this fact. Let us introduce the following constants:

$$k \equiv \Omega_0^2\left(\delta r^2 + \alpha R_0^2\right), \quad \lambda \equiv L/m. \tag{64}$$

Expanding in Cartesian coordinates $\vec{R} = x\hat{x} + y\hat{y}$, we obtain:

$$k\alpha\left(\delta r^2 + \alpha(x^2+y^2)\right) - \left(\lambda^2 + 2\lambda c y + c^2y^2\right) = 0, \tag{65}$$

$$k\alpha^2x^2 + (k\alpha^2 - c^2)y^2 - 2\lambda c y = \lambda^2 - k\alpha\delta r^2 \tag{66}$$

Since $k\alpha^2 > 0$, the trajectory is determined by the sign of $k\alpha^2 - c^2$:

$$\text{Ellipse if } k\alpha^2 > c^2. \tag{67}$$
$$\text{Parabola if } k\alpha^2 = c^2. \tag{68}$$
$$\text{Hyperbola if } k\alpha^2 < c^2. \tag{69}$$

2.3 Part C

C.1

Here it is given that $\Omega = const$. In addition, for the given mass distribution where the ball is filled up to half of its radius, the momentum of inertia becomes

$$I = \frac{mr^2}{10}. \tag{70}$$

. In the presence of vertical uniform magnetic field $\vec{B}$ and if the ball is charged with uniform surface density $\rho = Q/4\pi r^2$, the equation of motions are changed as follows:

$$m\dot{\vec{v}} = \vec{F}_f + Q\vec{v} \times \vec{B} \tag{71}$$
$$I\dot{\vec{\omega}} = -r\hat{z} \times \vec{F}_f + \vec{\tau}_s, \tag{72}$$

where $\tau_s = Qr^2\vec{\omega} \times \vec{B}/3$ is the torque due to spinning of the charged sphere and F_f is the friction force. Calculation of τ_s is essentially identical to the mechanical moment of inertia for thin spherical shell. The torque is calculated as:

$$\begin{aligned}
\vec{\tau}_s &= \int d\cos\theta d\phi \rho \vec{r} \times \left((\vec{\omega} \times \vec{r}) \times \vec{B} \right) && (73)\\
&= \int r^2 d\cos\theta d\phi \rho \left(\vec{\omega} \times \vec{r}\right) \left(\vec{r} \cdot \vec{B}\right) && (74)\\
&= \rho\vec{\omega} \times \int r^2 d\cos\theta d\phi \vec{r} \left(\vec{r} \cdot \vec{B}\right) && (75)\\
&= \rho B r^4 \vec{\omega} \times \int d\cos\theta d\phi \cos\theta \left(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}\right) && (76)\\
&= \rho B r^4 \vec{\omega} \times \hat{z} \int_{-1}^{1} d\cos\theta \, \cos^2\theta \int_0^{2\pi} d\phi && (77)\\
&= \frac{Qr^2}{3}\vec{\omega} \times \vec{B}. && (78)
\end{aligned}$$

C.2

In addition we have the non–slipping condition from which we get:

$$\vec{v} = \Omega\hat{z} \times \vec{R} + \omega \times \hat{z}r, \tag{79}$$
$$\dot{\vec{v}} = \Omega\hat{z} \times \vec{v} + \dot{\vec{\omega}} \times \hat{z}r \rightarrow \tag{80}$$
$$\dot{\vec{\omega}}r = \Omega\vec{v} - \dot{\vec{v}} \times \hat{z}. \tag{81}$$

Substituting these and F_f from the Newton's equation into the torque equation, one gets:

$$I\dot{\vec{\omega}} = -r\hat{z}\left(m\dot{\vec{v}} - Q\vec{v} \times \vec{B}\right) + \frac{Qr^2}{3}\vec{\omega} \times \vec{B} \tag{82}$$
$$I(\Omega\vec{v} + \hat{z} \times \dot{\vec{v}}) = -r^2\hat{z} \times \left(m\dot{\vec{v}} - Q\vec{v} \times \vec{B}\right) + \frac{Qr^2B}{3}\left(\vec{v} - \Omega\hat{z} \times \vec{R}\right) \tag{83}$$
$$\left(I + mr^2\right)\dot{\vec{v}} = \left(\frac{4Qr^2B}{3} - I\Omega\right)\vec{v} \times \hat{z} - \frac{Qr^2B}{3}\Omega\vec{R}. \tag{84}$$

The last equation maybe written as:

$$\frac{d^2\vec{R}}{dt^2} - \gamma\frac{d\vec{R}}{dt} \times \hat{z} + \beta\vec{R} = 0, \tag{85}$$

where

$$\beta \equiv \frac{Qr^2B}{3\left(I + mr^2\right)}, \tag{86}$$
$$\gamma \equiv \frac{4Qr^2B - 3I\Omega}{3\left(I + mr^2\right)} = \frac{4\beta}{\Omega} - \alpha\Omega. \tag{87}$$

C.3

Here we divide $\vec{R}$ and Ω respectively by 1 meter and 1 Hz, so we will deal with unitless quantities. Then, in terms of components $\vec{R} = \{x, y\}$, we have the following unitless equations:

$$\ddot{x} - \gamma\dot{y} + \beta x = 0, \tag{88}$$
$$\ddot{y} + \gamma\dot{x} + \beta y = 0. \tag{89}$$

Substituting the following coordinate transformation

$$x(t) = \rho(t)\cos(\eta(t)), \tag{90}$$
$$y(t) = \rho(t)\sin(\eta(t)), \tag{91}$$

in the component equation leads to

$$\ddot{\rho} + (\beta - \gamma\dot{\eta} - \dot{\eta}^2)\rho = 0, \tag{92}$$
$$\dot{\rho}(\gamma + 2\dot{\eta}) = 0. \tag{93}$$

The first equation comes from the requirement that the coefficients of $\cos\eta$ ($\sin\eta$) and the terms containing first time derivative $\dot\rho$ and η vanish separately. It is straightforward to see this is equivalent to both $\dot x$ and $\dot y$ terms vanish. From this we find:

$$\eta = -\frac{\gamma}{2}t + \phi, \tag{94}$$

$$\beta' \equiv \beta - \gamma\dot\eta - \dot\eta^2 = \beta + \frac{\gamma^2}{4}. \tag{95}$$

It is clear that for $\ddot\rho + \beta'\rho = 0$ one gets three distinct behavior for $\rho(t)$:

$$\beta' > 0, \text{ for harmonic oscillation} \tag{96}$$

$$\beta' < 0, \text{ for exponential run away} \tag{97}$$

$$\beta' = 0 \tag{98}$$

We examine the case $\beta' = 0$ in part C.4.
C.4

If $\beta' = 0$ we have $\beta = -\frac{\gamma^2}{4}$. Therefore, $\ddot\rho = 0$ and we have $\rho(t) = A + Dt$, where A and D are constants to be determined.

From the initial conditions

$$x(0) = 1,\ \ y = 0,\ \ v_x(0) = \dot x|_{t=0} = 1,\ \ , v_y(0) = \dot y|_{t=0} = -1. \tag{99}$$

we find:

$$A = 1,\ \ D = 1,\ \ \gamma = 2,\ \ \beta = -1. \tag{100}$$

Then the solution for the coordinates are:

$$x(t) = (1+t)\cos(t),\ \ y(t) = -(1+t)\sin(t). \tag{101}$$

From this, the length of $\vec R$ can be calculated:

$$R^2 = x(t)^2 + y(t)^2 = (1+t)^2. \tag{102}$$

Using the definitions of β and γ, the solutions for Ω are found as:

$$\Omega = -11 \pm \sqrt{77}. \tag{103}$$

Since the both solutions for $\Omega < 0$ and $B > 0$ ($\vec B$ is in $\hat z$ direction), from $\beta < 0$ we see that $Q < 0$.
C.6

From the solution we see that for every $t = 2\pi$ time $\vec{R}$ makes one revolution. After $N \gg 1$ rotations, $R^2 = (1+t)^2 = t^2$ or $R = 1 + t$ and we find the change in R per rotation to be $\Delta R = \Delta t = 2\pi$.

Scalar multiplying the acceleration by velocity and integrating it we obtain:

$$\vec{v} \cdot \dot{\vec{v}} = -\beta \vec{v} \cdot \vec{R} \rightarrow \tag{104}$$

$$v^2 - v_0^2 = -\beta \left(R^2 - R_0^2\right) = t^2. \tag{105}$$

Then the total and per rotation changes in the kinetic energy associated to the motion of the ball's center per rotation are:

$$\vec{v} \cdot \dot{\vec{v}} = \beta \dot{\vec{R}} \cdot \vec{R} \rightarrow \tag{106}$$

$$\Delta K = \frac{v^2 - v_0^2}{2} = \left(\frac{R^2 - R_0^2}{2}\right) = \frac{t^2}{2}, \tag{107}$$

$$\Delta K_N = \frac{v_{N+1}^2 - v_N^2}{2} = \Delta \left(\frac{R^2}{2}\right) = t\Delta t = 4\pi^2 N. \tag{108}$$

Now we estimate the change in the kinetic energy associated with the spinning of the ball. From non–slipping condition we get

$$\omega^2 = v^2 + \Omega^2 R^2 + 2\Omega \vec{v} \cdot \left(\hat{z} \times \vec{R}\right). \tag{109}$$

For our initial condition $\vec{v}_0 \cdot \left(\hat{z} \times \vec{R}_0\right) = -v_0 R_0$ and, for large N, $\vec{v}$ and $\vec{R}$ are approximately orthogonal to a very good approximation, so $\vec{v} \cdot \left(\hat{z} \times \vec{R}\right) = -vR$. Our calculated $\Omega < 0$, so we can write this term as $|\Omega| vR$. So the kinetic energy for spinning and its change are

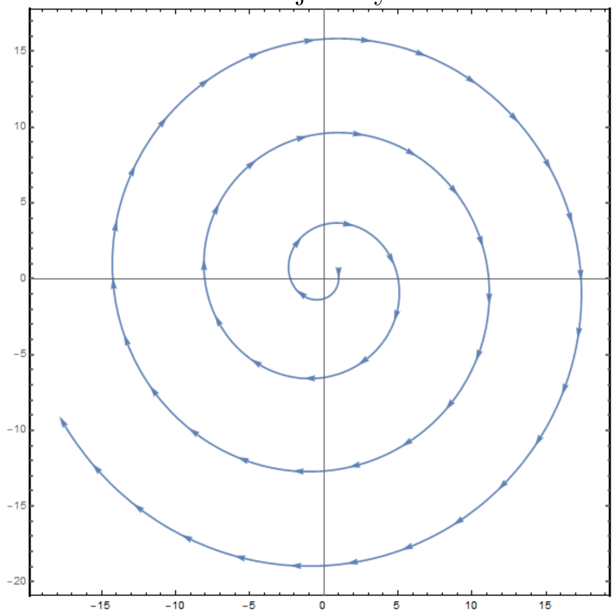
$$K_s = \frac{I\omega^2}{2} = \frac{I\left(v^2 + \Omega^2 R^2 + 2|\Omega| vR\right)}{2}, \tag{110}$$

$$\Delta K_s = \frac{I\left((v^2 - v_0^2 + \Omega^2 (R^2 - R_0^2) + 2|\Omega|(vR - v_0 R_0)\right)}{2}. \tag{111}$$

Finally, combining all the results we have:

$$\begin{aligned}
\Delta E &= \frac{I(\omega^2 - \omega_0^2)}{2} + \Delta K \simeq \frac{I\left(v^2 + \Omega^2 R^2 + 2|\Omega| vR\right)}{2} + \frac{t^2}{2}, && (112)\\
&= \frac{t^2}{2}\left(\frac{(1+|\Omega|)^2}{11} + 1\right), && (113)\\
\Delta E_N &= \frac{I(\omega_N^2 - \omega_{N-1}^2)}{2} + \Delta K_N && (114)\\
&= 4\pi N \left(\frac{(1+|\Omega|)^2}{11} + 1\right) \text{ with:} && (115)\\
|\Omega| &= |11 \pm \sqrt{77}|. && (116)
\end{aligned}$$

The sketch of the trajectory looks like



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[2] Luis Rodriguez, Comment on "A ball rolling on a freely spinning turntable" by Warren Weckesser AM. J. Phys. **66** (10), 927 (1998).

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Theoretical Problem 2: Ball on turntable
Q2 – Marking Scheme

Question part	Total marks	Partial marks	Explanation for partial marks and special cases
A.1	0.1	0.1	Kinematic constraint from non-slipping condition
A.2	0.2	0.1 0.1	Time derivative of the constraint equation Expression for the linear acceleration of the ball
A.3	0.2	0.1 0.1	Time integral of the acceleration with initial values Velocity formula in terms of initial values
A.4	0.5	0.2 0.3	Identification of trajectory radius Identification of the center of the trajectory
A.5	0.5	0.2 0.3	Calculate the return time on the same spot in general Calculate the return time on the same spot for spectial case
B.1	0.2	0.1 0.1	Velocity formula Acceleration
B.2	0.6	0.3 0.3	Angular acceleration of the turntable with the ball's velocity dependency and coordinates Angular acceleration of the turntable with the ball's coordinates dependency
B.3	0.6	0.3 0.3	Find scalar product of coordinate and velocity Integrate and obtain table's angular velocity depending on magnitude of position only
B.4	0.1	0.1	Maximum table's angular velocity
B.5	2.5	0.5 0.5 0.9 0.6	Calculate table's angular velocity using conservation of angular momentum Calculate table's angular velocity in the given coordinates Find the equation for the curve List the possible trajectories
C.1	0.5	0.2 0.3	Newton's equation Spinning torque due to magnetic field
C.2	0.5	0.2 0.3	Acceleration formula including spinning of the ball Acceleration formula and 2^{nd} order diff equation

C.3	1.0	0.5 0.3 0.2	Transformation to polar system removing the first time Identification of angle function and beta’ Classification of time behavior of rho(t)
C.4	0.9	0.5 0.3 0.1	Identification of the curve from the initial condition Find angular velocity of the table Identify the charge sign
C.5	1.6	0.4 0.4 0.3 0.3 0.2	Calculation of the total kinetic energy change from linear motion Calculation of the kinetic energy change from linear motion per revolution Calculation of the total kinetic energy change from spinning Calculation of the kinetic energy change from spinning per revolution Total energy change

CAVITATION: A POSSIBLE SOLUTION

TSOGTGEREL GANTUMUR

Note: The chosen units might be a bit different from the "official" problem statement. For instance, we might write $1\,\mu$m for 10^{-6} m. "Table 1" refers to the "notation table" from the statement.

A1. By performing a simple dimensional analysis, estimate the collapse time τ of a pure vapour bubble, in terms of bubble's initial radius R_0, water density ρ, water pressure p_∞, and the vapour pressure p_v. Evaluate the formula when $R_0 = 1$ mm and the quantities ρ, p_∞ and p_v take their typical values from Table 1. Assume no surface tension: $\sigma = 0$.

Solution. It is reasonable to expect that the bubble would not collapse if $p_\infty \leq p_v$. Hence we take $p_\infty > p_v$, and presume that the difference $p_\infty - p_v$ will feature in the final formula. The dimensions of the quantities are

$$[R_0] = \mathrm{m}, \qquad [\rho] = \mathrm{kg/m^3}, \qquad [p_\infty - p_v] = \mathrm{N/m^2} = \mathrm{kg/(m \cdot s^2)}. \tag{1}$$

The only combination of these that has the dimension of time is

$$\tau \sim R_0 \sqrt{\frac{\rho}{p_\infty - p_v}} \approx 0.1\,\mathrm{s}. \tag{2}$$

A2. Suppose that a micro-bubble consisting of air and vapour, with radius $R_0 = 10\,\mu$m, is in equilibrium when the external pressure $p_\infty = 100$ kPa. Find the partial pressure q_0 of air in the bubble. Now suppose that the external pressure p_∞ was gradually decreased, and that the air inside the bubble follows an isothermal process. Find the critical pressure p_c, defined by the condition that if $p_\infty < p_c$ the bubble size grows without bound. The quantities p_v and σ take their typical values from Table 1.

Solution. The pressure equilibrium condition is

$$p_v + q = p_\infty + \frac{2\sigma}{R}, \tag{3}$$

which, under $R = R_0$ and $q = q_0$, yields

$$q_0 = p_\infty - p_v + \frac{2\sigma}{R_0} \approx 170\,\mathrm{kPa}. \tag{4}$$

On the other hand, taking into account the isothermal law

$$q_0 R_0^3 = q R^3, \tag{5}$$

we get

$$p_\infty = p_v + \frac{q_0 R_0^3}{R^3} - \frac{2\sigma}{R}. \tag{6}$$

The critical pressure corresponds to the minimum of p_∞ as a function of R. The radius at the minimum is easily found to be

$$R_c = R_0 \sqrt{\frac{3 q_0 R_0}{2\sigma}} \approx 60\,\mu\mathrm{m}, \tag{7}$$

and the corresponding pressure is

$$p_c = p_v - 2q_0\big(\frac{R_0}{R_c}\big)^3 \approx 700\,\text{Pa}. \tag{8}$$

B1. Suppose that a single spherical bubble resides within water that fills space uniformly, and that the bubble may evolve in size without distorting its spherical shape, due to changes, e.g., in the external pressure p_∞. Derive an equation that relates the bubble radius $R(t)$ and its time derivatives $R'(t)$ and $R''(t)$, surface tension σ, water density ρ, the pressure far from the bubble p_∞, and the pressure inside the bubble p. Then split the pressure p into two terms, by assuming that the bubble has both vapour and air in it, and that the air follows an adiabatic process with exponent γ. To give a reference point, the partial air pressure must be q_0 when the bubble size equals R_0. Assume that evaporation, condensation, or transfer of air between the bubble cavity and the surrounding water has no effect on the water volume.

Solution. Let $v(r,t)$ denote the radial velocity of the fluid element at the distance r from the bubble center and at the time moment t. Let also $u(t) = v(R,t)$, that is,

$$u(t) = R'(t). \tag{9}$$

Then the incompressibility condition yields

$$v(r,t) = \frac{R^2 u(t)}{r^2}, \tag{10}$$

and hence

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\frac{R^2 u}{r^2} = \frac{2R}{r^2}u^2 + \frac{R^2}{r^2}u'. \tag{11}$$

The radial acceleration can now be computed as

$$a = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial r}v = \frac{R^2}{r^2}u' + \frac{2R}{r^2}u^2 - \frac{2R^4}{r^5}u^2. \tag{12}$$

With $p = p(r)$ denoting the pressure field (where possible time dependence is suppressed in the notation), Newton's law reads

$$\rho a = -p', \tag{13}$$

or

$$\frac{R^2}{r^2}u' + \big(\frac{2R}{r^2} - \frac{2R^4}{r^5}\big)u^2 = -\frac{p'}{\rho}. \tag{14}$$

We integrate it from $r = R$ to $r = \infty$, to get

$$Ru' + \frac{3}{2}u^2 = \frac{p(R) - p_\infty}{\rho}. \tag{15}$$

Since the pressure inside the bubble satisfies

$$p = p(R) + \frac{2\sigma}{R}, \tag{16}$$

we conclude that

$$RR'' + \frac{3}{2}(R')^2 + \frac{2\sigma}{\rho R} = \frac{p - p_\infty}{\rho}. \tag{17}$$

Finally, taking into account the fact that the bubble pressure p consists of the vapour pressure p_v and the partial air pressure q, as

$$p = p_v + q = p_v + q_0\big(\frac{R_0}{R}\big)^{3\gamma}, \tag{18}$$

we have

$$\rho R R'' + \frac{3}{2}\rho (R')^2 + \frac{2\sigma}{R} - \frac{q_0 R_0^{3\gamma}}{R^{3\gamma}} = p_v - p_\infty. \tag{19}$$

B2. A water tank under the external pressure $p_\infty^- = 100\,\mathrm{kPa}$, containing a nucleus of radius $R_0 = 10\,\mu\mathrm{m}$ initially in equilibrium, was exposed to vacuum, so that the system suddenly has $p_\infty = 0$. Estimate the terminal (asymptotic) value of the growth speed R', as well as the time it reaches this terminal value.

Solution. Putting $R'' = 0$ and $R \to \infty$ in (19) yields

$$R'(\infty) = \sqrt{\frac{2(p_v - p_\infty)}{3\rho}} = \sqrt{\frac{2p_v}{3\rho}} \approx 1.24\,\mathrm{m/s}. \tag{20}$$

The initial acceleration can also be found from (19) as

$$R''(0) = \frac{q_0 + p_v - 2\sigma/R_0}{\rho R_0} = \frac{p_\infty^-}{\rho R_0} \approx 10^7\,\mathrm{m/s^2}, \tag{21}$$

and hence the time for the speed to be stabilized can be estimated as

$$t = \frac{R'(\infty)}{R''(0)} \approx 0.1\,\mu\mathrm{s}. \tag{22}$$

B3. A water tank under the external pressure $p_\infty^- = 1600\,\mathrm{Pa}$, containing a gas bubble of radius $R_0 = 10\,\mu\mathrm{m}$ initially in equilibrium, was suddenly exposed to the atmospheric pressure $p_\infty = 100\,\mathrm{kPa}$. Estimate the minimum radius of the bubble before it rebounds.

Solution. Multiply (19) by $2R^2R'$ to get

$$\rho[R^3(R')^2]' + 4\sigma R R' - 2q_0 R_0^{3\gamma} R^{2-3\gamma} R' = 2(p_v - p_\infty) R^2 R', \tag{23}$$

or

$$\rho[R^3(R')^2]' + 2\sigma (R^2)' + \frac{2q_0 R_0^{3\gamma}(R^{3-3\gamma})'}{3(\gamma-1)} = \frac{2}{3}(p_v - p_\infty)(R^3)'. \tag{24}$$

This can easily be integrated, from $R(0) = R_0$ and $R'(0) = 0$ to $R(t) = R$ and $R'(t) = R'$, which yields

$$\rho R^3 (R')^2 + 2\sigma(R^2 - R_0^2) + \frac{2q_0 R_0^{3\gamma}(R^{3-3\gamma} - R_0^{3-3\gamma})}{3(\gamma-1)} = \frac{2}{3}(p_v - p_\infty)(R^3 - R_0^3), \tag{25}$$

or

$$\rho (R')^2 = \frac{2(p_v - p_\infty)}{3} - \frac{2\sigma}{R} + \frac{2R_0^3}{R^3}\left(\frac{\sigma}{R_0} + \frac{p_\infty - p_v}{3} + \frac{q_0}{3(\gamma-1)} - \frac{q_0 R_0^{3\gamma-3}}{3(\gamma-1)R^{3\gamma-3}}\right). \tag{26}$$

When $R \ll R_0$, it shows first that

$$R' \sim -R^{-3/2}, \tag{27}$$

and moreover that the rebound radius satisfies

$$\left(\frac{R_0}{R}\right)^{3(\gamma-1)} = \frac{(\gamma-1)(p_\infty - p_v + 3\sigma/R_0)}{q_0} + 1. \tag{28}$$

Taking into account that

$$q_0 = p_\infty^- - p_v + 2\sigma/R_0 \approx 10\,\mathrm{Pa}, \tag{29}$$

we conclude that the rebound radius is

$$R \approx 0.4\,\mu\mathrm{m}. \tag{30}$$

B4. If there is no gas other than water vapour present in a bubble, the bubble completely collapses in finite time. Determine the characteristic exponent α in

$$R(t) \sim (T-t)^\alpha, \tag{31}$$

where T is the collapse time.

Solution. We have found in the previous part that

$$R' \sim -R^{-3/2}. \tag{32}$$

Putting $R(t) \sim (T-t)^\alpha$ into it we get

$$(T-t)^{\alpha-1} \sim -(T-t)^{-3\alpha/2}, \tag{33}$$

or

$$\alpha = \frac{2}{5}. \tag{34}$$

B5. Based on the equation derived in Part 3, find the natural frequency of the spherical oscillation of a bubble of radius $R_0 = 0.1\,\mathrm{mm}$.

Solution. Introducing the new variable x by $R = R_0 x$, we write (19) as

$$\rho R_0^2 x x'' + \frac{3}{2}\rho R_0^2 (x')^2 + \frac{2\sigma}{R_0 x} - q_0 x^{-3\gamma} = p_v - p_\infty. \tag{35}$$

Now put $x = 1 + y$ and retain the terms up to linear in y, to have

$$\rho R_0^2 y'' + \frac{2\sigma}{R_0}(1-y) - q_0(1-3\gamma y) = p_v - p_\infty. \tag{36}$$

Taking into account the equilibrium condition

$$\frac{2\sigma}{R_0} - q_0 = p_v - p_\infty, \tag{37}$$

we infer

$$y'' + \frac{3\gamma q_0 - 2\sigma/R_0}{\rho R_0^2} y = 0. \tag{38}$$

Thus the bubble is unstable if

$$3\gamma q_0 \leq 2\sigma/R_0 = q_0 + p_v - p_\infty, \tag{39}$$

or equivalently, if

$$p_\infty \leq p_v - (3\gamma - 1)q_0. \tag{40}$$

On the other hand, if $p_\infty > p_v - (3\gamma-1)q_0$ then the bubble oscillates with the natural frequency

$$f_0 = \frac{1}{2\pi R_0}\sqrt{\frac{(3\gamma-1)q_0 + p_\infty - p_v}{\rho}} \approx 33\,\mathrm{kHz}. \tag{41}$$

B6. Suppose that the bubble described in the previous part is subjected to a standing sound wave along the x-axis, whose pressure field is given by

$$p(x,t) = p_0 + A\sin\big(\frac{2\pi f}{c}(x+a)\big)\sin(2\pi f t), \tag{42}$$

where f is the frequency, and c is the speed of sound. The parameters p_0, A, and a are constants, whose meanings may readily be deduced from the equation. Find the average force exerted upon the bubble. The bubble is situated at the origin of the xyz coordinate system, and its size is much smaller than the wavelength of the sound.

Solution. Small oscillation of the bubble is described by

$$\rho R_0^2(y'' + 4\pi^2 f_0^2 y) = A\sin\big(\frac{2\pi f}{c}a\big)\sin(2\pi f t). \tag{43}$$

Looking for the solution in the form

$$y(t) = B\sin(2\pi f t), \tag{44}$$

we find

$$B = \frac{A\sin(2\pi f a/c)}{4\pi^2\rho R_0^2(f_0^2 - f^2)}. \tag{45}$$

Since

$$R(t) = R_0 + R_0 y(t), \tag{46}$$

the volume of the bubble is

$$V(t) = \frac{4\pi}{3}R(t)^3 \approx \frac{4\pi}{3}R_0^3[1 + 3y(t)]. \tag{47}$$

Now recalling that the average force is

$$F = -\langle V\frac{\partial p}{\partial x}\rangle, \tag{48}$$

where the average is taken over time, and that the pressure gradient is

$$\frac{\partial p}{\partial x} = \frac{2\pi f A}{c}\cos\big(\frac{2\pi f}{c}a\big)\sin(2\pi f t), \tag{49}$$

we conclude

$$F = -\frac{f A^2 R_0}{2\rho c(f_0^2 - f^2)}\sin\big(\frac{4\pi f a}{c}\big). \tag{50}$$

C1. Suppose that a nucleus consisting of air and vapour, with radius $R_0 = 10\,\mu\text{m}$, is placed in water-air solution, in which the dissolved air is in equilibrium with the atmospheric pressure above the water. The partial pressure of air in the bubble is $q = 170\,\text{kPa}$, and the vapour pressure can be neglected. Estimate the time required for the bubble to be completely resorbed into water. The quantities p_∞, κ, δ and σ take their typical values from Table 1. Assume that the region surrounding the bubble in which air diffusion takes place immediately gets much larger than the bubble itself.

Solution. From Henry's law, the initial concentration of dissolved air in the body of water is

$$u_i = Hp_\infty \approx 0.024\,\text{kg/m}^3, \tag{51}$$

and the initial concentration of dissolved air in the immediate vicinity of the bubble is

$$u = Hq \approx 0.041\,\text{kg/m}^3. \tag{52}$$

Since $u > u_i$, there will be diffusive flux directed away from the bubble, and the bubble will start losing air. As a result, the bubble shrinks and the surface tension term $2\sigma/R$ increases. Since the external pressure p_∞ is constant, the partial air pressure q increases, which leads to more diffusive flux. In the end, the bubble will get completely resorbed into water.

To quantify the diffusion of air, let us consider the region enclosed by concentric spheres of radii r and $r + \Delta r$ with Δr small. The rate of change of air mass in this region is

$$\frac{dm}{dt} = 4\pi r^2\Delta r\frac{\partial u}{\partial t}, \tag{53}$$

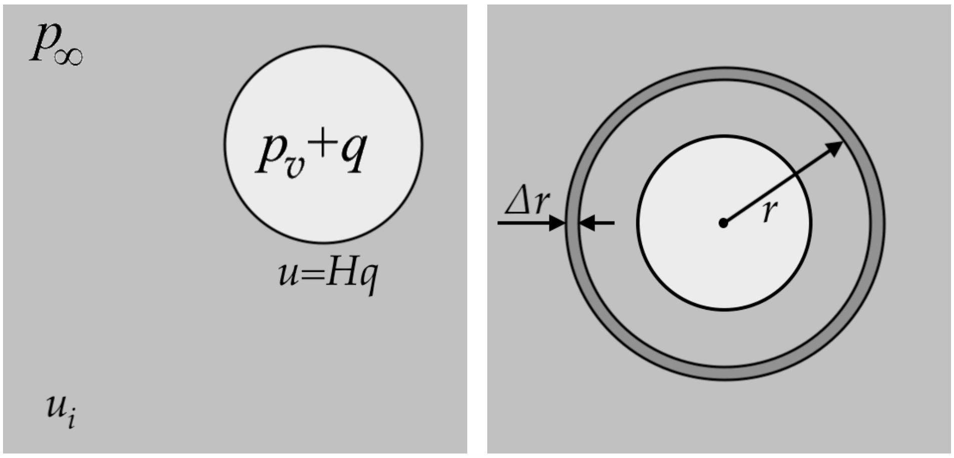

FIGURE 1. (a) Initial configuration of the system (b) The geometry used in the solution

where $4\pi r^2\Delta r$ is the volume of the region and the air concentration $u = u(r,t)$ is a function of r and t. On the other hand, the same quantity can be computed as the difference between the diffusive fluxes through the concentric spheres:

$$\begin{aligned}\frac{dm}{dt} &= 4\pi(r+\Delta r)^2 J(r+\Delta r) - 4\pi r^2 J(r)\\ &= 4\pi(r+\Delta r)^2 \kappa\frac{\partial u}{\partial r}(r+\Delta r) - 4\pi r^2\kappa\frac{\partial u}{\partial r}(r)\\ &\approx 4\pi r^2\kappa\Delta r\frac{\partial^2 u}{\partial r^2}(r) + 8\pi r\kappa\Delta r\frac{\partial u}{\partial r}(r),\end{aligned} \tag{54}$$

where we have taken into account the directions of the fluxes, and the fact that

$$\frac{\partial u}{\partial r}(r+\Delta r) \approx \frac{\partial u}{\partial r}(r) + \Delta r\frac{\partial^2 u}{\partial r^2}(r). \tag{55}$$

Comparing the two equations, we infer

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial r^2} + \frac{2\kappa}{r}\frac{\partial u}{\partial r}. \tag{56}$$

The dissolved air concentration in water at the initial time moment $t = 0$ is uniformly u_i, and assuming that the radius R of the bubble remains *constant*, the air concentration in the immediate vicinity of the bubble should be equal to Hq:

$$\begin{cases} u(r,0) = u_i & \text{for } r > R,\\ u(R,t) = Hq & \text{for } t > 0.\end{cases} \tag{57}$$

Introducing a new variable $v = r(u - qH)$, and a new time parameter $\tau = \kappa t$, the diffusion equation (56) becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial r^2}, \tag{58}$$

with

$$\begin{cases} v(r,0) = r(u_i - qH) & \text{for } r > R,\\ v(R,\tau) = 0 & \text{for } \tau > 0.\end{cases} \tag{59}$$

We can go further by introducing

$$\xi = r - R \qquad \text{and} \qquad w(\xi, \tau) = v(R + \xi, \tau), \tag{60}$$

to write (56) as

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial \xi^2}, \tag{61}$$

with

$$\begin{cases} w(\xi, 0) = (R + \xi)(u_i - qH) & \text{for } \xi > 0, \\ w(0, \tau) = 0 & \text{for } \tau > 0. \end{cases} \tag{62}$$

The solution to this problem is

$$w(\xi, \tau) = \frac{u_i - qH}{\sqrt{4\pi\tau}} \int_0^\infty \big(e^{-(\xi-\eta)^2/(4\tau)} - e^{-(\xi+\eta)^2/(4\tau)}\big)(\eta + R)d\eta. \tag{63}$$

Since

$$u(r, t) = qH + \frac{v(r, \kappa t)}{r} = qH + \frac{w(r - R, \kappa t)}{r}, \tag{64}$$

we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial w}{\partial \xi} - \frac{w}{r^2}, \tag{65}$$

and hence

$$\frac{\partial u}{\partial r}\Big|_{r=R} = \frac{1}{R}\frac{\partial w}{\partial \xi}\Big|_{\xi=0} = (u_i - qH)\big(\frac{1}{R} + \frac{1}{\sqrt{\pi\kappa t}}\big). \tag{66}$$

The second term corresponds to the width of the diffusion layer surrounding the bubble, which we assume to be much larger than the bubble itself. Hence the rate of change of the mass of the bubble can be estimated as

$$\frac{dm}{dt} = 4\pi R^2 \kappa \frac{\partial u}{\partial r}\Big|_{r=R} \approx 4\pi\kappa R(u_i - qH). \tag{67}$$

On the other hand, the mass of the bubble is related to the air density, which in turn is proportional to the pressure:

$$m = \frac{4\pi}{3}R^3\delta = \frac{4\pi}{3}R^3 \cdot \frac{\delta_0 q}{p_\infty}, \tag{68}$$

where $\delta_0 = 1.2\,\text{kg/m}^3$ is the air density at the atmospheric pressure $p_\infty = 10^5\,\text{Pa}$. Furthermore, neglecting vapour pressure, we have the mechanical equilibrium condition

$$q = p_\infty + \frac{2\sigma}{R}, \tag{69}$$

leading to

$$m = \frac{4\pi}{3}R^3 \cdot (\delta_0 + \frac{\varepsilon}{R}), \tag{70}$$

with

$$\varepsilon = \frac{2\sigma\delta_0}{p_\infty} \approx 1.73 \cdot 10^{-6}\,\text{kg/m}^2. \tag{71}$$

After taking the derivative of (70) with respect to t, we equate it to (67), and get

$$4\pi\kappa R(u_i - p_\infty H - \frac{2\sigma H}{R}) = 4\pi\delta_0 R^2\frac{dR}{dt} + \frac{8\pi}{3}\varepsilon R\frac{dR}{dt}. \tag{72}$$

Since $u_i - p_\infty H = 0$, we have

$$-2\sigma\kappa H dt = \delta_0 R^2 dR + \frac{2}{3}\varepsilon R dR, \tag{73}$$

and a direct integration yields

$$2\sigma\kappa Ht = \frac{\delta_0 R_0^3}{3} + \frac{4\varepsilon R_0^2}{3}, \tag{74}$$

finally giving

$$t = \frac{(\delta_0 R_0 + 4\varepsilon)R_0^2}{6\sigma\kappa H} \approx 9\,\mathrm{s}. \tag{75}$$

Thus the nucleus collapses in a matter of seconds.

C2. Consider a conical crevice in the wall of a water container, with an aperture angle α. A small amount of air and vapour is trapped within the cone. Write down the condition of mechanical and diffusive equilibrium. Determine when the pocket of air stays in the crevice without disappearing.

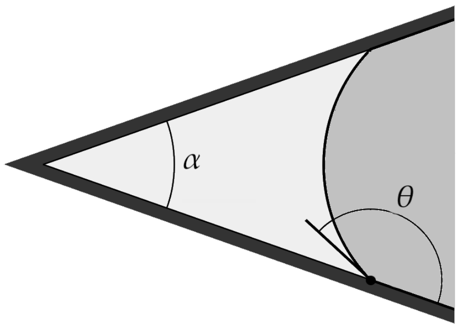

FIGURE 2. Conical crevice

Solution. The equilibrium conditions are

$$p_v + q = p_\infty \pm \frac{2\sigma}{R} \qquad \text{and} \qquad u = Hq. \tag{76}$$

The plus sign corresponds to the situation where the water surface is concave, and the surface tension tends to squeeze the air out of the crevice through diffusion. When the water surface is convex, we have the opposing sign. Supposing that initially the partial air pressure q is large, q will decrease as the region of trapped air shrinks due to diffusion. At some point, a diffusive equilibrium will be found. Thus the pocket of air does not disappear when the water surface is convex, meaning that

$$\pi + \alpha < 2\theta. \tag{77}$$

Theoretical Problem 3: Cavitation
Q3 - Marking Scheme

Question part	Total marks	Partial marks	Explanation
A1	0.5	0.4	Derivation of the formula
		0.1	Numerical answer
A2	1	0.3	Pressure equilibrium condition
		0.1	Numerical answer for partial air pressure
		0.1	Equilibrium condition with isothermal law
		0.4	Formula for critical pressure
		0.1	Numerical answer for critical pressure
B1	1.5	0.3	Incompressibility condition
		0.5	Computation of radial acceleration
		0.3	Taking into account Newton’s second law
		0.2	Final equation with p, where surface tension enters
		0.2	Final equation with p split and adiabatic law taken into account
B2	1	0.4	Formula for terminal speed
		0.1	Numerical answer for terminal speed
		0.4	Formula for time or for initial acceleration
		0.1	Numerical answer for time
B3	1	0.5	Formula for speed R’
		0.4	Formula for rebound radius
		0.1	Numerical answer for rebound radius
B4	0.5	0.2	Writing down the asymptotic equation for R’
		0.2	Substitution of the form into the equation

Question part	Total marks	Partial marks	Explanation
		0.1	Numerical answer for alpha
B5	1	0.4	Writing down the linear second order equation
		0.2	Recognition of the stability condition
		0.3	Formula for the natural frequency
		0.1	Numerical answer for the natural frequency
B6	1	0.2	Writing down the forced oscillator equation
		0.3	Solution of the equation
		0.1	Computation of volume
		0.1	Computation of pressure gradient
		0.3	Final answer
C1	2	0.1	Initial concentration of air in water
		0.1	Initial concentration of air near bubble
		0.1	Mass change rate in a shell in terms of concentration change rate
		0.1	Mass change rate in a shell in terms of flux
		0.1	Use of Fick's law
		0.2	Diffusion equation in terms of r and t
		0.2	Initial and boundary conditions
		0.2	Reformulation and solution
		0.2	Computing the air flux at the bubble wall
		0.1	Simplification due to smallness of the bubble
		0.2	Mass of the bubble in terms of pressure with ideal gas law taken into account
		0.1	Mass balance
		0.2	Final formula for resorption time

Question part	Total marks	Partial marks	Explanation
		0.1	Numerical answer for resorption time
C2	0.5	0.2	Condition of equilibrium
		0.3	Final condition on angles