General instructions for Theory

Before the exam

- You must not open the envelopes containing the problems before the sound signal indicating the beginning of the competition. You can open the envelopes five minutes before the scheduled time i.e. @11:55 IST. The beginning and end of the examination will be indicated by a sound signal.
- There will be announcements by the supervisor at 16:30 IST (half an before the end of the examination) and at the end of the exam at 17:00 IST.

During the exam

- Dedicated answer sheets are provided for writing your answers. Write your answers into the appropriate tables, boxes or graphs on the corresponding answer sheet (marked A). For every problem, there are extra blank working sheets for carrying out detailed work (marked W). Be sure to always use the working sheets that belong to the problem you are currently working on (check the problem number in the header). If you have written something on any sheet which you do not want to be graded, cross it out. Only use the front side of every page.
- In your answers, try to be as concise as possible: use equations, logical operators and sketches to illustrate your thoughts whenever possible. Avoid the use of long sentences.
- Please give an appropriate number of significant figures when stating numbers.
- Often, you may be able to solve later parts of a problem without having solved the previous ones.
- A list of physical constants is given on the next page.
- You are not allowed to leave your working place without permission. If you need any assistance, please draw the attention of a team guide by raising one of your placard ("Bio-break/washroom").

At the end of the exam

- At the end of the examination you must stop writing immediately.
- For every problem, sort the corresponding sheets in the following order: answer sheets (A), working sheets (W).
- Put all the answer sheets and working sheets belonging to one problem into the envelope for that question, including any blank working sheets. Also, put the Cover Sheet, Question papers and general instructions (G) into the remaining separate envelope. You are not allowed to take any sheets of paper out of the examination area.
- Leave your writing equipment on the table.
- Wait at your table in silence until your envelopes are collected. Once all envelopes are collected your guide will escort you out of the examination area.

Physical constants

Acceleration due to gravity	g	=	$9.81 m \cdot s^{-2}$
Boltzmann constant	k_B	=	$1.38 \times 10^{-23} J \cdot K^{-1}$
Current Mass of the Sun	M_s	=	$2.00 \times 10^{30} kg$
Current Radius of the Sun	R_s	=	$7.00 \times 10^{8} m$
Magnitude of the electron charge	e	=	$1.60 \times 10^{-19} C$
Mass of the electron	m_e	=	$9.11 \times 10^{-31} kg$
Mass of the proton	m_p	=	$1.67 \times 10^{-27} kg$
Mass of the neutron	m_n	=	$1.67 \times 10^{-27} kg$
Permeability of free space	μ_0	=	$1.26 \times 10^{-6} T \cdot m \cdot A^{-1}$
Permittivity of free space	ϵ_0	=	$8.85 \times 10^{-12} F \cdot m^{-1}$
Planck's constant	h	=	$6.63 \times 10^{-34} J \cdot s$
Avogardo Constant	N_A	=	$6.02 \times 10^{23} mol^{-1}$
Speed of light in vacuum	c	=	$3 \times 10^{8} m \cdot s^{-1}$
Stefan-Boltzmann constant	σ	=	$5.67 \times 10^{-8} W \cdot m^{-2} \cdot K^{-4}$
Universal gas constant	R	=	$8.31 J \cdot mol^{-1} \cdot K^{-1}$
Universal Gravitational constant	G	=	$6.67 \times 10^{-11} N \cdot m^{2} \cdot kg^{-2}$
Wien's constant	b	=	$2.90 \times 10^{-3} m \cdot K$
ln 2	$\approx$	=	0.69
ln 3	$\approx$	=	1.10
ln 10	$\approx$	=	2.30
Base of the Napierian logarithm e	$\approx$	=	2.72

The Stern-Gerlach Experiment[1]

The Stern-Gerlach experiment was performed in 1922 and it eventually led to the determination of the magnetic moment of the electron. In this experiment a beam of silver atoms each of mass $m = 1.80 \times 10^{-25}$ kg emerges from an oven kept at temperature T=1.20 $\times$ 10^3 K (see Figure 1). Assume that on emerging from the oven all the atoms have the same momentum along the direction of the beam (z-direction). Ignore gravity.

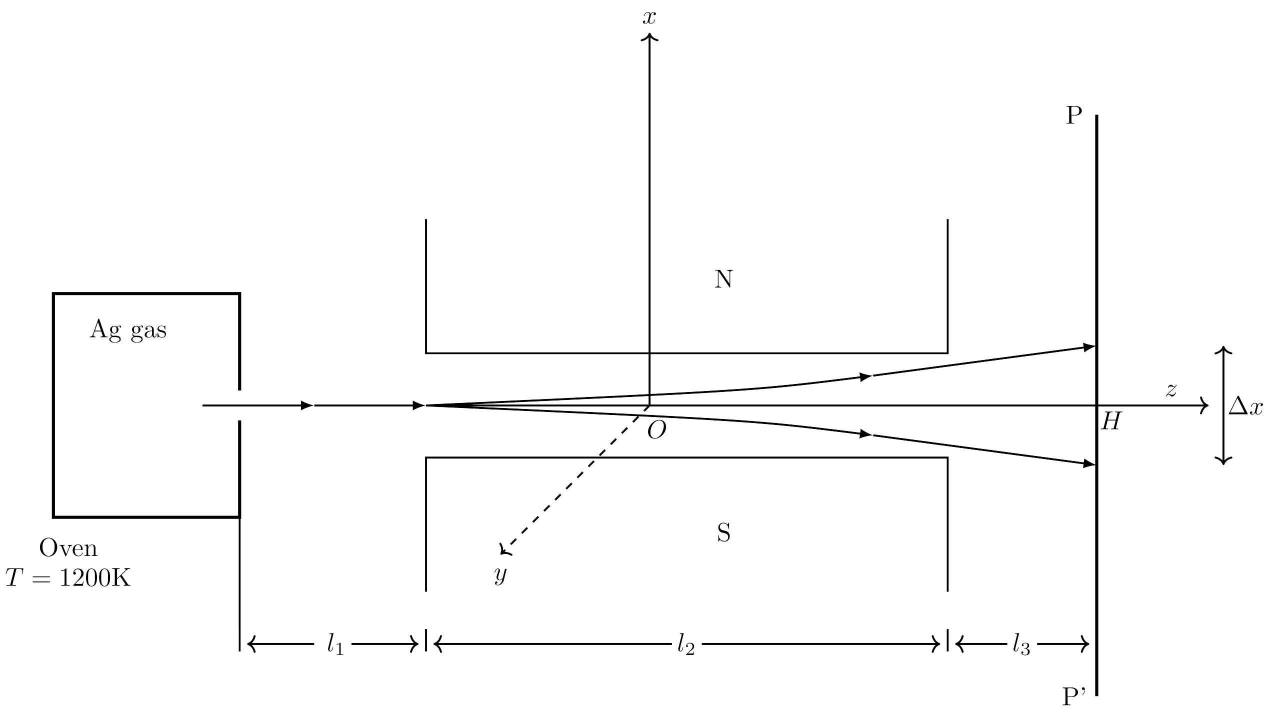

Figure 1: Schematic diagram of the Stern-Gerlach setup.

A.1	**Speed of the Silver Atoms:** The speed v_z of the silver atoms emerging from the oven can be estimated to be $\sqrt{3k_BT/m}$ using the equipartition theorem. Calculate this value.	0.5pt

[1] H. S. Mani (former Director, HRI, Prayagraj) and Gautam Datta (DAIICT, Gandhinagar) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group, and the International Board are gratefully acknowledged.

B.1	**The Basic Expression:** After emerging from the oven the silver atoms move along the z- direction over a distance $l_1 = 0.25$ m. Next, the silver atoms pass between two magnets over a distance $l_2 = 0.5$ m. The magnets produce inhomogeneous magnetic field B in the x-direction with constant gradient dB/dx. Assume that the silver atom has a magnetic moment pointing either in + x-direction or - x-direction, i.e. $\vec{\mu}_s = \pm\mu_s\hat{\imath}$. After passing through the magnets the silver atoms pass through a further distance $l_3 = 0.25$ m before striking the screen PP'. The distance between the two striking beams on the screen is Δx. Derive the expression for the splitting distance Δx on the screen.	2pt

The Inhomogeneous Magnetic Field:

This part is concerned with the setup to create the inhomogenous magnetic field ($dB/dx \neq 0$). It consists of a number of sub-parts. Two very long wires parallel to the z-axis carry currents of magnitude I_0 and are located at $A_1\ (0, -a, z)$ and $A_2\ (0, a, z)$ (see figure below). The direction of the current passing through $y = -a$ is $-\hat{k}$ and for the one passing through $y = a$ is $\hat{k}$. The entire system is inside a medium of high relative magnetic permeability μ_r. We take μ = $\mu_0\mu_r$. The wires are insulated and no current leaks into the medium.

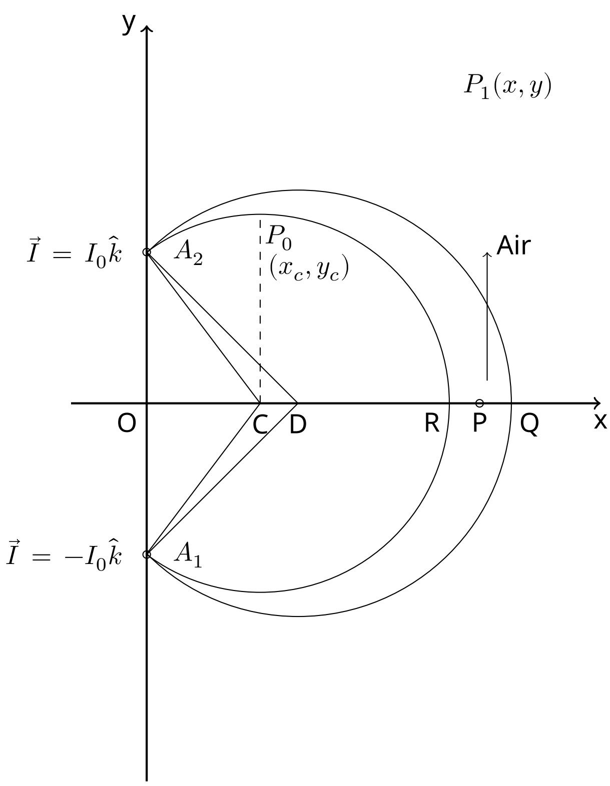

Figure 2: The arrangement for the inhomogeneous magnetic field

C.1	Derive the expression for the magnetic field vector at a point $P_1\,(x, y, 0)$ in the x-y plane (see Fig. 2).	1.5pt

C.2	Consider a circle with its centre C on the x-axis ($x_c, 0$) and with radius equal to the distance AC (see Figure 2). Obtain the direction of the magnetic field at circumferential points R (on the x axis) and at P_0 (CP_0 is parallel to the y axis).	0.5pt

C.3	Now a small slice of the high magnetic permeability medium between the circles with radii AC and AD is removed and replaced with air at very low pressure (see Fig. 2). One can show from continuity considerations that the magnetic field in this gap region is given by the same expression as if the magnetic medium were not removed. (We shall assume this and you are not required to prove this). Hence state the expression for the magnetic field at the point $(x, 0)$ in the air gap region.	0.5pt

D.1	**The Force:** As mentioned earlier the silver atoms are travelling in the $(x, 0, z)$ plane with their velocities parallel to the z-axis and given by $\vec{v} = v_z\hat{k}$. Recall also that the magnetic dipole of the silver atom is $\vec{\mu_s} = \pm\mu_s\hat{\imath}$. Obtain the expression for the magnitude of the force F_x acting on a silver atom along the x-direction in terms of μ_s , I_0 , a μ and relevant coordinates.	0.5pt

E.1	**The Magnetic Field and its Gradient:** We assume that this same force F_x acts over a small distance l_2 along the z axis (Figure 1). Assume also that the silver atoms pass through the mid-point P of RQ (Figure 2). The following experimental values are given: $$\frac{\mu}{\mu_0} = 10^4\,; \quad a = 0.60\,cm\,; \quad OC = 0.60cm\,; \quad OD = 0.80\,cm\,; I_0 = 2.00A$$ Here μ_o is the magnetic permeability of free space. Obtain the numerical value of the magnitudes of the magnetic field B_P and its gradient dB_P/dx at this mid-point in S.I. units.	2.0pt

F.1	**The Magnetic Moment of the Silver Atom:** If $v_z = 500\,\mathrm{m{\cdot}s^{-1}}$ and magnetic field is calculates as above, the Stern-Gerlach experiment yields a split of $\Delta x = 0.20$ cm. Obtain the value of the magnetic moment of the silver atom μ_s in S.I. units.	1.5pt

G.1	**The spread of the line:** The silver atoms may not all have the same speed. Let there be a spread of 20 % in the beam speed. What would be the consequent spread of the dot δx on the screen?	0.5pt

H.1	**The error in the magnetic moment:** What is the consequent error bar on the evaluation of the magnetic moment $\delta \mu_s$?	0.5pt

A Mechanical Model for Phase Transitions[1]

A ring of radius R has a bead of negligible size and mass m threaded on it. The ring is set rotating about its vertical diameter with angular velocity ω as shown in Fig. 1. Alongside this, there is an opposing force on the bead, F_f, which is proportional to the normal reaction N, and is given by $F_f = fkN$ where f is 1 or -1. You are advised to employ polar coordinates {r, θ}. As far as possible express your answers in terms of $\omega_c = \sqrt{g/R}$ where g is the magnitude of the acceleration due to gravity.

In a phase transition the free energy, $V(\mathcal{M})$ [mechanical equivalent is potential energy] depends on the magnetization $\mathcal{M}$ as follows

$$V(\mathcal{M}) = a(T)\mathcal{M}^2 + b(T)\mathcal{M}^4$$

where $T\rightarrow$ temperature, $b(T) > 0$ and $a(T)$ changes sign with temperature. We attempt to understand the phenomenon of phase transition using the above mentioned model.

Note 1: For circular motion of radius R, the velocity in polar coordinates is $\dot{\vec{r}} = R\dot{\theta}\hat{\theta}$ and the acceleration is $\ddot{\vec{r}} = -R\dot{\theta}^2\hat{r} + R\ddot{\theta}\hat{\theta}$. Here $\hat{r}$ and $\hat{\theta}$ are unit vectors in the radial and the tangential directions respectively.

Note 2: The direction of the opposing force say $\vec{F}_f$ will be denoted by f. Here f = +1 if the bead is moving in the counter-clockwise direction (of increasing) θ and f = -1 if the bead is moving in the clockwise direction θ, e.g. f = $sgn(\dot{\theta})$ where sgn is +1 or -1 depending on whether its argument is positive or negative.

Note 3: You may find the expansions

$$\begin{aligned}
\sin(\theta) &= \theta - \theta^3/6 + .. \\
\cos(\theta) &= 1 - \theta^2/2 + \theta^4/24 + .. \\
(1+x)^n &= 1 + nx + n(n-1)x^2/2 + n(n-1)(n-2)x^3/6 + ..
\end{aligned}$$

(where θ is in radians and $|x| << 1$) useful for some parts of the problem.

[1] Sitikantha Das (IIT Kharagpur) and Pramendra Ranjan Singh (Principal, Narayan College, J.P. University) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group, and the International Board are gratefully acknowledged.

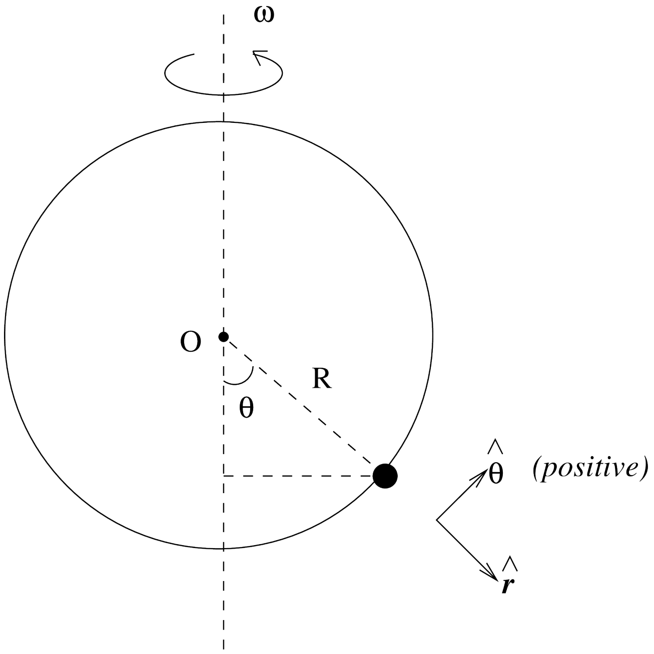

Fig. 1: The bead on a rotating ring.

In what follows we shall understand the dynamics of the bead in the frame of the rotating ring and for angles in the range $-\pi/2 < \theta < \pi/2$. The free body diagram of the bead is shown in Fig.2. Neglect all forces other than the ones shown in the free body diagram.

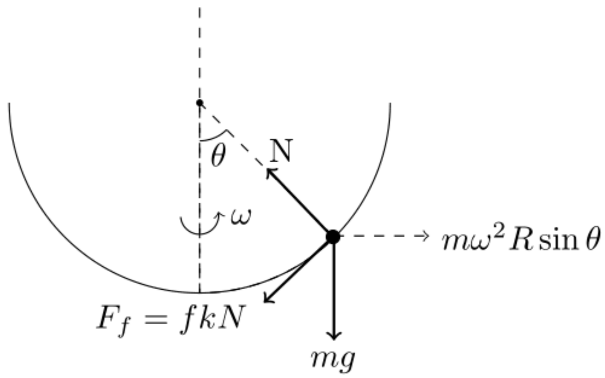

Fig. 2: The free body diagram.

A.1	Write down the equations of motion for the radial F_r and the tangential F_θ components of the force on the bead. Assume θ increasing in the counter-clockwise direction.	0.5pt

For the following parts B.1 to B.9 assume $k = 0$.

B.1	State the relation between the equilibrium angle(s) θ_0 in terms of { $\omega,\ \omega_c$}.	1.0pt

B.2	Qualitatively sketch θ_0 (y-axis) as a function of ω/ω_c (x-axis).	0.5pt

B.3	Qualitatively sketch the magnitude of the normal reaction force on the bead as a function of ω/ω_c at stable equilibrium.	0.5pt

B.4 We define the potential energy corresponding to the tangential force F_θ, namely **1.0pt**

$$F_\theta = -\frac{1}{R}\frac{d}{d\theta}V(\theta)$$

with the zero of potential energy at $\theta = 0$. If $V(\theta)$ is expressed as $P + Q\cos(\theta) + S\sin^2(\theta)$, obtain P, Q, S.

B.5	One can expand $V(\theta)$ for small θ and express it as $V(\theta) = a(\omega)\,\theta^2 + b(\omega)\,\theta^4$. Obtain the coefficients $a(\omega)$ and $b(\omega)$.	1.0pt

B.6	Make representative plots of $V(\theta)$ versus θ for values of ω/ω_c just less than 1.0 (e.g. say 0.9) and ω/ω_c large, say 5.0. Note that only qualitative sketches and no detailed calculations for the plots are required.	1.0pt

B.7 Landau theory of second order phase transitions can be used to demarcate simple magnetic systems into two phases. For temperatures T greater than the critical temperature T_c the system is paramagnetic. For $T < T_c$ the system is ferromagnetic and the magnetization $\mathcal{M}$ is given by **1.0pt**

$$\mathcal{M}(T) = \mathcal{M}_0(1 - T/T_c)^{1/2} \quad T < T_c$$

Let us denote the exponent $1/2$ by β. Compare this behaviour with the bead problem discussed above. What are the analougues of $\mathcal{M}$, T_c, T/T_c in our case? What is the equivalent value of β in our case?

B.8 Determine the angular frequency of oscillation Ω_0 of the bead when it is disturbed from its *"equilibrium"* position θ_0. Note that for small oscillations **1.0pt**

$$\Omega_0 = \frac{1}{R}\sqrt{\frac{V''(\theta_0)}{m}}$$

B.9	Qualitatively sketch Ω_0 as a function of ω.	1.0pt

For the following parts C.1 to C.2 $k \neq 0$.

C.1 Take f = 1 and express $k = \tan\alpha$. We may express the condition for the equilibrium angle(s) θ_0 as 1.0pt

$$\left(\frac{\omega}{\omega_c}\right)^2 = \frac{\tan(x)}{\sin(y)}$$

Obtain x and y.

C.2 It is given that f =1 and $k = 0.05$. Obtain the equilibrium angles θ_0, if any, for the following cases: 0.5pt

1. ω/ω_c = 0.50
2. ω/ω_c = 0.70

Maxwell, Rayleigh and Mount Everest: THE PROBLEM[1]

Lord Rayleigh visited Darjeeling, India in 1897. On viewing Mount Everest at a distance of 170 km he was reminded of a query by James Maxwell 26 years earlier on the attenuation of light by air and the visibility of far-off peaks. Lord Rayleigh authored a famous research paper[2] two years later in 1899 in which he investigated this problem. In the present problem we attempt to reconstruct, at least partially, a modern version of Lord Rayleigh's line of reasoning.

Oscillation of the electron cloud: We model a typical neutral air molecule by a stationary positive charge q surrounded by a spherical uniform charge cloud of mass m, radius r and charge $-q$. The natural angular frequency of vibration of the molecule is ω_0. Light is incident on it and the negative cloud oscillates maintaining its spherical shape with angular frequency ω as,

$$y = y_0 \cos(\omega t), \tag{1}$$

under the influence of the electric field

$$\vec{E}(t) = E_0 \cos(\omega t)\, \hat{y}, \tag{2}$$

of the light wave. Here y represents the separation between the stationary positive charge and the centre of the negative charge cloud of the molecule.

A.1	Set up the equation for the motion for y. Take the contribution to the magnitude of the acceleration due to the electric field to be $E(t)\, q/m$.	0.5pt

A.2	Based on the information provided above, solve the equation for y. Find the amplitude.	0.5pt

A.3	Find the magnitude $p(t)$ of the dipole moment of the air molecule as a function of time for $\omega \ll \omega_0$.	0.5pt

A.4	Obtain an expression for ω_0 in terms of q, m and r.	0.5pt

Power radiated: The sinusoidal time dependent dipole radiates electromagnetic radiation. The power radiated depends on the amplitude of the dipole moment $p_0 = qy_0$, the permittivity of vacuum ϵ_0, the speed of light c, and the frequency of oscillation ω.

B.1	Use dimensional analysis to express the average power s radiated in terms of these quantities.	1pt

[1] Amitabh Virmani (CMI, Chennai) and A. C. Biyani (retired Govt. Nagarjuna P.G. College of Science. Raipur) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group, and the International Board are gratefully acknowledged.

[2] Philosphical Mag. “On the transmission of light through an atmosphere ... and the origin of the blue of the sky”, Vol. 47, pg 375-384 1899

B.2	Take the proportionality constant to be $1/12\pi$. Express the answer for the power radiated s in terms of E_0, ω_0, ω and related quantities. Take $\omega \ll \omega_0$.	0.2pt

Attenuation of the Intensity $I(x)$: Recall that the intensity of EM waves is

$$\frac{1}{2}c\epsilon_0 E_0^2. \tag{3}$$

The intensity decreases along the path of light because the power

$$S = n_0 s \tag{4}$$

is lost per unit volume. Here n_0 is the number of molecules per unit volume.

C.1	Set up the differential equation for the intensity $I(x)$ as a function of the distance x.	1pt

C.2	Obtain the expression for the intensity $I(x)$ as a function of x in terms of the characteristic length scale L associated with the drop in intensity. Take the initial intensity to be I_0.	0.5pt

C.3 Take m to be the mass of the electron (typically only one electron constitutes the charge cloud) and take (0.3pt)

$$n_0 = 2.54 \times 10^{25}\ \text{m}^{-3}, \tag{5}$$

$$\omega_0 = 1.25 \times 10^{16}\ \text{rad} \cdot \text{s}^{-1}, \tag{6}$$

$$\omega = 3.25 \times 10^{15}\ \text{rad} \cdot \text{s}^{-1}. \tag{7}$$

Obtain the numerical value of L in kilometers.

D.1 **Height H' of the Mountains as seen by an observer:** In the figure the point P denotes Darjeeling, a hill station in the eastern Himalayas at height h = 2042 m above the sea level. The line BS denotes Mt Everest which is d = 170 km away from Darjeeling and is of height H = 8848 m. Another peak Mount Kanchenjunga (not shown in the figure) is 75 km away from Darjeeling and is of height 8586 m. Obtain an expression and the numerical values of the vertical height H' of these mountains as seen by an observer from Darjeeling in terms of the above-mentioned quantities. Assume that the observer is unable to see below the local horizon. Draw an appropriate figure. Take the radius R of the Earth to be 6378 km. 2pt

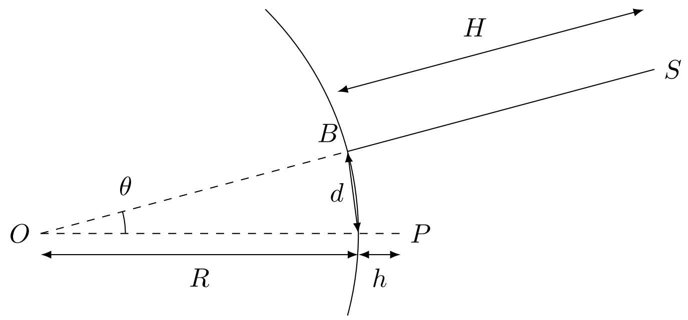

Figure 1. Great circle on which lie the mountain BS at height H and the observer P at height h. Note that the figure is not to scale.

E.1 Take the intensity of Mt Kanchenjunga at Darjeeling to be the reference value. What would be the intensity of Mt Everest relative to Mt Kanchenjunga as seen from Darjeeling? In this problem we ignore the variation in the number density of air molecules with height. If the intensity is 5 % or more of the reference value the mountain is said to be visible. Will Mt Everest be visible from Darjeeling? 1pt

F.1 **Attenuation length L_p due to aerosol pollution:** Above, we calculated the characteristic length scale L associated with the drop in intensity due to scattering with air molecules. We are now interested in the characteristic length L_p associated with the drop in intensity due to scattering with *aerosol* particles (pollution). L_p depends on the number density n of particles and the cross sectional area πr^2 of the aerosol particle of radius r. Obtain this dependence using physical insight and dimensional analysis. Take the dimensionless constant to be 1/8. Given mild pollution the average aerosol density at Darjeeling is $\rho_p = 5\ \mu\mathrm{g/m}^3$, and their average radius is 500 nanometres. What is the length L_p? Let the density of an individual aerosol particle be $\rho = 3\ \mathrm{g/cm}^3$. Note 1 μg = 10^{-9} kg and 1 nm = 10^{-9} m. 1pt

G.1	**Relative intensity and Visibility of Mt. Kanchenjunga and Mt. Everest :** Estimate the relative intensity of Mt Kanchenjunga and Mt Everest with the above level of pollution with respect to the reference value. Which, if any of these peaks will be visible from Darjeeling? Assume that pollution is uniform throughout the path the light travels.	1pt