General Grading Guidelines

When student's solutions are correct and s/he also show how solutions were obtained, the stduent gets full credit. The scheme oulined below is helpful if the student's answers are partially correct. Attention will be paid to the detailed solution so, if the final answer is correct but it is obtained by incorrect method(s) then no credit will be given. Alternative solutions may exist and will be given due credit.

Partial or full outcomes obtained for later sections in the problem which are incorrect solely because of errors being carried forward from previous sections, but are otherwise reasonable, will not be further penalized. For example a dimensioanlly wrong answer when carried forward will not get any credit in the subsequent sections. A numerically wrong evaluation when carried forward will get credit in subsequent sections unless the numerical answer is patently wrong (e.g. the value of g is 981 m/sec^2!)

Incorrect or no labeling of an axis is penalized by -0.1 points

The numerical answer (i) must be correct to +/- 10% AND (ii) must respect significant figures.

It maybe noted that NO micro-marking scheme takes care of all contingencies. A certain amount of discretion rests with and a certain level of judgement is invested in the academic committee.

The Stern-Gerlach Experiment: THE SOLUTION[1]

A.1 Speed of the Silver Atoms: 0.5pt

We employ the equipartition theorem. Let $\overline{v^2}$ be the mean square speed of the silver atoms in the oven kept at 1200 K. Then

$$\frac{m\overline{v^2}}{2} = \frac{3k_BT}{2}$$

where k_B is the Boltzmann constant. This yields the root mean square speed to be 5.255×10^2 m·s^{-1}.

[0.5]

[1] H. S. Mani (former Director, HRI, Prayagraj) and Gautam Datta (DAIICT, Gandhinagar) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group, and the International Board are gratefully acknowledged.

B.1 **The Basic Expression** 2pt

The length l_1 is irrelevant and will not be part of the expression.
The magnitude of the acceleration a of the silver atoms in the region defined by l_2 is

$$a = \frac{\mu_s}{m}\frac{dB}{dx}$$

[0.4]

and it will be either in the +x or -x direction. At the same time it has a constant horizontal velocity v_z. It traverses the region l_2 in time l_2/v_z. Thus after traversing the inhomogeneous region the deflection in say the +x direction is

$$\delta_1 = \frac{1}{2}\frac{\mu_s}{m}\frac{dB}{dx}\frac{l_2^2}{v_z^2}$$

[0.6]

For the remaining part of the flight the atom will have a constant hoirzontal speed v_z and a constant vertical speed $v_{x0} = (\mu_s dB/dx)(l_2/mv_z)$. On account of the v_x component the atom will acquire an additional deflection

$$\delta_2 = l_3 v_{x0}/v_z$$

This yields

$$\delta_2 = l_3 l_2 \frac{\mu_s}{m v_z^2}\frac{dB}{dx}$$

[0.4]

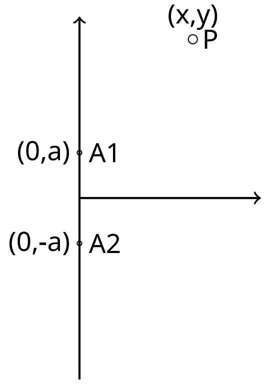
The total deflection in the +x direction is $\delta_1 + \delta_2$. The splitting seen on the screen in this idealized case is twice this amount, e.g. 2 $(\delta_1 + \delta_2)$. Thus we obtain

$$\Delta x = 2\frac{\mu_s}{m}\frac{dB}{dx}\frac{l_2}{v_z^2}(l_2/2 + l_3)$$

-0.3 if factor of 2 is missing.

[0.6]

C.1 **The Inhomogeneous Magnetic Field:** 1.5pt



Let $\vec{A_1P} = \vec{r}_1 = x\hat{i} + (y - a)\hat{j}$ and $\vec{A_2P} = \vec{r}_2 = x\hat{i} + (y + a)\hat{j}$. This gives for $\vec{B}(x, y)$

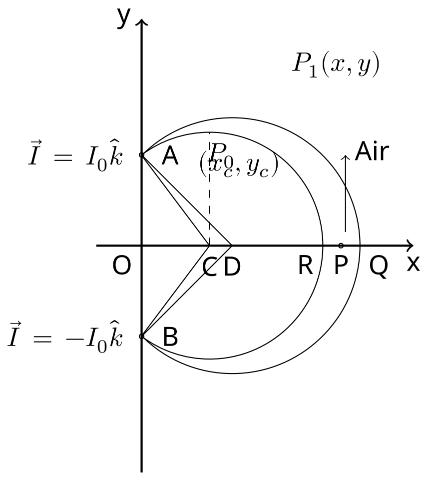
$$\frac{\mu I_0}{2\pi}\left[\frac{\hat{k}\times(x\hat{i} + (y-a)\hat{j}}{r_1^2} - \frac{\hat{k}\times(x\hat{i} + (y+a)\hat{j}}{r_2^2}\right] \tag{1}$$

[0.4+0.4]

$$= \frac{\mu I_0}{2\pi r_1^2 r_2^2}\left[(x\hat{j} - (y-a)\hat{i})(x^2 + (y+a)^2) - (x\hat{j} - (y+a)\hat{i})(x^2 + (y-a)^2)\right]$$

$$= \frac{\mu I_0\, a}{\pi r_1^2 r_2^2}\left[2xy\hat{j} + (x^2 - y^2 + a^2)\hat{i}\right] \tag{2}$$

[0.7]



Writing the final expression as any correct function of x and y will get full marks. If collecting all the terms component-wise not done, then penalize by -0.1. If an error has been made in simplification then penalize by -0.1.

C.2 **Direction at point R:** Field at the point R ($(x_c+\sqrt{x_c^2+a^2},0)$) is given by substituting $y=0$. On simple inspection the $\hat{j}$ component vanishes. Thus $\vec{B}(x,0)\ \propto\ \hat{i}$ — 0.5pt

[0.2]

Direction at point P_0: Field at P_0 ($(x_c, y_c=(x_c^2+a^2)^{1/2})$) is given, using Eq.(2)

$$\frac{\mu I_0}{\pi r_1^2 r_2^2}(2x_c(x_c^2+a^2)^{1/2}\hat{j}\,+\,(x_c^2\,-\,x_c^2\,-a^2\,+a^2)\hat{i})$$

The $\hat{i}$ component is zero. Thus $\vec{B}(x_c,(x_c^2+a^2)^{1/2})\ \propto\ \hat{j}$

[0.3]

First Alternative Solution
We can show in general that the field at **any** point on the circle will be radial (i.e. normal to the circle). We will confine our discussion to the z=0 plane. Consider a point (x_c, y) with radius $\sqrt{x_c^2\,+\,a^2}$. The equation of a circle with $(x_c, 0)$ as centre and $\sqrt{x_c^2\,+\,a^2}$ as radius is

$$(x\,-\,x_c)^2\,+\,y^2\,=\,x_c^2\,+\,a^2$$

or

$$x^2\,-\,2\,x\,x_c\,+\,y^2\,=\,a^2 \tag{3}$$

If at the point (x_c, y_c) the magnetic field is along $\hat{j}$ then the component along $\hat{i}$ is zero. $(x_c,0)$ is identified with the point C on the figure. The point y_c is then,

$$x_c^2\,-y_c^2\,+\,a^2\,=0$$

or

$$y_c^2\,=\,x_c^2\,+\,a^2 \tag{4}$$

Now consider a line joining C $(x_c,0)$ to any point $P_C(x,y)$ lying on the circle given by eq.(3). The radial vector is $\vec{CP}_C\,=\,(x-x_c)\hat{i}\,+\,y\hat{j}$. The magnetic field at P_C is

$$\propto\ \vec{B}(x,y,0)\,=\,\left(\frac{\mu I_0}{\pi}\right)\left(2xy\hat{j}\,+\,(x^2\,-\,y^2\,+a^2)\hat{i}\right)$$

To show that they are in the same direction, we evaluate the cross product, $\vec{CP}_C\times\vec{B}$. The cross product is proportional to $\hat{n}$ which is a unit vector along the direction which is normal to both CP_C and $\vec{B}$ and is along $\hat{k}$

$$\vec{CP}_C\times\vec{B}\ \propto\ (2\,x\,y(x\,-\,x_c)\,-\,y(x^2\,-\,y^2\,+a^2))\,\hat{k}$$

which simplifies to

$$y(x^2\,-2x\,x_c\,+\,y^2\,-\,a^2)\hat{k}$$

Using eq.(3), this is zero, proving the result.

C.2 (cont.)

Second Alternative Solution
To show that the field lines are radial over the circe one may merely show the proportionality of the components of the field and the radius vector. The radius vector is $(x-x_c)\hat{i}+y\hat{j}$ while the magnetic field is proportional to $(x^2-y^2+a^2)\hat{i}+2xy\hat{j}$. Thus

$$\frac{y}{2xy} = \frac{1}{2x}$$

and

$$\frac{x-x_c}{x^2-y^2+a^2} = \frac{1}{2x}$$

The last step is obtained by observing that the equation of the circle is $(x-x_c)^2+y^2=x_c^2+a^2$.

C.3 Field in the airgap because of the argument presented in the problem continues to be given by Eq.(2). So the field ($y = 0$), is again — 0.5pt

$$\vec{B} = \frac{\mu I_0\, a}{\pi(x^2\,+\,a^2)}\hat{i}$$

[0.5]

D.1 The force F_x on a magnetic dipole along the x- direction is — 0.5pt

$$F_x = -\mu_s\frac{\partial B_x}{\partial x} = \frac{\mu_s\mu\, I_0}{\pi}\times\frac{2\,a\,x}{(x^2\,+\,a^2)^2} \quad (5)$$

[0.5]

E.1 2.0pt

$$\frac{\mu}{\mu_0} = 10^4; \qquad a = 6.00\times 10^{-3} m; \qquad OC = 6.00\times 10^{-3} m; \qquad OD = 8.00\times 10^{-3} m;$$

and

$$I_0 = 2.00\, A$$

and so at the midpoint P,

$$y = 0;$$
$$x_P = OP = ((1+\sqrt{2})\times .6 + 1.8)/2 = 1.624\times 10^{-2} m$$

[0.5]

where we have used $OD = .8\times 10^{-2} m$ and $DA = 10^{-2} m$. This gives for $B_x(x_P, 0)$

$$\frac{\mu}{\mu_0}\frac{\mu_0}{\pi}\frac{I_0\, a}{(x_P^2 + a^2)} = \frac{10^4\times 4\times 10^{-7}\times 2\times .6\times 10^{-2}}{(1.624^2 + .6^2)\times 10^{-4}}$$
$$= 0.16\, T$$

[1]

We also have

$$\left(\frac{\partial B_x}{\partial x}\right)_{x_P} = \frac{2\times x_p}{x_p^2 + a^2}\times B_x(x_P,0) = \frac{2\times 1.624\times 10^{-2}}{(1.624^2 + .6^2)\times 10^{-4}}\times .16 = 17.34\, T\cdot m^{-1}$$

[0.5]

F.1 **The magnetic moment of the silver atom:** 1.5pt

We use

$$\Delta x = \frac{2\mu_s}{m}\left(\frac{\partial B}{\partial x}\right)_{x_P}\frac{l_2}{v_z^2}\left(\frac{l_2}{2} + l_3\right)$$

to rewrite

$$\mu_s = \frac{m\Delta x}{2(\frac{\partial B_x}{\partial x})_{x_P}}\times\frac{1}{[\frac{l_2}{v_z^2}(\frac{l_2}{2} + l_3)]}$$

[0.5]

$$= \frac{1.8\times 10^{-25}\times 2\times 10^{-3}}{2\times 17.34}\times 10^6 = 1.04\times 10^{-23} J\cdot T^{-1}$$

[1]

G.1 **The spread in the line:** The two lines on the screen are separated symmetrically about the centre by Δx. So the upper (lower) line is at $\Delta x/2$ from the centre. From Part (2) 0.5pt

$$\Delta x/2 = \frac{\mu_s}{m}\frac{dB}{dx}\frac{l_2}{v_z^2}(l_2/2 + l_3)$$

This depends on the beam speed v_z. The spread in this speed leads to a consequent spread in the splitting.

$$\begin{aligned}\delta(\Delta x/2) &= |\frac{\partial \Delta x/2}{\partial v_z}|\delta v_z \\ &= 2(\Delta x/2)\frac{\delta v_z}{v_z} \\ &= 2(\Delta x/2) \times 0.2 \\ &= 0.04cm\end{aligned}$$

[0.3]

Hence the spread in the line from the centre is 0.1 - 0.04 = 0.06 cm to 0.1 + 0.04 = 0.14 cm.
1. Credit will also be given if 20% is interpreted as 10% on each side
2. Answer reported in terms of percentages receive full credit

[0.2]

H.1 **Error in the evaluation of the magnetic moment:** 0.5pt

From the previous part we have that the splitting ranges from 0.12 cm to 0.28 cm whereas earlier it was 0.2 cm. The relationship between the splitting and the magnetic moment is linear. So the magnetic moment ranges from (0.12/0.2) to (0.28/0.2) the original value. This yields 0.62 $\times$ 10^{-23} J·T^{-1} to 1.46 $\times$ 10^{-23} J·T^{-1}. The total spread is 0.84 $\times$ 10^{-23} J·T^{-1} about the mean value of 1.04 $\times$ 10^{-23} J·T^{-1}

[0.3]

or in other words

$$\mu_s = 1.04 \pm 0.42\,\text{J}\cdot\text{T}^{-1}$$

[0.2]

A Mechanical Model for Phase Transitions[1]

General Grading Guidelines

When student's solutions are correct and s/he also show how solutions were obtained, the stduent gets full credit. The scheme oulined below is helpful if the student's answers are partially correct. Attention will be paid to the detailed solution so, if the final answer is correct but it is obtained by incorrect method(s) then no credit will be given. Alternative solutions may exist and will be given due credit.

Partial or full outcomes obtained for later sections in the problem which are incorrect solely because of errors being carried forward from previous sections, but are otherwise reasonable, will not be further penalized. For example a dimensioanlly wrong answer when carried forward will not get any credit in the subsequent sections. A numerically wrong evaluation when carried forward will get credit in subsequent sections unless the numerical answer is patently wrong (e.g. the value of g is 981 m/sec^2!)

Incorrect or no labeling of an axis is penalized by -0.1 points

The numerical answer (i) must be correct to +/- 10% AND (ii) must respect significant figures.

It maybe noted that NO micro-marking scheme takes care of all contingencies. A certain amount of discretion rests with and a certain level of judgement is invested in the academic committee.

A.1 (0.5 pt)
Equations of motion

The radial component F_r yields:

$$mR\dot{\theta}^2 = N - mg\cos(\theta) - mR\sin^2(\theta)\omega^2 \quad (1)$$

[0.2]

The tangential component F_θ

$$mR\ddot{\theta} = mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) - \mathrm{sgn}\left(\dot{\theta}\right) kN \quad (2)$$

OR

$$mR\ddot{\theta} = mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) - fkN \quad (f=1) \quad (3)$$

[0.3]

No points if equations not written using radial and tangential components.

[1] Sitikantha Das (IIT Kharagpur) and Pramendra Ranjan Singh (Principal, Narayan College, J.P. University) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group, and the International Board are gratefully acknowledged.

B.1 (1.0 pt)
Equilibrium angle(s)

We set $k = 0$ in the equation for the tangential component of the force. Thus

$$mR\ddot{\theta} = mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) \quad (4)$$

For equilibrium we set $\ddot{\theta}_0 = 0$ in the above equation. Then $\theta_0 = 0$ is an equilibrium angle for all values of ω

[0.4]

The other values are given by

$$\cos\theta_0 = \frac{g}{\omega^2 R} = \frac{\omega_c^2}{\omega^2} \quad (5)$$

[0.3]

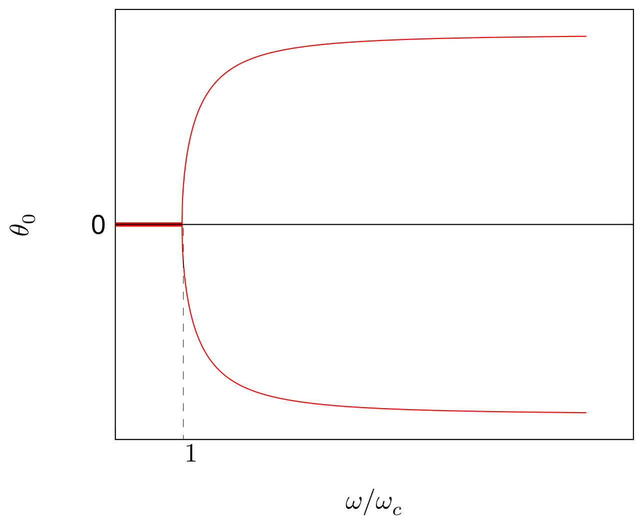
$$\theta_0 = \pm\left|\cos^{-1}\frac{\omega_c^2}{\omega^2}\right| \quad (6)$$

[0.1]

with values of θ_0 between -$\pi/2$ to $\pi/2$. The $\pm$ indicates that there are two equivalent positions. () The bead could rise on either side of the axis shown in the figure depicted in the problem. Note that for $\omega < \omega_c$, Eq. (5) implies $\cos\theta_0 > 1$. This is clearly unphysical. A little reflection will convince us that $\theta_0 = 0$ for $\omega < \omega_c$.

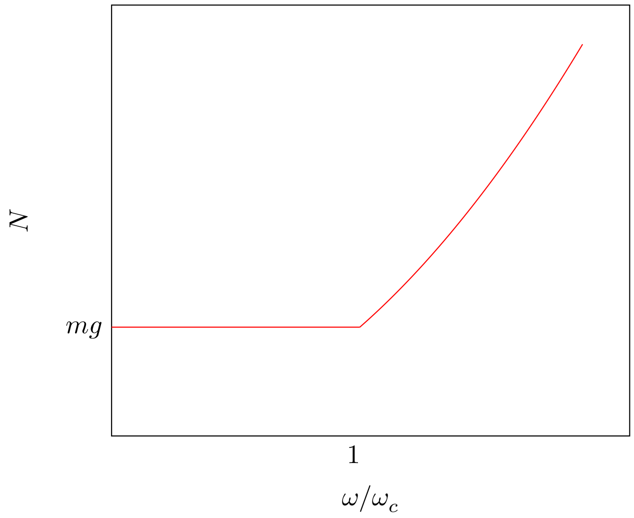
[0.2]

B.2 (0.5 pt)
Sketch of θ_0.



[shape correct: 0.2]
[only if both branches: 0.3]

B.3 (0.5 pt)
Sketch of the magnitude of the normal reaction



[$\omega < \omega_c$: **0.2**]
[$\omega > \omega_c$: **0.3**]

If the shape of the plot is wrong, in that case, the following would be used. If in the detailed work, it is shown that $N = mg$ for $\omega < \omega_c$, **0.1** points would be provided. If it is shown that $N = m\omega^2 R$ for $\omega \geq \omega_c$, **0.1** points would be provided.

B.4 (1.0 pt)
The potential $V(\theta)$

Solution 1: Using direct integration
Given that

$$F_\theta = -\frac{1}{R}\frac{dV(\theta)}{d\theta} \tag{7}$$

and taking $V(\theta = 0) = 0$, we obtain on integrating Eq. (4) that

$$-R\int_0^\theta F_\theta d\theta = \int_0^V dV = V - 0$$

[0.3]

the left hand side is

$$\begin{aligned} -R\int_0^\theta F_\theta d\theta &= \frac{-m\omega^2R^2}{2}\int_0^\theta \sin(2\theta) + mgR\int_0^\theta \sin(\theta)d\theta \\ &= \frac{m\omega^2R^2(\cos(2\theta)-1)}{4} - mgR(\cos(\theta)-1) \end{aligned} \tag{8}$$

[0.4]

Noting that $cos(2(\theta)-1) = -2\sin^2(\theta)$ and $\omega_c^2 = g/R$ we obtain

$$V(\theta) = mgR\left[(1-\cos\theta) - \frac{\omega^2}{2\omega_c^2}\sin^2\theta\right] \tag{9}$$

[0.3]

We can also verify the above equation by substitution into Eq. (7)
$P = mgR$
$Q = -mgR$
$S = -\frac{\omega^2 mgR}{2\omega_c^2}$
Solution 2: Differentiating $V = P + Q\cos(\theta) + S\sin^2(\theta)$
$F_\theta = mR\sin\theta\cos\theta\omega^2 - mg\sin\theta = -\frac{1}{R}\frac{dV(\theta)}{d\theta} = \frac{Q}{R}\sin(\theta) - 2\frac{S}{R}\sin\theta\cos\theta$

[0.3]

Comparing, we get,
$Q = -mgR$
$S = -\frac{\omega^2 mgR}{2\omega_c^2}$

[0.2,0.2]

Also, as $V(0) = 0$, we have $P + Q = 0$. Hence, $P = mgR$

[0.3]

B.5 (1.0 pt)
The coefficients

We use the expansions for the trigonmetric functions $\sin(\theta)$ and $\cos(\theta)$ in Eq. (10). We shall keep terms upto and inculding order θ^4. Thus

$$\begin{aligned} V(\theta) &\approx mgR\left[1 - 1 + \theta^2/2 - \theta^4/24 - \frac{\omega^2}{2\omega_c^2}(\theta - \theta^3/6)^2\right] \\ &\approx \frac{mgR}{2}\left[1 - \frac{\omega^2}{\omega_c^2}\right]\theta^2 + \frac{mgR}{6}\left[\frac{\omega^2}{\omega_c^2} - \frac{1}{4}\right]\theta^4 \end{aligned}$$

Thus

$$a(\omega) = \frac{mgR}{2}(1 - \frac{\omega^2}{\omega_c^2})$$

[0.5]

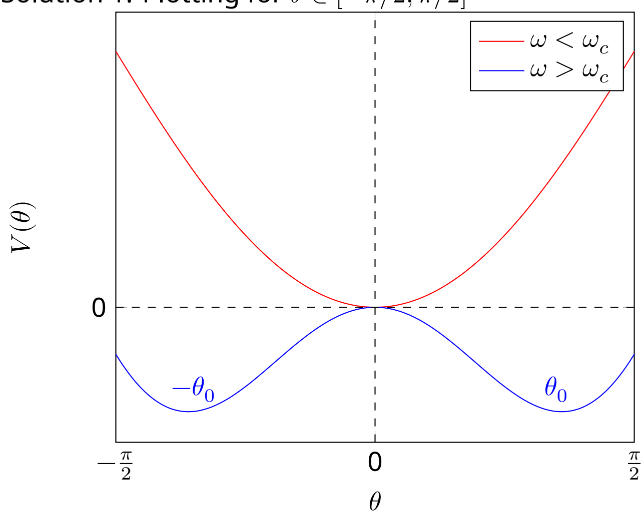
$$b(\omega) = \frac{mgR}{6}(\frac{\omega^2}{\omega_c^2} - \frac{1}{4})$$

[0.5]

Note: no penalty if the 1/4 term is missed.
One observes that if one incorrectly expands $\sin\theta \approx \theta$, in that case, only $a(\omega)$ will turn out to be correct.

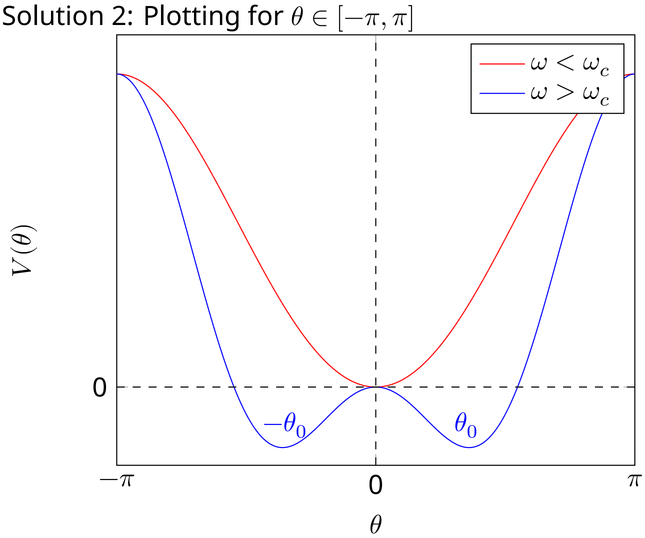
B.6 (1.0 pt)
Representative plots of the potential

Solution 1: Plotting for $\theta \in [-\pi/2, \pi/2]$



[$\omega < \omega_c$: **0.5**]
[$\omega > \omega_c$: **0.5**]

Solution 2: Plotting for $\theta \in [-\pi, \pi]$



[$\omega < \omega_c$: **0.5**]
[$\omega > \omega_c$: **0.5**]

B.7 (1.0 pt)

Bead analogues

Solution 1:

For $\omega \to \omega_c^+$, θ_0 is close to zero. Hence on expanding the cosine term in Eq. (5),

$$1 - \frac{\theta_0^2}{2} = \frac{\omega_c^2}{\omega^2}$$

$$\theta_0 = \pm\sqrt{2}\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2} \tag{10}$$

Also note from Eq. (5) that as $\omega \to \infty$, $\theta_0 \to \pm\pi/2$. This plot also has an analogue in phase transition. The magnetization $\mathcal{M}$ goes to zero as T goes to T_c in a similar fashion. Thus the role of $\mathcal{M}$ is played by θ_0 and temperature is inversely related to ω. Increasing temperature is equivalent to decreasing ω. Summarizing,

$$\mathcal{M} \longrightarrow \theta$$

[0.4]

$$T_c \longrightarrow 1/\omega_c^2$$

$$T/T_c \longrightarrow \omega_c^2/\omega^2$$

[0.4]

Equivalent value of β for bead is = 1/2.

[0.2]

Solution 2:

For $\omega > \omega_c$, $\cos\theta_0 = \omega_c^2/\omega^2$. Hence on writing $\sin^2\theta_0 = 1 - \cos^2\theta_0$ and substituting the value of $\cos\theta_0$, one gets $\sin\theta_0 = (1 - \frac{\omega_c^4}{\omega^4})^{1/2}$. This plot also has an analogue in phase transition. The magnetization $\mathcal{M}$ goes to zero as T goes to T_c in a similar fashion. Thus the role of $\mathcal{M}$ is played by $\sin\theta_0$ (or equivalently θ_0 in the small angle limit) and temperature is inversely related to ω^4. Increasing temperature is equivalent to decreasing ω. Summarizing,

$$\mathcal{M} \longrightarrow \sin\theta$$

[0.4]

$$T_c \longrightarrow 1/\omega_c^4$$

$$T/T_c \longrightarrow \omega_c^4/\omega^4$$

[0.4]

Equivalent value of β for bead is = 1/2.

[0.2]

[Note: The critical exponent is $1/2$ in our case and also in Landau theory. However experimentally and in more elaborate theories the exponent of vanishing magnetization is $1/3$].

B.8 (1.0 pt)
Oscillation frequency

The frequency of oscillation Ω_0 of the bead about the "equilibrium" position θ_0 is

$$\Omega_0 = \frac{1}{R}\sqrt{\frac{V''(\theta)}{m}}$$

We take the second order derivative of the potential as given in Eq. (10)

$$V''(\theta) = mgR\cos\theta\left[1 - \frac{\omega^2}{\omega_c^2}\cos\theta\right] + mgR\,\frac{\omega^2}{\omega_c^2}\,\sin^2\theta \tag{11}$$

For $\theta = \theta_0 = \pm\cos^{-1}\left(\omega_c^2/\omega^2\right)$

$$V''(\theta_0) = mgR\,\frac{\omega^2}{\omega_c^2}\left(1 - \frac{\omega_c^4}{\omega^4}\right) > 0 \qquad \text{if } \omega > \omega_c \tag{12}$$

For $\omega < \omega_c$, $\theta_0 = 0$, and we obtain from Eq. (12) that

$$\Omega_0 = (\omega_c^2 - \omega^2)^{1/2} \tag{13}$$

[0.5]

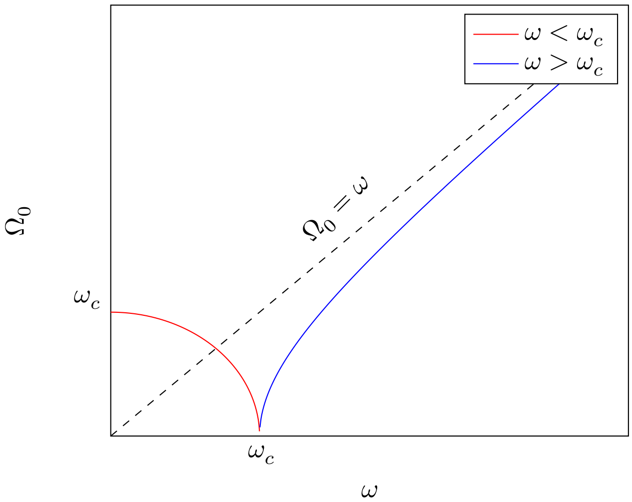
Similarly for $\omega > \omega_c$, using Eq. (13) we obtain

$$\Omega_0 = \omega\left(1 - \frac{\omega_c^4}{\omega^4}\right)^{1/2} \tag{14}$$

[0.5]

No credit will be provided is small angle approximation of $V(\theta)$ is used.

B.9 (1.0 pt)
Sketch of Ω_0



[$\omega < \omega_c$: **0.5**]
[$\omega > \omega_c$: **0.5**]

In the case of wrong expression of Ω_0 derived in the previous part, marks would be awarded based on the plot of expression obtained, and physicality of the plots.

C.1 (1.0 pt)
Condition for equilibrium angles

We substitute the expression for the normal reaction (Eq.(1)) in the angular part (Eq.(3)) to obtain

$$mR\ddot{\theta} \;=\; mR\sin(\theta)\cos(\theta)\omega^2 - mg\sin(\theta) - fk(mg\cos(\theta) + mR\sin^2(\theta)\omega^2 + mR\dot{\theta}^2)$$

Noting that $\omega_c^2 = g/R$ and rearranging terms we have

$$\ddot{\theta} \;=\; \omega_c^2\left[(\sin(\theta))\,(\cos(\theta) - fk\sin(\theta))\left(\frac{\omega}{\omega_c}\right)^2 - \sin(\theta) - fk\cos(\theta) - fk\left(\frac{\dot{\theta}}{\omega_c}\right)^2\right]$$

[0.2]

At equilibrium, $\dot{\theta} = 0,\ \ddot{\theta} = 0$ and $f = \text{sgn}\left(\dot{\theta}\right) = \pm 1$ depending on how this equilibrium was attained, iė., depending on the value of $\dot{\theta}$ just before equilibrium was attained. Thus we obtain the expression for the equilibrium angle θ_0,

$$\boxed{\sin(\theta_0)\,(\cos(\theta_0) - fk\sin(\theta_0))\left(\frac{\omega}{\omega_c}\right)^2 = \sin(\theta_0) + fk\cos(\theta_0) \text{ with } \theta_0 \in (-\pi/2, \pi/2)}$$

[0.4]

For f = 1 and $k = \tan(\alpha)$ we may express the above as

$$\begin{aligned}\left(\frac{\omega}{\omega_c}\right)^2 &= \frac{\sin(\theta_0) + \tan(\alpha)cos(\theta_0)}{\sin(\theta_0)(\cos(\theta_0) - \tan(\alpha)\sin(\theta_0))} \\ &= \frac{\tan(\theta_0 + \alpha)}{\sin(\theta_0)} \qquad (15)\end{aligned}$$

[0.4]

In case of algebraic error leading to $x = \theta_0 - \alpha$, only 0.1 points would be deducted.

C.2 (0.5 pt)
Representative values for θ_0
We are given the expansions for the trignometric functions in the problem. We notice that the coefficient of the opposing force k is small (=0.05). Thus $k = \alpha$. We then have

$$\sin(\theta_0) \approx \theta_0$$

$$\tan(\theta_0 + \alpha) \approx \theta_0 + \alpha$$

[0.2]

Thus

$$\left(\frac{\omega}{\omega_c}\right)^2 \approx 1 + \frac{k}{\theta_0}$$

Simple calculations yield
(a) θ_0 = -0.07 radians
(b) θ_0 = -0.1 radians

[0.3]

The plot will no longer be symmetric.

General Grading Guidelines

When student's solutions are correct and s/he also show how solutions were obtained, the stduent gets full credit. The scheme oulined below is helpful if the student's answers are partially correct. Attention will be paid to the detailed solution so, if the final answer is correct but it is obtained by incorrect method(s) then no credit will be given. Alternative solutions may exist and will be given due credit.

Partial or full outcomes obtained for later sections in the problem which are incorrect solely because of errors being carried forward from previous sections, but are otherwise reasonable, will not be further penalized. For example a dimensioanlly wrong answer when carried forward will not get any credit in the subsequent sections. A numerically wrong evaluation when carried forward will get credit in subsequent sections unless the numerical answer is patently wrong (e.g. the value of g is 981 m/sec^2!)

Incorrect or no labeling of an axis is penalized by -0.1 points

The numerical answer (i) must be correct to +/- 10% AND (ii) must respect significant figures.

It maybe noted that NO micro-marking scheme takes care of all contingencies. A certain amount of discretion rests with and a certain level of judgement is invested in the academic committee.

Maxwell, Rayleigh and Mount Everest: THE SOLUTION[1]

Oscillation of the electron cloud:

A.1 (0.5 pt)
$\vec{E}(t)$ is the electric field at the location of the molecule. The equation of motion of the charge in the absence of $\vec{E}(t)$ would be

$$\ddot{y} = -\omega_0^2 y, \tag{1}$$

and under forced oscillations

$$\ddot{y} = -\omega_0^2 y - \frac{qE_0}{m}\cos\omega t. \tag{2}$$

[0.5]

[a sign mistake or a term missing −0.3]

[1] Amitabh Virmani (CMI, Chennai) and A. C. Biyani (retired Govt. Nagarjuna P.G. College of Science. Raipur) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group, and the International Board are gratefully acknowledged.

A.2 (0.5 pt)
In equation (2) we substitute $y = y_0 \cos \omega t$ to obtain

$$-\omega^2 y_0 = -\omega_0^2 y_0 - \frac{qE_0}{m}. \quad (3)$$

[0.2]

This implies that the amplitude of oscillation is

$$y_0 = \frac{qE_0/m}{\omega^2 - \omega_0^2}. \quad (4)$$

[0.3]

[a sign mistake or a term missing $-$0.1]

A.3 (0.5 pt)
Since y is the separation between the positive and negative charge clouds, the magnitude $p(t)$ of the dipole moment is

$$p(t) = qy(t) \approx \frac{q^2 E_0}{m\omega_0^2} \cos \omega t. \quad (5)$$

[0.5]

[sign mistake $-$0.1]
[answer without approximation $-$0.2]

A.4 (0.5 pt)
We model the atom as a stationary positive point charge q surrounded by a spherical negative charge cloud of total charge $-q$, radius r and mass m. Now let the charge cloud be displaced by a small distance y. The electrostatic force on the electron cloud by the central positive charge is (see Figure)

$$\vec{F}_{\rm el} = m\ddot{y}\hat{y} = -\frac{q^2}{4\pi\epsilon_0 r^3} y\hat{y} = -m\omega_0^2 y\hat{y} \quad (6)$$

[0.4]

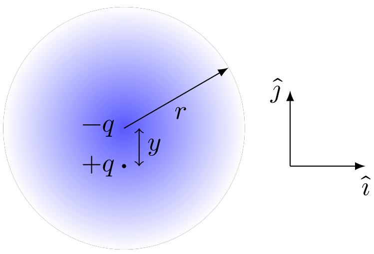


Figure 1. Model of the atom with a central positive charge and a displaced spherical electron cloud of radius r

Thus, the natural frequency of oscillation is

$$\omega_0 = \frac{q}{\sqrt{4\pi\epsilon_0 m r^3}} \quad (7)$$

[0.1]

Power radiated:

B.1 (1 pt)

Dimension of power is

$$[s] = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3} \tag{8}$$

[0.1]

Dimension of dipole moment is

$$[p_0] = \mathrm{C} \cdot \mathrm{m}. \tag{9}$$

[0.1]

We are using SI units. C stands for Coloumb. Dimension of ω is

$$[\omega] = \mathrm{s}^{-1}. \tag{10}$$

[0.1]

Dimension of ϵ_0 is

$$[\epsilon_0] = \mathrm{C}^2 \cdot \mathrm{N}^{-1} \cdot \mathrm{m}^{-2} = \mathrm{C}^2 \cdot \mathrm{kg}^{-1} \cdot \mathrm{m}^{-3} \cdot \mathrm{s}^2. \tag{11}$$

[0.1]

Dimension of speed of light c is

$$[c] = \mathrm{m} \cdot \mathrm{s}^{-1}. \tag{12}$$

[0.1]

Let us take the ansatz $s = p_0^\alpha \omega^\beta \epsilon_0^\gamma c^\delta$, i.e.,

$$[s] = [p_0]^\alpha [\omega]^\beta [\epsilon_0]^\gamma [c]^\delta. \tag{13}$$

[0.1]

We get four equations for the four variables,

$$\alpha + 2\gamma = 0, \qquad \gamma = -1, \tag{14}$$

$$-\beta + 2\gamma - \delta = -3 \qquad \alpha - 3\gamma + \delta = 2. \tag{15}$$

This gives

$$\alpha = 2, \qquad \beta = 4, \qquad \gamma = -1, \qquad \delta = -3. \tag{16}$$

Implying,

$$s = k\frac{p_0^2 \omega^4}{\epsilon_0 c^3} \tag{17}$$

[0.4]

B.2 (0.2 pt)
We have

$$s = \frac{1}{12\pi}\frac{p_0^2\omega^4}{\epsilon_0 c^3} = \frac{1}{12\pi}\frac{q^2 y_0^2\omega^4}{\epsilon_0 c^3} = \frac{1}{12\pi}\frac{q^4 E_0^2}{m^2\epsilon_0 c^3}\frac{\omega^4}{\omega_0^4}. \tag{18}$$

[0.2]

Attenuation of the Intensity $I(x)$:

C.1 (1 pt)
Recall that the intensity is the power incident per unit area. Consider a horizontal column of the atmosphere of cross-sectional area A and length Δx. Let the incident intensity be $I(x)$. Let the transmitted intensity be $I(x+\Delta x)$. The drop in the intensity is due to the scattering of light by the air molecules. If the number density of air molecules is n_0 then the total power radiated per unit volume is $n_0 s$. Therefore,

$$I(x)A - I(x+\Delta x)A = n_0 s(A\Delta x). \tag{19}$$

[0.8]

This gives

$$-\frac{dI}{dx} = n_0 s. \tag{20}$$

[0.2]

C.2 (0.5 pt)
Since $s \propto E_0^2$ and $I \propto E_0^2$ we have

$$-\frac{dI}{dx} = \frac{I}{L}, \tag{21}$$

[0.2]

where

$$L = \frac{6\pi\epsilon_0^2 m^2 c^4}{n_0 q^4}\left(\frac{\omega_0}{\omega}\right)^4. \tag{22}$$

[0.2]

The solution to the differential equation as a function of x is

$$I(x) = I_0 e^{-x/L}, \tag{23}$$

with L given above.

[0.1]

C.3 (0.3 pt)
Substituting the numbers we find

$$L = \frac{6\pi\epsilon_0^2 m^2 c^4}{n_0 q^4}\left(\frac{\omega_0}{\omega}\right)^4. \tag{24}$$

$$L \approx 130\ \text{km}. \tag{25}$$

[points are for numerical calculation 0.3]

Height H' of the Mountains as seen by an observer:

D.1 (2 pt)

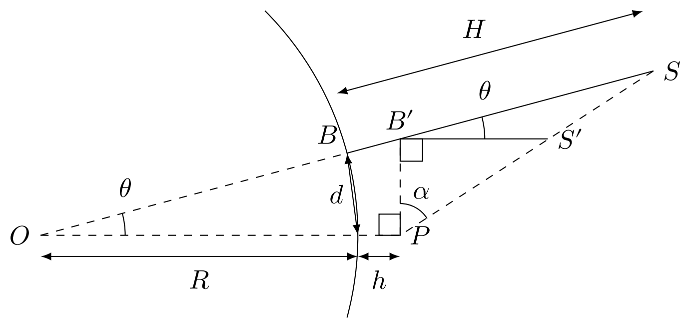


Figure 2. Great circle on which lie the mountain BS at height H and the observer P at height h. The figure is not to scale.

[0.7]

In $\Delta OPB'$,

$$OB' = OP\,\text{sec}(\theta) = (R+h)\,\text{sec}(\theta) \tag{26}$$

Now, $\angle OSP = \angle B'SS'$ and $\angle SOP = \angle SB'S'$, hence ΔOSP and $\Delta B'SS'$ are similar. Thus

$$\frac{B'S'}{OP} = \frac{B'S}{OS} = \frac{OS - OB'}{OS} = 1 - \frac{OB'}{OS}. \tag{27}$$

Noting that $B'S' = H', OP = R+h, OS = R+H$ and using Eq. (26), we obtain

$$\frac{H'}{R+h} = 1 - \frac{(R+h)\,\text{sec}(\theta)}{R+H} \tag{28}$$

Or

$$H' = R + h - \frac{(R+h)^2}{R+H}\,\text{sec}(\theta). \tag{29}$$

[0.8]

Noting that $\cos(\theta) \approx 1 - \theta^2/2$ and $\theta = d/R$ we get,

$$H' \simeq R + h - \frac{(R+h)^2}{R+H}\left(1 + \frac{d^2}{2R^2}\right). \tag{30}$$

Alternative answers such as

$$H' = H - h - \frac{d^2}{2R} \tag{31}$$

are given credit.

The numerical values are H' = 6096 m for Mt Kanchenjunga and H' = 4534 m for Mt Everest.

[0.5]

E.1 (1 pt)
In Eq.(23) I_0 represents the intensity of the source which would have been perceived by an observer at that location if attenuation effects were absent. If the power of the source is taken to be P_0, then $I_0 = P_0/4\pi d^2$ for the location at distance d.

$$I = \frac{P_0}{4\pi d^2} \exp\left[-\frac{d}{L}\right] \tag{32}$$

[points only if $1/d^2$ is recognised 0.5]

The relative intensity of Mt Everest as seen from Darjeeling would be

$$\frac{I_{\text{Everest}}}{I_{\text{Kanchenjunga}}} = \frac{d^2_{\text{Kanchenjunga}}}{d^2_{\text{Everest}}} \exp\left[-\frac{d_{\text{Everest}}}{L} + \frac{d_{\text{Kanchenjunga}}}{L}\right] \tag{33}$$

$$= 0.093 \tag{34}$$

[0.3]

Yes, Mt Everest is visible.

[0.2]

Attenuation length L_p due to aerosol pollution :

F.1 (1 pt)
From the information given in the problem we have

$$L_p = \frac{1}{8n\pi r^2} \tag{35}$$

[0.3]

$$n = \frac{\rho_p}{m} \tag{36}$$

$$m = \frac{4\pi}{3} r^3 \rho \tag{37}$$

This yields

$$L_p = \frac{r\rho}{6\rho_p} = 50 \text{ km}, \tag{38}$$

[0.2 (expression)]
[0.5 (evaluation)]

where,

$$r = 500 \times 10^{-9} \text{ m}, \qquad \rho = 3 \times 10^3 \text{ kg/m}^3, \qquad \rho_p = 5 \times 10^{-9} \text{ kg/m}^3. \tag{39}$$

Relative intensity and Visibility of Mt. Kanchenjunga and Mt. Everest :

G.1 (1 pt)
The new relation for the intensity attenuation is

$$I = \frac{P_0}{4\pi d^2} \exp\left[-\frac{d}{L} - \frac{d}{L_p}\right]. \tag{40}$$

For Mt Kanchenjunga

$$\frac{I_K}{I_{\text{ref}}} = \exp\left[-\frac{d_K}{L_p}\right] = 0.22. \tag{41}$$

[0.3 (expression) + 0.1 (numerical answer)]

The drop in intensity is to 22 % of the reference value. Mt Kanchenjunga will be visible from Darjeeling. For Mt Everest

[0.1]

$$\frac{I_E}{I_{\text{ref}}} = 0.093 \exp\left[-\frac{d_E}{L_p}\right] = 0.093 \times 0.033 = 0.003. \tag{42}$$

[0.3 (expression) + 0.1 (numerical answer)]

The drop in intensity is to 0.3 % of the reference value. Mt Everest will not be visible from Darjeeling.

[0.1]