Water-Hammer Effect

Introduction

This problem studies variations of fluid pressure caused by pressure waves in a flow pipe. Proposed tasks mainly deal with the water-hammer effect arising from both fast and slow closings of a flow-control valve in the pipe.

We consider only nonviscous liquids and liquid flows which are essentially one-dimensional. All pipes including their valves are assumed to be rigid, but liquids are not always considered to be incompressible. If a liquid element of volume V_0 at equilibrium under pressure P_0 is subjected to a change of pressure ΔP , the change of its volume ΔV is assumed to be proportional to ΔP so that

$$\Delta P = -B\,\frac{\Delta V}{V_0} \tag{1}$$

The constant of proportionality B represents the bulk modulus of the liquid. For water, take $\rho_0 = 1.0 \times 10^3$ kg/m^3 as its equilibrium density and $B = 2.2$ GPa.

Part A. Excess Pressure and Propagation of Pressure Wave (2.2 points)

In a uniform cylindrical pipe of length L, water is flowing steadily along the $+x$ direction with horizontal velocity v_0, density ρ_0, and pressure P_0. As shown in Fig. 1, the pipe is connected to a reservoir at a depth h and opens into the atmosphere at pressure P_a.

Suppose the flow-control valve T at the end of the pipe is then shut instantly so that the oncoming liquid element next to the valve suffers both a pressure change $\Delta P_\mathrm{s} \equiv P_1 - P_0$ and a velocity change $\Delta v = v_1 - v_0$ with $v_1 \leq 0$. This causes a longitudinal wave of excess pressure ΔP_s to travel upstream in the $-x$ direction with a speed of propagation c.

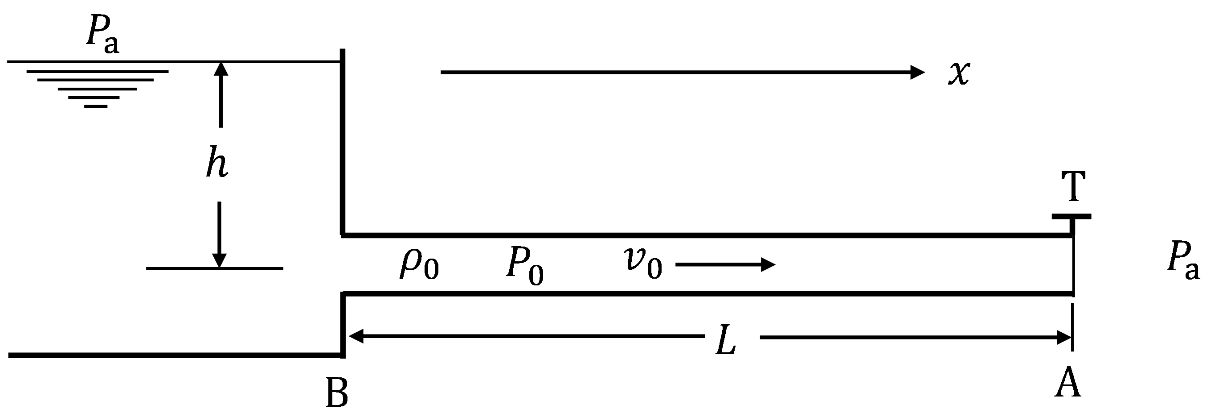

Fig. 1: Steady flow in a uniform pipe.

A.1	The excess pressure ΔP_s is related to the velocity change Δv by $\Delta P_\mathrm{s} = \alpha \rho_0 c \Delta v$. The speed of propagation c is given by $c = \beta + \sqrt{\gamma B/\rho_0}$. Find α, β, and γ.	1.6pt

A.2	Calculate values of c and ΔP_s for the case of water flow with $v_0 = 4.0$ m/s and $v_1 = 0$.	0.6pt

Part B. A Model for the Flow-control Valve (1.0 points)

Fig. 2 shows a model for control valve T and the liquid flow through it. The valve is taken to be a short section of length ΔL and inner radius R near the end A of the pipe. Its cone-shaped outlet has an orifice of radius r and opens into the atmosphere at pressure P_a. Effects of gravity on the efflux are to be neglected.

The liquid is to be regarded as incompressible and the flow as steady with liquid element at the valve inlet having velocity v_{in}, pressure P_{in}, and density ρ_0. In Fig. 2, stream lines and normal lines are drawn only as an aid for visualizing the flow pattern.

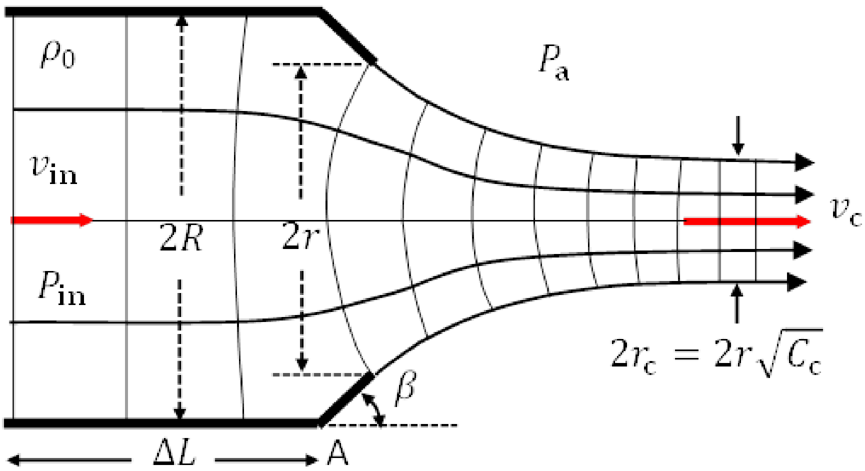

Fig. 2: Valve dimensions and contraction of jet.

It is known that, after leaving the valve into the atmosphere, the cross section of the flow will contract until it reaches a minimum where the stream lines are again parallel. At this point of minimum, the flow velocity is v_c and the cross section of the flow has a radius $r_c = r\sqrt{C_c}$. Here C_c, called the **contraction coefficient**, depends on the ratio r/R and the cone angle β as shown in Table 1.

r/R	$C_c(\beta = 45^\circ)$	$C_c(\beta = 90^\circ)$
0.00	0.746	0.611
0.20	0.747	0.616
0.30	0.748	0.622
0.40	0.749	0.631
1.00	1.000	1.000

Table 1. Contraction Coefficients for Orifices

B.1	Find the excess pressure $\Delta P_{\text{in}} = P_{\text{in}} - P_{\text{a}}$ at the valve inlet where the stream lines are parallel. Give your answer in terms of ρ_0, v_{in}, r, R, and C_{c}.	1.0pt

For all tasks in Part C and Part D, we consider the reservoir-pipe system in Fig. 1 and make the following assumptions:

- Speed of propagation c and density ρ_0 of liquid are given constants independent of flow velocity. The ambient atmospheric pressure P_{a} and the acceleration of gravity g are constant.
- Initially, the valve is fully open and the flow in the pipe is steady with fluid pressure P_0 and velocity v_0.
- As in Fig. 1 and Fig. 2, the pipe has length L and radius R. The valve T is a circular opening of variable radius r with angle $\beta = 90^\circ$ and its length ΔL is negligible so that the valve inlet is effectively at the end A of the pipe. Effects of gravity on the efflux are negligible.
- Liquid in the reservoir is quasi-static so that its pressure P_h near the pipe entrance B remains constant and we assume that the variation of fluid pressure across the pipe is negligible so that the flow is one-dimensional throughout the pipe.
- The model outlined in Part B may be used to determine the excess pressure $\Delta P_{\text{in}} = P_{\text{in}} - P_{\text{a}}$ at the valve inlet.

Part C. Water-Hammer Effect due to Fast Closure of Flow Control Valve (1.8 points)

Refer to the reservoir-pipe system in Fig. 1. When liquid flow in the pipe is obstructed by complete or partial closure of the valve, a pressure wave starts traveling upstream. It gets reflected at the reservoir end of the pipe and travels back to the valve and gets reflected there. Then another pressure wave is generated and the process just described is repeated. This causes a sequence of sudden pressure surges and dips for liquid element next to the valve and is referred to as **water-hammering**.

C.1	Refer to Fig. 1 and Fig. 2. Find the pressure P_0 and velocity v_0 of the steady flow in the pipe when valve T is fully open ($r = R$). Give answers in terms of ρ_0, g, h, and P_{a}.	0.6pt

C.2	Consider the same steady flow as in task C.1 with pressure P_0 and flow velocity v_0. Now, at $t = 0$, the valve is closed ($r = 0$) instantly. A pressure wave heads toward the reservoir with speed of propagation c. Take note $P_h = P_0 + \rho_0 g h$. Let $\tau = 2L/c$. What are the fluid pressure $P(t)$ and flow velocity $v(t)$ in the pipe when t is getting very close to each of the instants $\tau/2$ and τ?	1.2pt

Part D. Water-Hammer Effect due to Slow Closure of Flow Control Valve (5.0 points)

Consider again the same steady flow as in task C.1 with pressure P_0 and flow velocity v_0. Now we close the valve slowly and adopt a finite-step approach to simulate the closing process.

Starting at time $t = 0$, the instant reduction of the radius r of the valve (see Fig. 2) is carried out sequentially at a time interval $\tau = 2L/c$. Immediately after each instant reduction of radius, the flow in the valve region is approximated to be steady as in Part B. The pressure and velocity at the valve are then different from those of the rest of the flow in the pipe.

For each closing step n, its duration and the radius r_n of the valve opening are specified in Table 2 along with the symbols used to represent the corresponding fluid pressure P_n and flow velocity v_n at the valve.

closing step n	time interval of step n	ratio r_n/R	pressure at valve when $t=(n-1)\tau$	flow velocity at valve when $t=(n-1)\tau$
$n=0$	$t<0$	1.00	P_0	v_0
$n=1$	$0\le t<\tau$	0.40	P_1	v_1
$n=2$	$\tau\le t<2\tau$	0.30	P_2	v_2
$n=3$	$2\tau\le t<3\tau$	0.20	P_3	v_3
$n=4$	$3\tau\le t<4\tau$	0.00	P_4	$v_4=0$

Table 2. Valve closing steps

Take fluid density ρ_0 and speed of propagation c as constants. Let $n=0,1,2,3,4$. Define $\Delta P_n = P_n - P_0$ and $\Delta v_n = v_n - v_0$. Make sure to enforce the approximation $P_h = P_0$.

D.1 Obtain an equation which expresses $\Delta P_n/(\rho_0 c)$ in terms of $\Delta P_{n-1}/(\rho_0 c)$, v_{n-1}, and v_n. It must be valid for all steps $n>0$ specified in Table 2. For $n=1,2,3$, obtain also an equation which allows v_n to be computed if both v_{n-1} and $\Delta P_{n-1}/(\rho_0 c)$ are known. 3.0pt

D.2 Apply the result of task D.1 to water flow with $v_0 = 4.0$ m/s. Use the graph paper provided in the Answer Sheet to make all plots of ΔP versus $\rho_0 c v$. Be sure to draw lines and curves intersecting at points having coordinates which give the values of $\rho_0 c v_n$ and ΔP_n for steps $n=1,2,3,4$. On the plot, label each point of intersection $(\rho_0 c v_n, \Delta P_n)$ with the value of n to which it corresponds. From the graph, estimate values of $\rho_0 c v_n$ and ΔP_n (both in units of MPa) for $n=1,2,3,4$. 2.0pt

Ray tracing and generation of entangled light

Useful formula:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

Introduction

Let $\vec{E}$ represent the electric field, $\vec{H}$ the magnetic field, $\vec{D}$ the electric displacement, and $\vec{B}$ the magnetic induction. We have $\vec{D}=\epsilon_0\vec{E}+\vec{P}$, with $\vec{P}$ being the polarization of the medium and ϵ_0 being the permittivity of free space. Only nonmagnetic dielectric media are considered in this problem, hence $\vec{B}=\mu_0\vec{H}$, with μ_0 being the permeability of free space. The energy density and energy flow density associated with the electromagnetic field are given by $u_{em}=\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})$ and Poynting's vector $\vec{S}=\vec{E}\times\vec{H}$, respectively.

In homogeneous dielectric media, a monochromatic plane wave of light can be described by its angular frequency ω, wave vector $\vec{k}$, $\vec{D}$, and $\vec{B}$. According to Maxwell's equations, we have $\vec{k}\times\vec{E}=\omega\vec{B}$ and $\vec{k}\times\vec{H}=-\omega\vec{D}$. For such a wave, variations of $\vec{D}$ and $\vec{B}$ with position $\vec{r}$ and time t are given by sinusoidal functions of the phase $(\vec{k}\cdot\vec{r}-\omega t)$.

Part A. Light propagation in isotropic dielectric media (1.0 points)

If the medium is isotropic, we have $\vec{P}=\chi\epsilon_0\vec{E}$ and $\vec{D}=\epsilon\vec{E}$, with χ and $\epsilon=\epsilon_0(1+\chi)$ being the electric susceptibility and permittivity, respectively, of the medium. For a light wave of angular frequency ω in such a medium, a given phase will propagate in the direction $\vec{k}$ with a velocity (called *phase velocity*) $v_p=c/n$. Here c is the speed of light in vacuum and n is the refractive index of the medium. One can also use rays to represent a train of light waves. The propagation of a light ray is characterized by the direction and speed v_r of the electromagnetic energy flow.

Consider a plane wave of light with angular frequency ω and wave vector $\vec{k}$ in a homogeneous isotropic dielectric medium.

A.1	Express its phase velocity v_p in terms of ϵ and μ_0.	0.4pt

A.2	What is the refractive index n of the dielectric medium for the wave?	0.2pt

A.3	What are the direction $\hat{S}\equiv\vec{S}/S$ and speed v_r of its electromagnetic energy flow?	0.4pt

Part B. Light propagation in uniaxial dielectric media (4.8 points)

We now assume the dielectric medium to be uniaxial, i.e, it is electrically anisotropic along a special direction fixed in the medium, called the *optic axis*, which we presently call it the z direction. In such a case, the displacement $\vec{D}$ and the electric field $\vec{E}$ are related by $D_x=\epsilon E_x$, $D_y=\epsilon E_y$, and $D_z=\epsilon' E_z$, where x, y, and z axes are mutually orthogonal. Consequently, the phase velocity of a light wave is anisotropic and depends additionally on the directions of $\vec{k}$ and $\vec{D}$. Let $n_o=c\sqrt{\mu_0\epsilon}$ and $n_e=c\sqrt{\mu_0\epsilon'}$, answer the followings questions: **B.1**, **B.2**, and **B.3**.

B.1	Suppose the wave vector $\vec{k}$ of a monochromatic plane light wave is in the xz plane so that $\vec{k} = k(\sin\theta, 0, \cos\theta)$. At each angle θ, what directions of $\vec{D}$ and $\vec{B}$ are permissible for the light wave? Find all possible refractive indices and express the refractive indices in terms of θ, n_o, and n_e. Find the angle θ for which only one value is permitted for the refractive index.	1.5pt

B.2	The polarization of a light wave, i.e., the direction of its electric field $\vec{E}$, can be either perpendicular (called an *ordinary wave or ray*) or parallel (called an *extraordinary wave or ray*) to the xz plane. For each of the light waves you found in **B.1**, specify its polarization as a unit vector and indicate whether it is an ordinary or extraordinary wave. Also compute $\tan\alpha$, where α is the angle between $\vec{E}$ and $\vec{D}$ (α is positive when going from $\vec{E}$ to $\vec{D}$ is clockwise).	0.8pt

B.3	Extend the results in **B.1** and **B.2** to the general case when the angle between $\vec{k}$ and the positive z direction is still θ, but $\vec{k}$ is not in the xz plane. Find all possible values of the refractive indices and the corresponding polarizations.	0.6pt

In a uniaxial medium, the direction of $\vec{k}$ of a light wave may differ from the direction of the light ray. The phase velocity of the wave is still given by c/n with n being the refractive index along $\vec{k}$, while the ray velocity is defined jointly by the direction and the rate of energy flow.

B.4	Following problems **B.1-3**, consider a light wave with $\vec{k} = k(\sin\theta, 0, \cos\theta)$. Let the angle between $\hat{k} \equiv \vec{k}/k$ and the direction of the ray, $\hat{S}$, be α_r (α_r is positive when going from $\hat{S}$ to $\hat{k}$ is clockwise). Find all possible values of $\tan\alpha_r$, speed v_r of the ray and $\hat{S}$. Using these results, express the ray index $n_s = c/v_r$ in terms of $\hat{S}$, $\hat{x}$, $\hat{z}$, n_o, and n_e.	0.8pt

Consider the propagation of a light ray from A to B through an interface between an isotropic medium, labelled 1, and an anisotropic medium, labelled 2, as shown in Fig. 1. The interface coincides with the yz plane, while the plane of incidence is the xz plane. Let the angle of incidence be θ_1. The refractive index of medium 1 is n, while the refractive indices of medium 2 for axes z_2, y_2, x_2 are n_e, n_o, and n_o, respectively. Here y_2 axis coincides with y axis. Fermat's principle states that the propagation time for the path that the light ray goes from A to B is a minimum. For light with polarization parallel to xz plane and incident at the angle θ_1, Fermat's principle leads to the following equation:

$$\bar{A}(\tan\theta_2)^2 + \bar{B}\tan\theta_2 + \bar{C} = 0 \tag{1}$$

B.5	Find $\bar{A}$, $\bar{B}$, and $\bar{C}$ in terms of P_1, P_2, P_3, and $n\sin\theta_1$, where $P_1 = n_o^2\cos^2\phi + n_e^2\sin^2\phi$, $P_2 = n_o^2\sin^2\phi + n_e^2\cos^2\phi$, and $P_3 = (n_o^2 - n_e^2)\sin\phi\cos\phi$. From Eq. (1), find corresponding $\tan\theta_2$ to two special orientations: $\phi = 0$ and $\phi = \pi/2$.	1.1pt

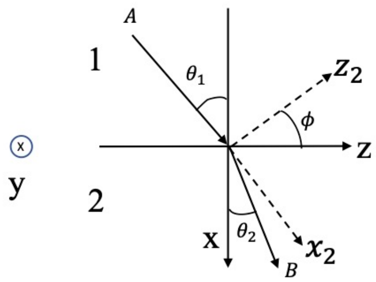

Fig. 1: Propagation of light from A to B through an interface between an isotropic medium 1 and an anisotropic medium 2.

Part C. Entanglement of light (4.2 points)

In a nonlinear medium, the electric field $\vec{E}$ is related to the polarization $\vec{P}$ by $P_i = (\epsilon - \epsilon_0)E_i + \sum_j \sum_k \chi^{(2)}_{ijk} E_j E_k$. Here i, j, k each can be any one of the three components x, y, z, and $\chi^{(2)}_{ijk}$ are constants representing the second-order nonlinear susceptibility of the medium. Non-vanishing of $\chi^{(2)}_{ijk}$ implies that as a light wave travels through a nonlinear medium, it can split into two light waves.

Suppose that because $\chi^{(2)}_{ijk}$ are not all zero, the electric field in the medium is made up of a superposition of three plane waves of angular frequencies ω, ω_1, and ω_2, propagating with wave vectors $\vec{k}$, $\vec{k}_1$, and $\vec{k}_2$, respectively. Assume $\omega \geq \omega_2$ and $\omega_1 \geq \omega_2$.

C.1	Find all possible relations (known as the *phase matching conditions*) between these angular frequencies and wave vectors. Viewing light as composed of photons, what kinds of conservation laws do these conditions imply for the three photons involved? Write down equations expressing these conservation laws for the case that a photon with angular frequency ω and wave vector $\vec{k}$ being split into two photons of angular frequencies $\omega_1$and ω_2, propagating with wave vectors $\vec{k}_1$and $\vec{k}_2$, respectively.	0.8pt

C.2	Consider a light wave in a uniaxial medium. Denote an ordinary ray as **o** and an extraordinary ray as **e**. There are 8 possible ways of splitting for the light wave: **o** → **o + o**, **o** → **e + o**, **o** → **o + e**, **o** → **e + e**, **e** → **o + o**, **e** → **e + o**, **e** → **o + e**, and **e** → **e + e**. Assume that the refractive indices n_o and n_e are both increasing functions of ω. Using the same notations for wave vectors as in problem **C.1** and considering the case that $\vec{k}$, $\vec{k}_1$, and $\vec{k}_2$ are collinear, indicate which of the 8 ways of splitting are not possible.	0.8pt

Consider an incoming **e** ray traveling along z' direction with wave vector $\vec{k}$ and $\omega = \Omega_p$ in an uniaxial medium with refractive index $n_e < n_o$. Suppose that, in a collinear splitting **e** → **e + o**, the phase-matching conditions are realized with $k_1 = K_e$, $\omega_1 = \Omega_e$, $k_2 = K_o$, and $\omega_2 = \Omega_o$. Here subscripts 1 and 2 refer to **e** ray and **o** ray. $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}$ all point in the z'direction. As shown in Fig. 2(a), the optic axis (OA) of the medium lies in the $x'z'$ plane and makes an angle $\theta < \pi/2$ with the z' axis. Therefore, n_e is a function of ω and θ, i.e., $n_e = n_e(\omega, \theta)$. For the same incoming **e** ray with wave vector $\vec{k}$ and $\omega = \Omega_p$, suppose its non-collinear splitting into **e + o** rays causes the latter two rays to separate but remain on two cones with $\omega_1 = \omega_2 = \Omega$, $k_1 = k_2$, as shown in Fig. 2(b). Note that in the collinear splitting, Ω_e is already close to Ω_o, and here Ω is only slightly less than Ω_e. In a plane perpendicular to $\vec{k}$, two circles on the cones for $\vec{k}_1$ and $\vec{k}_2$ intersect at points a and b with the line $\overline{ab}$ parallel to y'axis. As shown in Fig. 2(a), $\vec{k}_\alpha (\alpha = 1, 2\,)$ makes an angle θ_α

with the optic axis and has angular coordinates $(\psi_\alpha, \phi_\alpha)$ with $\vec{k}_{\alpha\perp}$ being its projection in the $x'y'$ plane. Each vector $\vec{k}_\alpha$ deviates from z' axis only slightly so that $|(\Omega-\Omega_e)/\Omega_e| \ll 1$, $|\vec{k}_{\alpha\perp}|/k_\alpha \ll 1$ and $|\theta_\alpha - \theta| \ll 1$. Using approximations which agree with the z' component of $\vec{k}_\alpha$ to terms of the order $k_{\alpha\perp}^2$ and the angle θ_α to $(\theta_\alpha-\theta)^2$, one finds that $\vec{k}_{2\perp} = (q_{x'}, q_{y'})$ must satisfy $M(q_{x'}+N)^2 + Mq_{y'}^2 = L$.

C.3	Let $M>0$. Evaluate M, N, and L in terms of Ω, Ω_e, Ω_o, K_e, K_o and $N_e(\omega,\theta) = \frac{1}{n_e(\omega,\theta)}\frac{dn_e(\omega,\theta)}{d\theta}$ and the group velocities $u_o = \frac{dw_2}{dk_2}$ and $u_e = \frac{d\omega_1}{dk_1}$ for the **o** and **e** rays. Estimate the angle between the axis of the cone and z', and also the angle of the cone in terms of L, M, N and K_o.	1.3pt

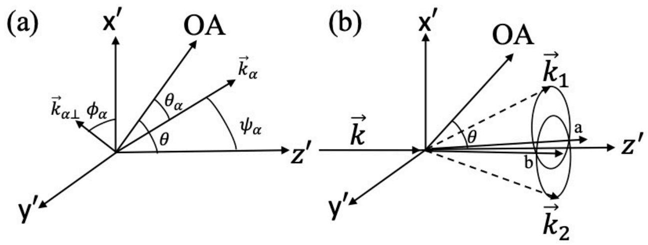

Fig. 2: (a) Vector $\vec{k}_\alpha$ has angular coordinates $(\psi_\alpha, \phi_\alpha)$ in the $x'y'z'$ coordinate system with $\vec{k}_{\alpha\perp}$ being its projection in the $x'y'$ plane. Note that $\vec{k}_\alpha$ makes an angle θ_α with OA. (b) Non-collinear splitting of an **e** ray into **e + o** rays that form two cones. Line $\overline{ab}$ is parallel to the y' axis.

Problem **C.3** shows that a photon may split into two photons which when passing through points a and b are polarized in perpendicular directions. These two photons are called *entangled photon pair* because if one photon that passes a (called a-photon) is polarized in a direction $\hat{x}'$, the other that passes b (called b-photon) will be polarized in the direction $\hat{y}' \perp \hat{x}'$, and if the a-photon is polarized in $\hat{y}'$, then the b-photon will be polarized in $\hat{x}'$. The entangled photon-pair state can be prepared experimentally. It is a superposition of the above two alternative states and can be expressed as $\frac{1}{\sqrt{2}}(|\hat{x}'_a\rangle|\hat{y}'_b\rangle + |\hat{y}'_a\rangle|\hat{x}'_b\rangle)$. Here $|\hat{x}'_a\rangle|\hat{y}'_b\rangle$represents the state when a-photon is polarized in $\hat{x}'$ direction and b-photon is polarized in $\hat{y}'$ direction; similar meaning applies to $|\hat{y}'_a\rangle|\hat{x}'_b\rangle$. The coefficient $1/\sqrt{2}$ can be viewed as the product of electric field amplitudes (expressed in suitable units) of a- and b-photons. As illustrated in Fig. 3, two linear polarizers 1 and 2 have transmission axes at angles α and β respectively with respect to $\hat{x}'$. We may use them to perform coincidence measurement on the two photons that pass a and b. Let the probability of simultaneously finding two photons passing through polarizers 1 and 2 be $P(\alpha,\beta)$. Alternatively, $P(\alpha,\beta)$ can also be regarded as being proportional to the product of intensities (after appropriate superpositions) of light passing through the two polarizers. Denote $\alpha+\pi/2$ and $\beta+\pi/2$ by $\alpha_\perp$ and $\beta_\perp$ respectively.

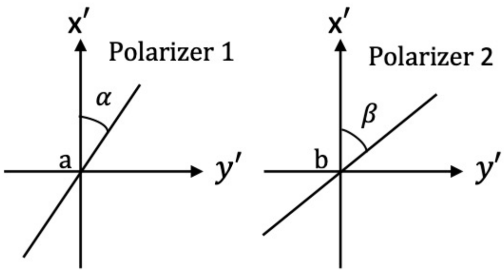

Fig. 3: Two linear polarizers 1 and 2 for coincidence measurement of photons that pass a and b.

C.4	Consider the total electric field projected by linear polarizers. Find the probabilities $P(\alpha, \beta)$, $P(\alpha, \beta_{\perp})$, $P(\alpha_{\perp}, \beta)$, and $P(\alpha_{\perp}, \beta_{\perp})$.	0.8pt

C.5	Assign $\sigma_a = 1$ when polarizer 1 with angle α finds an a-photon and $\sigma_a = -1$ when polarizer 1 with angle $\alpha_{\perp}$ finds an a-photon. Similarly, $\sigma_{\beta} = 1$ or -1 is assigned when polarizer 2 with angle β or $\beta_{\perp}$ finds a b-photon. If $E(\alpha, \beta)$ denotes the average of $\sigma_a \sigma_b$, the quantity $S = \lvert E(\alpha, \beta) - E(\alpha, \beta')\rvert + \lvert E(\alpha', \beta) + E(\alpha', \beta')\rvert$ has important meaning. For classical theories of light, $S \leq 2$. This is a variant form of Bell's inequality (the Clauser-Horne-Shimony-Holt inequality). Find the expression of S and evaluate S for the case $\alpha = \frac{\pi}{4}$,$\alpha' = 0$, $\beta = -\frac{\pi}{8}$, $\beta' = \frac{\pi}{8}$. Indicate if S is consistent with the classical theories.	0.5pt

Magnetic Levitation

Useful Information

(1) Directional derivative of a spatial function $f(\vec{r})$, given by $\vec{\nabla} f(\vec{r})$, has
$\vec{\nabla} f \equiv (\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z})f(\vec{r})$, where $\frac{\partial}{\partial x} f(\vec{r})$ denotes a partial derivative of $f(\vec{r})$ with respect to x while keeping y and z unchanged.

(2) Integral:

$\int_0^\infty dt \, \frac{(a+pt)}{[(a+pt)^2+(b+qt)^2]^{3/2}} = \frac{1}{bp-aq}(\frac{b}{\sqrt{a^2+b^2}} - \frac{q}{\sqrt{p^2+q^2}})$.

Introduction

We intend to study the motion of a small magnetic dipole in the vicinity of a conducting thin film. In the problem text, the terms dipole and monopole are to be regarded, respectively, as synonymous with magnetic dipole and magnetic monopole.

A dipole consisting of a spherical permanent magnet with a uniform magnetization $\vec{M}$ (magnetic dipole moment per unit volume) and a uniform mass density ρ_0 may be treated as a point-like object when its radius R is small. Such a dipole representation is good for describing the magnetic field that the dipole produces everywhere outside of its sphere. The representation is also a good approximation for the force acting on the dipole from an applied magnetic field, whenever distances of field sources from the dipole are much larger than R.

A point-like dipole can be considered as a pair of monopoles carrying negative and positive magnetic charges $-q_{\text{m}}$ and q_{m} respectively. The pair has a vanishingly small separation, but possesses a finite magnetic dipole moment $\vec{m} = q_{\text{m}}\vec{\delta}_{\text{m}}$. Here $\vec{\delta}_{\text{m}}$ is the displacement vector from the south monopole $(-q_{\text{m}})$ to the north monopole $(+q_{\text{m}})$. The position of the point-like dipole is chosen to be that of the north monopole.

The magnetic field $\vec{B}_{\text{mp}}$ from a monopole q_{m} is assumed to have a Coulombic form, given by

$$\vec{B}_{\text{mp}} = \frac{\mu_0 q_{\text{m}}}{4\pi r^2}\,\hat{r}, \tag{1}$$

where $\vec{r}$ is the displacement vector from q_{m} to the observation point (or field point), $\hat{r}$ is the unit vector $\hat{r} = \vec{r}/r$, and μ_0 is the free-space permeability. The force exerted by an applied magnetic field $\vec{B}'$ on q_{m} is given by $\vec{F} = q_{\text{m}}\vec{B}'$. It follows, from extending the concept of the monopole field just described in Eq.(1), that the magnetic field $\vec{B}$ from a point-dipole is derivable from a scalar potential Φ , given by the form $\vec{B} = -\vec{\nabla}\Phi$. The scalar potential Φ is also called the magnetic potential.

The conducting thin film is uniform with thickness d in the z direction (Fig. 1). It extends horizontally in x and y directions to infinity and its upper surface is located at a distance h from either a point monopole or a dipole. We consider only the case $h \gg d$. This allows us to take the electric current density induced in the film to be independent of z. We also assume that the displacement current effect to be negligible.

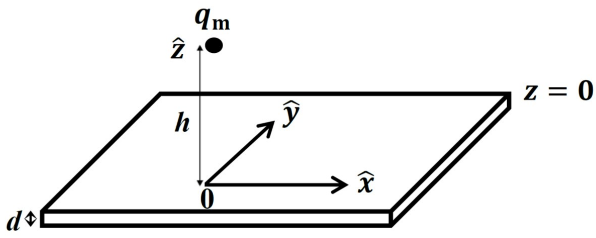

Fig.1 A monopole q_{m} appears at a distance h from a conducting thin film of thickness d. The origin of the coordinates is located on the upper surface.

The problem is divided into three parts. In Part A, the system consists of a monopole and a thin film, while in Parts B and C, a moving dipole and a thin film.

We choose the $z = 0$ plane to coincide with the upper surface of the thin film. The vector $\vec{\rho} = x\hat{x} + y\hat{y} = \rho\hat{\rho}$ denotes the in-plane position vector.

Part A. Sudden appearance of a magnetic monopole: initial response and subsequent time evolution of the response in the thin film (3.0 points)

We first focus on the initial response of the conducting thin film when at time $t = 0$ a north monopole q_{m} appears suddenly at the position $\vec{r}_{\mathrm{mp}} = h\hat{z}$ $(h > 0)$, as is shown in Fig. 1. The monopole remains stationary in all later times ($t > 0$).

Our interest here is the initial total magnetic field $\vec{B}(\vec{\rho}, z)$ in regions $z \geq 0$ and $z \leq -d$, and the induced electric current density in the thin film. The total magnetic field $\vec{B} = \vec{B}_{\mathrm{mp}} + \vec{B}'$, where magnetic fields $\vec{B}_{\mathrm{mp}}$ and $\vec{B}'$ are, respectively, due to the monopole and the induced current in the thin film. The initial $\vec{B}(\vec{\rho}, z)$ we refer to is at the time t_0, which falls within the interval $h/c \leq t_0 \ll \tau_{\mathrm{c}}$. Here τ_{c} is a time constant characterizing the subsequent response of the thin film, and c is the speed of light in vacuum. In this problem, we take the limit $h/c \to 0$ and hence let $t_0 = 0$.

The calculation of the initial total magnetic field $\vec{B}(\vec{\rho}, z)$ (at $t_0 = 0$) is facilitated by introducing an image monopole. For $\vec{B}(\vec{\rho}, z)$ in the region $z \geq 0$, the image monopole has a magnetic charge q_{m} and is located at $z = -h$. On the other hand, for $\vec{B}(\vec{\rho}, z)$ in the region $z \leq -d$, the image monopole has a magnetic charge $-q_{\mathrm{m}}$ and is located at $z = h$.

Initial response

A.1	Obtain the initial total magnetic field $\vec{B}(\vec{\rho}, z)$ in $z \geq 0$ at $t_0 = 0$.	0.4pt
A.2	Obtain the initial total magnetic field $\vec{B}(\vec{\rho}, z)$ in $z \leq -d$ at $t_0 = 0$.	0.2pt
A.3	Find the initial magnetic flux Φ_{B} through surfaces at $z = 0$, and at $z = -d$.	0.4pt
A.4	Obtain the initial induced electric current density $\vec{j}(\vec{\rho})$ in the conducting thin film at $t_0 = 0$.	0.6pt

For $t > 0$, the total magnetic field $\vec{B}$ becomes $\vec{B}(\vec{\rho}, z; t) = \vec{B}_{\text{mp}}(\vec{\rho}, z) + \vec{B}'(\vec{\rho}, z; t)$, by superposition, with $\vec{B}'(\vec{\rho}, z; t)$ due to the induced electric current in the thin film. You are required below to obtain an equation for $B'_z(\rho, z; t)$ near the $z = 0$ thin film surface. The time-evolution behavior of B'_z would reveal a moving image-monopole picture for the description of the $\vec{B}'$ field near $z \approx 0$ in $t > 0$.

The equation for B'_z inside the thin film is given below,

$$\frac{\partial^2 B'_z(\rho, z; t)}{\partial z^2} = \mu_0 \sigma \frac{\partial B'_z(\rho, z; t)}{\partial t}. \qquad (2)$$

This equation has been obtained from imposing inside the thin film the Maxwell equation and the Ohmic behavior of the conducting thin film ($\vec{j} = \sigma \vec{E}$, where σ is the electrical conductivity) while neglecting the displacement-current effect. Term being neglected on the left-hand side of Eq.(2) is $\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \frac{\partial B'_z}{\partial \rho})$, based on the $h \gg d$ condition.

Subsequent response

A.5	Obtain from Eq. (2) an equation of $B'_z(\rho, z; t)$ near $z \approx 0$. The equation contains first partial derivatives of $B'_z(\rho, z; t)$ with respect to z, and, separately, to t.	0.6pt

A.6	Solve for the general form of $B'_z(\rho, z; t)$ near $z \approx 0$ in $t > 0$.	0.4pt

A.7	Show that your solution in **A.6** reveals a moving image-monopole picture for the magnetic field $B'_z(\rho, z \approx 0; t)$, with a downwardly moving velocity. Find the speed v_0 of the image monopole in terms of known parameters from the problem text.	0.4pt

Part B. Magnetic force acting on a point-like dipole moving with a constant velocity and at a constant h (4.0 points)

The moving image-monopole concept developed in **A.7** for B'_z near $z \approx 0$ can be assumed to hold also for the $\vec{B}'$ field in the $z \geq 0$ region. This assumption is good as long as the time evolution is sufficiently slow in the conducting thin film response.

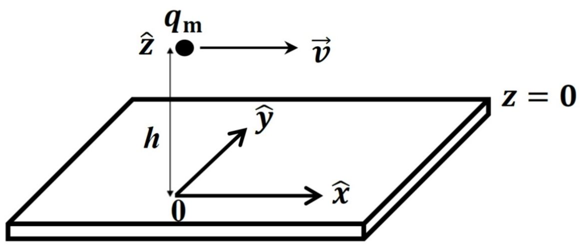

Fig. 2 A monopole q_{m} moves with a constant velocity $\vec{v}$ and a constant height h from the conducting thin film. As shown are its coordinates at $t = 0$.

A monopole q_{m} (Fig. 2) is caused to move in a constant velocity $v\hat{x}$, with $v \ll c$, and a constant height, at $z = h$, motion up to the present moment ($t = 0$). Its present coordinates (x, y) are $(0, 0)$. Our focus is on the magnetic potential Φ_+ due to all image monopoles generated by this moving monopole along its trajectory.

By splitting q_m's trajectory into discrete time steps (a very small time step τ), we replace the motion of the q_m by a hopping at the beginning moment of each time step. The hopping is represented by a simultaneous removal and creation of the monopoles. The position of the created monopole coincides with a point on its trajectory right at the beginning moment of this time step. Thus the position of the removed monopole coincides with its trajectory position at the beginning moment of the previous time step. This is achieved by a simultaneous sudden appearance of two magnetic monopoles: q_m and $-q_\mathrm{m}$ at, respectively, the trajectory positions corresponding to the beginning moments of this and the previous time step. The two positions are separated by a hopping distance $\Delta x = v\tau$. This time-step approach facilitates the determination of all the image magnetic monopoles, and their positions, that are generated in all the time steps.

A moving monopole

B.1	Write down the present $(t = 0)$ positions of all the image monopoles of the types q_m and $-q_\mathrm{m}$. The beginning moments of the time steps are at $t = -n\tau$, where $n \geq 0$.	0.8pt

B.2	Find the summation form of the magnetic potential $\Phi_+(x,z)$ at $t = 0$ from all the image monopoles in **B.1**. Calculate $\Phi_+(x,z)$.	0.7pt

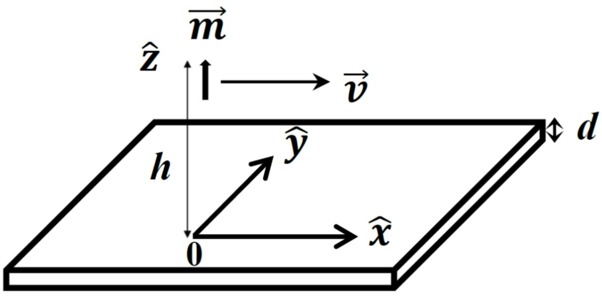

Fig. 3 A dipole with an **upward-pointing** magnetic dipole moment $\vec{m}$ moves with a constant $\vec{v}$ and a constant height h from the conducting thin film. As shown are its coordinates at $t = 0$.

Now consider a point-like moving magnetic dipole as shown in Fig. 3. The dipole, with a dipole moment $\vec{m} = m\hat{z}$, is caused to move in a constant velocity $v\hat{x}$, and a constant height ($z = h$) motion up to the present moment ($t = 0$), where its present coordinates are at $(0, 0)$. The point-like dipole can be represented by two slightly displaced monopoles as has been mentioned in the Introduction section. The location of the magnetic dipole is chosen to be that of the north monopole, and $\vec{m}$ is assumed kept fixed.

A moving dipole

B.3	Find the force $\vec{F}$ acting upon the point-like magnetic dipole by the conducting thin film at $t = 0$.	1.5pt

Relation between v_0 and v

For the numerical evaluation in this **Part** below, we consider a conducting thin film that is made of copper, such that $\sigma = 5.9 \times 10^7\ \Omega^{-1}\text{m}^{-1}$, $d = 0.50$ cm, and $h = 5.0$ cm.

B.4	Calculate the value of v_0, the speed of the image dipole as according to **A.7**.	0.3pt

It is known that the penetration depth δ (called skin depth), which distance an electromagnetic wave can penetrate into a conducting slab, depends on the angular frequency ω of the wave. The dependence is given by

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}. \tag{3}$$

For the consideration below, we take $\omega = v_\text{L}/h$, where v_L equals the larger velocity of v and v_0.

B.5	Obtain the v dependence of $v_0(v)$ in both the small and the large v regimes.	0.4pt

B.6	Obtain the critical velocity $v = v_\text{c}$ at which the two regimes in **B.5** meet.	0.3pt

Part C. Motion of the magnetic dipole when the conducting thin film is superconducting (3.0 points)

The consideration above can be applied to the case of type-I superconductors, where magnetic fields are completely repelled from the superconductors (the Meissner effect) at all times, by taking the limit that electrical conductivity $\sigma \to \infty$.

Here we consider a point-like magnetic dipole with a **horizontal** magnetic dipole moment $\vec{m} = m\hat{x}$, a mass M_0, and located at $(x, y, z) = (0, 0, h)$. We focus on vertical motions of the magnetic dipole under the action of a gravitational field, with gravitational acceleration $\vec{g} = -g\hat{z}$. Weak coupling between the given dipole orientation and its center-of-mass motion is assumed and is neglected. As such, we fix the magnetic dipole moment, as is given above, for our considerations below. In addition, we assume an ultra-high vacuum environment so that no damping to the motion from the residual air needs to be considered.

C.1	Find the equilibrium distance h_0 of the dipole from the superconducting thin film.	1.2pt

C.2	Find the dipole angular frequency Ω of oscillations about the equilibrium.	0.8pt

Physical parameters for a spherical permanent magnet are as follows: radius $R = 1.0\ \mu$m, mass density $\rho_0 = 7400$ kg m^{-3}, $g = 9.8$ m s^{-2}, $\mu_0 = 4\pi \times 10^{-7}$ TA^{-1}m, and magnetization $|\vec{M}| = 75 \times 10^{-2}$ T$/\mu_0$.

C.3	Calculate the value of h_0.	0.7pt

C.4	Calculate the value of Ω .	0.3pt