Solution

Water Hammer

Part A. ***Excess Pressure and Propagation of Pressure wave***

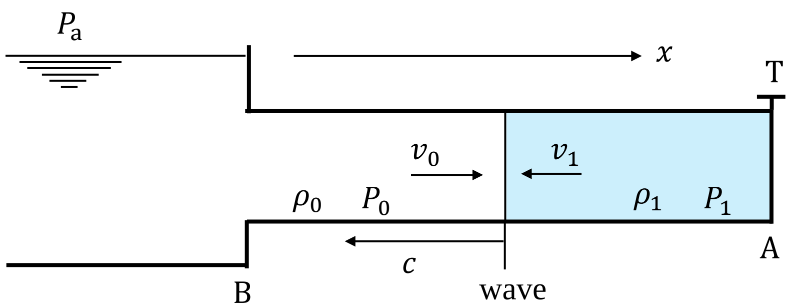


Fig. S1. Pressure wave (shaded) with speed c

A.1 (1.6 pt) Excess pressure and speed of propagation of the pressure wave

When the valve opening is suddenly blocked, fluid pressure at the valve jumps from P_0 to $P_1 = P_0 + \Delta P_s$, thus sending a pressure wave traveling upstream (to the left) with speed c and amplitude ΔP_s. Taking positive x direction as pointing to the right, the velocity of fluid particles next to the valve changes from v_0 to v_1 $(v_1 \leq 0)$. Thus the velocity change is $\Delta v = v_1 - v_0$.

In a frame moving to left (along $-x$ direction) with speed c, i.e., riding on the wave (see Fig. S1), velocity of fluid in the pressure wave is $c + v_1$, while that of the incoming fluid in the steady flow ahead of the wave is $c + v_0$. Let ρ_1 be the density of fluid in the pressure wave. From conservation of mass, i.e., equation of continuity, we have

$$\rho_0(c + v_0) = \rho_1(c + v_1) \tag{a1}$$

or, by letting $\Delta\rho \equiv \rho_1 - \rho_0$,

$$\frac{\Delta\rho}{\rho_1} = 1 - \frac{\rho_0}{\rho_1} = \frac{v_0 - v_1}{c + v_0} = \frac{-\Delta v}{c + v_0} \tag{a2}$$

Moreover, impulse imparted to the fluid must equal its momentum change. Thus, in a short time interval τ after the valve is closed, we must have

$$\rho_0(c + v_0)\tau[(c + v_1) - (c + v_0)] = -\tau\Delta P = (P_0 - P_1)\tau \tag{a3}$$

or

$$\Delta P_s = -\rho_0 c\left(1 + \frac{v_0}{c}\right)(v_1 - v_0) = -\rho_0 c\left(1 + \frac{v_0}{c}\right)\Delta v \quad \Rightarrow \quad \alpha = -\left(1 + \frac{v_0}{c}\right) \tag{a4}$$

If $v_0/c \ll 1$, we have

$$\Delta P_s = -\rho_0 c \Delta v \tag{a5}$$

Note that the *negative* sign in Eqs. (a4) and (a5) follows from the fact that the direction of propagation is opposite to the positive direction for x axis (and velocity). Otherwise the sign should be *positive*. Note also that for a compressional wave

($\Delta P_s > 0$), the velocity imparted to the fluid particle is in the direction of propagation, while for an extensional wave ($\Delta P_s < 0$), the velocity imparted is in the opposite direction of propagation.

Eqs. (a2) and (a4) can be combined to give

$$\Delta P_s = \rho_0 c^2 \left(1 + \frac{v_0}{c}\right)^2 \frac{\Delta\rho}{\rho_1} \quad \text{(a6)}$$

From the definition of the bulk modulus B, which is assumed to be constant, it follows

$$\Delta P_s = B\frac{V_0 - V_1}{V_0} = B\frac{1/\rho_0 - 1/\rho_1}{1/\rho_0} = B\frac{\Delta\rho}{\rho_1} \quad \text{(a7)}$$

From Eqs. (a6) and (a7), we obtain

$$\rho_0 c^2 \left(1 + \frac{v_0}{c}\right)^2 = B \quad \text{(a8)}$$

Thus

$$c = \sqrt{\frac{B}{\rho_0}} - v_0 \qquad \Rightarrow \quad \gamma = 1 \qquad \beta = -v_0 \quad \text{(a9)}$$

However, if in the definition of bulk modulus one uses the fractional change of density $\Delta\rho/\rho_0$ instead of $-\Delta V/V_0$, the result is then $\gamma = 1 + \Delta P_s/B$.* Either result is considered valid.

If $v_0/c \ll 1$, we have

$$c = \sqrt{\frac{B}{\rho_0}} \quad \text{(a10)}$$

*The result (a7) is pointed out by Dr. Jaan Kalda.

A.2 (0.6 pt) Values of c and ΔP_s for water flow

Ans:

From Eqs. (a5) and (a10), we have

$$c = \sqrt{B/\rho_0}$$
$$\Delta P_s = \rho_0 c v_0 = v_0\sqrt{\rho_0 B}$$

Putting in the given values $v_0 = 4.0\ \mathrm{m/s}$, $v_1 = 0$, $\rho_0 = 1.0 \times 10^3\ \mathrm{kg/m^3}$, and $B = 2.2 \times 10^9$ Pa, we have

$$c = \sqrt{B/\rho_0} = 1.5 \times 10^3\ \mathrm{m/s} \quad \text{(b1)}$$
$$\Delta P_s = v_0\sqrt{\rho_0 B} = 5.9\ \mathrm{MPa} \quad \text{(b2)}$$

so that ΔP_s is nearly 59 times the standard pressure.

Note that $v_0/c \sim 10^{-3}$ so that the use of approximate formulas (a5) and (a10) is justified when solving tasks in this problem.

Part B. *A Model for the Flow-Control Valve*

(B.1) (1.0 pt) Excess pressure at valve inlet

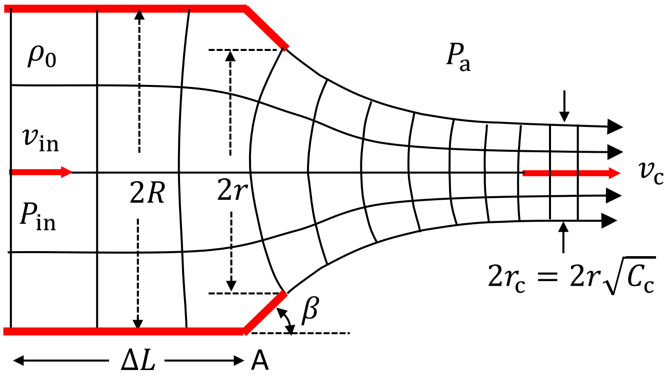


Fig. 2. Valve dimensions and contraction of jet.

Ans:

The model assumes the fluid to be incompressible. Neglecting effects of gravity, Bernoulli's principle gives us

$$\frac{1}{2}\rho_0 v_{in}^2 + P_{in} = \frac{1}{2}\rho_0 v_c^2 + P_a \quad (c1)$$

Equation of continuity and definition of contraction coefficient imply that

$$\pi R^2 v_{in} = \pi r_c^2 v_c = \pi r^2 C_c v_c$$

Therefore

$$v_c = \frac{1}{C_c}\left(\frac{R}{r}\right)^2 v_{in} \quad (c2)$$

From Eqs. (c1) and (c2), we obtain

$$\Delta P_{in} = P_{in} - P_a = \frac{1}{2}\rho_0 v_{in}^2\left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] = \frac{k}{2}\rho_0 v_{in}^2 \quad (c3)$$

This may be cast into a form involving only dimensionless variables:

$$\frac{\Delta P_{in}}{\rho_0 c^2} = \frac{1}{2}\left(\frac{v_{in}}{c}\right)^2\left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] = \frac{k}{2}\left(\frac{v_{in}}{c}\right)^2 \quad (c4)$$

where

$$k = \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right] \quad (c5)$$

Thus we see from eq. (c4) that ΔP_{in} is a quadratic function of v_{in}.

Part C. *Water-Hammer Effect due to Fast Closure of Flow-Control Valve*

(C.1) (0.6 pt) Pressure P_0 and velocity v_0 when the valve is fully open

Ans:

According to Bernoulli's theorem and the definition of P_h, we have

$$\frac{1}{2}\rho_0 v_0^2 + P_0 = \frac{1}{2}\rho_0 v_c^2 + P_a = 0 + P_a + \rho_0 gh = P_h \qquad \text{(d1)}$$

From the second equality in the preceding equation, it follows

$$v_c = \sqrt{2gh}$$

Furthermore, from continuity equation and $C_c(r = R) = 1.0$, we have

$$\pi R^2 v_0 = \pi (C_c R)^2 v_c = \pi R^2 v_c \ \Rightarrow \ v_0 = v_c = \sqrt{2gh} \qquad \text{(d2)}$$

Therefore

$$P_0 = P_a = P_h - \rho_0 gh \qquad \text{(d3)}$$

(C.2) (1.2 pt) Pressure $P(t)$ and flow velocity $v(t)$ just before $t = \frac{\tau}{2} = \frac{L}{c}$ and $t = \tau$

Ans:

When the valve is open, the flow in the pipe is steady with velocity v_0 and pressure P_0. The sudden closure of the valve causes an excess pressure ΔP_s on the fluid element next to the valve, causing it to stop with velocity $v_1 = 0$. The velocity change is thus $\Delta v = v_1 - v_0 = -v_0$. Thus, according to Eq. (a5), the excess pressure on the fluid is given by

$$\Delta P_s = -\rho_0 c \Delta v = \rho_0 c v_0 \qquad \text{(e1)}$$

At time $t = \tau/2 = L/c$, the pressure wave reaches the reservoir. The velocity of fluid in the length of the pipe has all changed to $v(\tau/2) = v_1 = v_0 + \Delta v = 0$ and the fluid pressure is $P(\tau/2) = P_1 = P_0 + \Delta P_s = P_0 + \rho_0 c v_0$.

At the reservoir end of the pipe, fluid pressure reduces to the constant hydrostatic pressure $P_h = P_0 + \rho_0 gh$. Equivalently, we may say that the reservoir acts as a free end for the pressure wave and, in reducing its excess pressure to P_h, causes a compression wave to be reflected as an expansion wave. Relative to the hydrostatic pressure P_h, the amplitude of the incoming pressure wave is $\Delta P_{1r} = P_1 - P_h$, hence the reflected expansion wave will have an amplitude $\Delta P_1' = -\Delta P_{1r}$ and we have

$$\Delta P_1' = -\Delta P_{1r} = P_h - P_1 = (P_0 + \rho_0 gh) - (P_0 + \rho_0 c v_0) = -\rho_0 c(v_0 - gh/c) \qquad \text{(e2)}$$

(Here we allow the pressure amplitude to have both signs with negative amplitude signifying an expansion wave.) This will cause the fluid at the reservoir end of the pipe to suffer a velocity change (keeping in mind that the direction of propagation is now the same as the $+x$ axis)

$$\Delta v_{1r} = +\Delta P_1'/(\rho_0 c) = -(v_0 - gh/c)$$

Consequently, its velocity changes to

$$v_{1r} = v_1 + \Delta v_{1r} = 0 - \left(v_0 - \frac{gh}{c}\right) \qquad \text{(e3)}$$

Ahead of the front of the reflected wave, conditions are unchanged and the particle velocity is still $v_1 = 0$ and the fluid pressure is still $P_1 = P_0 + \Delta P_s$, but behind the wave front the particle velocity now becomes $v_{1r} = -(v_0 - gh/c)$ and the pressure becomes

$$P_1 + \Delta P_1' = (P_0 + \rho_0 c v_0) - \rho_0 c\left(v_0 - \frac{gh}{c}\right) = P_0 + \rho_0 gh \tag{e4}$$

Therefore, just moment before $t = \tau = 2L/c$ when the front of the reflected wave reaches the valve, the fluid in the whole length of the pipe will be under the pressure $P(\tau) = P_0 + \rho_0 gh = P_h$ as given in Eq. (e4) , and all fluid particles in the pipe will move, as given in Eq. (e3), with velocity $v(\tau) = v_{1\mathrm{r}} = -v_0 + gh/c,$ i.e., the fluid in the pipe is expanding and flowing toward the reservoir.

Part D. *Water-Hammer Effect due to Slow Closure of Flow-Control Valve*

(D.1) (3.0 pt) Recursion relations for ΔP_n and v_n

Ans:

Enforcing the approximation $P_h = P_0 + \rho_0 gh \approx P_0$ is equivalent to putting $h = 0$ in all of the results obtained in task (e).

(1) Partial closing $n = 1$

At the valve, immediately after partial closing $n = 1$, fluid pressure jumps from P_0 to P_1, causing flow velocity to change from v_0 to v_1. The pressure and velocity changes are related by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_1 - P_0) = -(v_1 - v_0) \tag{f1}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_1 and velocity v_1. After reflection by the reservoir, i.e., a free end, and before the start of valve closure $n = 2$, the fluid in the entire pipe has pressure (Eq. (e4) with $h = 0$)

$$P_1 - (P_1 - P_0) = P_0$$

and velocity

$$v_1' = v_1 + \frac{-(P_1 - P_0)}{\rho_0 c} = v_1 + (v_1 - v_0)$$

(2) Partial closing $n = 2$

Immediately after partial closing $n = 2$, valve pressure changes from P_0 to P_2, causing flow velocity to change from v_1' to v_2. The pressure and velocity changes are given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_2 - P_0) = -(v_2 - v_1') = -v_2 + v_1 + (v_1 - v_0) \tag{f2}$$

Using Eq. (f1), we may rewrite the preceding equation as

$$\frac{1}{\rho_0 c}(P_2 - P_0) = -(v_2 - v_1) - \frac{1}{\rho_0 c}(P_1 - P_0) \tag{f3}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_2 and velocity v_2. After reflection by the reservoir and before valve closure $n = 3$, the fluid in the entire pipe has pressure

$$P_2 - (P_2 - P_0) = P_0$$

and velocity

$$v_2' = v_2 + (v_2 - v_1')$$

(3) Partial closing $n = 3$

Immediately after partial closing $n = 3$, valve pressure changes from P_0 to P_3, causing flow velocity to change from v_2' to v_3. The pressure and velocity changes are given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_3 - P_0) = -(v_3 - v_2') = -v_3 + v_2 + (v_2 - v_1') \qquad \text{(f4)}$$

Using Eq. (f2), we may rewrite the preceding equation as

$$\frac{1}{\rho_0 c}(P_3 - P_0) = -(v_3 - v_2) - \frac{1}{\rho_0 c}(P_2 - P_0) \qquad \text{(f5)}$$

Just before reflection by the reservoir, the fluid in the entire pipe has pressure P_3 and velocity v_3. After reflection by the reservoir and before valve closure $n = 4$, the fluid in the entire pipe has pressure

$$P_3 - (P_3 - P_0) = P_0$$

and velocity

$$v_3' = v_3 + (v_3 - v_2')$$

(4) Partial closing $n = 4$

When the valve is fully shut at valve closing $n = 4$, the valve becomes a fixed end, so the fluid velocity at the valve changes from v_3' to $v_4 = 0$. The pressure P_4 at the valve is then given by Eq. (a5):

$$\frac{1}{\rho_0 c}(P_4 - P_0) = -(v_4 - v_3') = -v_4 + v_3 - \frac{1}{\rho_0 c}(P_3 - P_0) \qquad \text{(f6)}$$

Finally, if we take note of the fact that $\Delta P_0 = 0$ and $v_4 = 0$, then all equations obtained above relating excess pressures and velocity changes after valve closings all have the same form:

$$\frac{\Delta P_n}{\rho_0 c} = -(v_n - v_{n-1}) - \frac{\Delta P_{n-1}}{\rho_0 c} \qquad (n = 1,2,3,4) \qquad \text{(f7)}$$

To solve for $\Delta P_n = P_n - P_0$, we note that, from Eqs. (c3) and (c5), we have another relation between ΔP_n and v_n:

$$\Delta P_n = \frac{1}{2} k_n \rho_0 {v_n}^2 \qquad (n = 1,2,3) \qquad \text{(f8)}$$

where C_n represents C_c for $r = r_n$ and

$$k_n = \left[\frac{1}{C_n^2}\left(\frac{R}{r_n}\right)^4 - 1\right] \qquad (n = 1,2,3) \qquad \text{(f9)}$$

Combining Eqs. (f7) and (f8), we have a quadratic equation for v_n:

$$\frac{1}{2} k_n \left(\frac{v_n}{c}\right)^2 + \frac{v_n}{c} + \left(\frac{\Delta P_{n-1}}{\rho_0 c^2} - \frac{v_{n-1}}{c}\right) = 0 \qquad (n = 1,2,3) \qquad \text{(f10)}$$

which can be solved readily using the formula

$$\frac{v_n}{c} = \frac{-1 + \sqrt{1 + 2k_n\left(\frac{v_{n-1}}{c} - \frac{\Delta P_{n-1}}{\rho c^2}\right)}}{k_n} \qquad (n = 1,2,3) \qquad \text{(f11)}$$

If both $\Delta P_{n-1}/(\rho c^2)$ and (v_{n-1}/c) are known, Eq. (f11) may be used to compute v_n/c and then find $\Delta P_n/(\rho c^2)$ by using Eq. (f8). Therefore, Eq. (f7) may be solved iteratively starting with $n = 1$ until $n = 3$. For $n = 4$, we know $v_n = 0$, so Eq. (f7) may be used directly to find ΔP_n.

Note that, from Eq. (f8), ΔP_{n-1} is a quadratic function of v_{n-1}, so that if v_{n-1} is known, then v_n may be computed using Eq. (f11) and then ΔP_n may again be computed using Eq. (f8).

(D.2) (2.0 pt) Estimating ΔP_n and $\rho_0 c v_n$ by graphical method

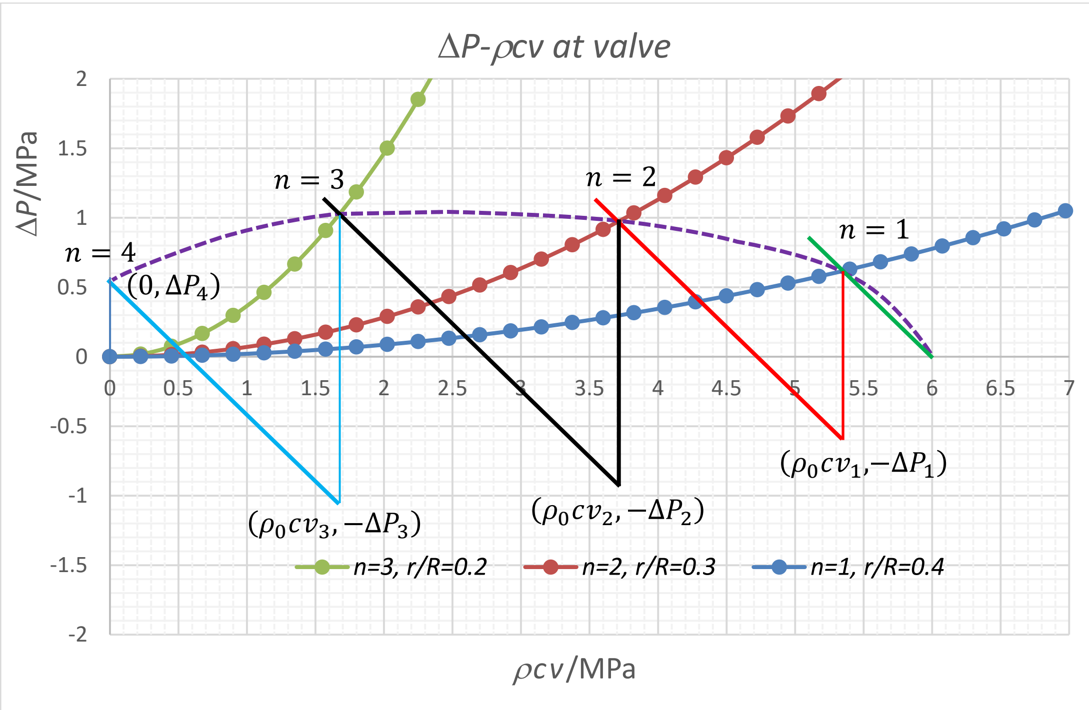
Ans:

To solve Eqs. (f7) and (f8) using graphical method, we rewrite them as follows:

$$\Delta P_n = -(\rho_0 c v_n - \rho_0 c v_{n-1}) - \Delta P_{n-1} \qquad (n = 1,2,3,4) \qquad \text{(g1)}$$

$$\Delta P_n = \frac{k_j}{2\rho_0 c^2}(\rho_0 c v_n)^2 \qquad (n = 1,2,3,4) \qquad \text{(g2)}$$

In a plot of ΔP vs. $\rho_0 c v$, Eq. (g1) and Eq. (g2) correspond to a line passing through the point $(\rho_0 c v_{n-1}, -\Delta P_{n-1})$ with slope -1 and a parabola passing through the origin, respectively. Thus one may readily obtain the solutions for each step of valve closing by locating their points of intersection, starting with $n = 1$. The result is shown in the following graph.



Excess Pressures and particle velocities at the valve for slow closing							
n	r_n/R	C_n	k_n	$v_n/(\mathrm{m/s})$	$\rho_0 c v_n/\mathrm{MPa}$	$\Delta P_n/(\mathrm{MPa})$	$\Delta P_n/(\rho_0 c v_0)$
0	1.00	1.00	0.0	4.0	6.0	0.0	0.0
1	0.40	0.631	97.1	3.6	5.8	0.62	10 %
2	0.30	0.622	318.	2.5	3.8	1.0	17 %
3	0.20	0.616	1646.	1.1	1.7	1.1	18 %

4	0.00			0.0	0.0	0.64	11 %

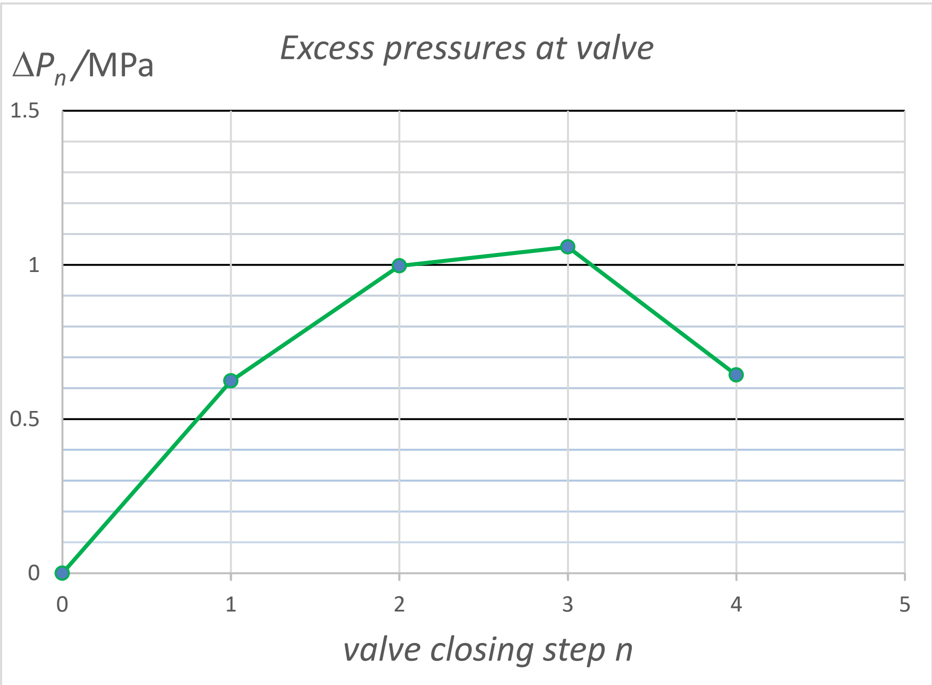
$\rho_0 c = 1.50 \times 10^6 \text{ kg m}^{-2} \text{ s}^{-1}$ $v_0 = 4.0 \text{ m/s}$

Appendix

(The following table and graph are for reference only, not part of the task.)

For $v_0 = 4.0\ \mathrm{m/s}$, $c = 1.5 \times 10^3\ \mathrm{m/s}$, and $\rho = 1.0 \times 10^3\ \mathrm{kg/m^3}$, the results for v_n and ΔP_n are shown in the following table and graph. They are computed according to equations given in task (f). Note that for a sudden full closure of the valve, we have $\Delta P_{\mathrm{sudden}} = \rho c v_0 = 6.0\ \mathrm{MPa}$.

Excess Pressures and particle velocities at the valve for slow closing							
n	r_n/R	C_n	k_n	$v_n/(\mathrm{m/s})$	$\rho c v_n/\mathrm{MPa}$	$\Delta P_n/(\mathrm{MPa})$	$\Delta P_n/(\rho c v_0)$
0	1.00	1.00	0.0	4.0	6.0	0.0	0.0
1	0.40	0.631	97.1	3.58	5.37	0.624	10 %
2	0.30	0.622	318.	2.50	3.75	0.997	17 %
3	0.20	0.616	1646.	1.13	1.695	1.06	18 %
4	0.00			0.0	0.0	0.643	11 %



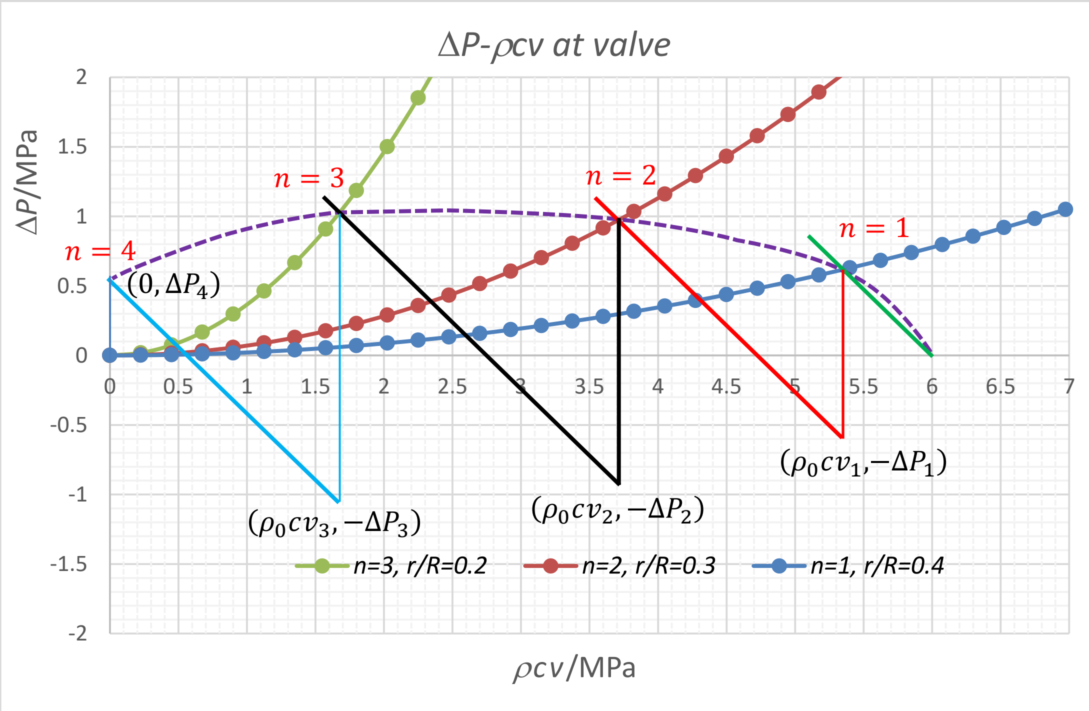
[Marking Scheme] *Theoretical Question 1*

Water Hammer

total	(Task) points	Marking Scheme for Answers to the Problem
Part A 2.2	(A.1) 1.6	*Excess pressure of pressure wave* $\alpha = -(1 + v_0/c)$ ➢ 0.1 expression for impulse. ➢ 0.1 expression for momentum change. ➢ 0.1 equating impulse to momentum change ➢ 0.2 correct equation of continuity for compressible fluid. ➢ {0.1 solving by use of energy conservation} ➢ 0.2 negative sign of α ➢ 0.3 correct magnitude $\lvert\alpha\rvert = 1 + v_0/c$ ➢ {0.1 for $\lvert\alpha\rvert \approx 1$} *Speed of propagation* $\beta = -v_0,\ \gamma = 1 \approx (1 + \Delta P_s/B)$ ➢ 0.1 realizing $-\Delta V/V_0 = \Delta\rho/\rho_1 \approx \Delta\rho/\rho_0$ ➢ 0.1 negative sign of β ➢ 0.2 correct magnitude $\lvert\beta\rvert = v_0$ ➢ 0.2 $\gamma = 1 \approx (1 + \Delta P_s/B)$ ➢ {0.1 if $]\gamma[\approx 1$}
	(A.2) 0.6	*Numerical values of c and ΔP_s for water flow.* ➢ 0.2 + 0.1 for magnitude and unit of $c = 1.5 \times 10^3$ m/s. ➢ 0.2 + 0.1 for magnitude and unit of $\Delta P_s = 5.9$ MPa. ➢ {0.1 + 0.1 for correct order of magnitude for c and ΔP_s}
Part B 1.0	(B.1) 1.0	*Excess Pressure at valve inlet.* $\Delta P_{in} = \frac{k}{2}\rho_0 v_{in}^2,\ k = \left[\frac{1}{C_c^2}\left(\frac{R}{r}\right)^4 - 1\right]$ ➢ 0.2 using inlet and vena contracta in Bernoulli theorem. ➢ 0.1 correct equation of continuity for incompressible fluid ➢ 0.1 deduce $r_c^2 = r^2 C_c$. ➢ 0.1 deduce $v_c = \frac{1}{C_c}\left(\frac{R}{r}\right)^2 v_{in}$. ➢ 0.5 obtain $\Delta P_{in} = \frac{k}{2}\rho_0 v_{in}^2$ with correct k. ➢ {0.2 for $\Delta P_{in} \propto v_{in}^2$.}
Part C 1.8	(C.1) 0.6	*Pressure and velocity when valve fully open.* $P_0 = P_a$ $v_0 = \sqrt{2gh}$ ➢ 0.1 correct equation of Bernoulli theorem. ➢ 0.1 correct equation of continuity. ➢ 0.1 realizing $C_c(r = R) = 1.0$ ➢ 0.1 $v_0 = \sqrt{2gh}$ ➢ 0.2 $P_0 = P_a$.
	(C.2) 1.2	*Pressure $P(t)$ and flow velocity $v(t)$ as $t \to \tau/2$ and $t \to \tau$.* ➢ 0.3 for $P(\to \tau/2) = P_0 + \rho_0 c v_0$ ➢ {0.1 for $P(\to \tau/2) = \rho_0 c v_0$} ➢ 0.3 for $v(\to \tau/2) = 0$ ➢ 0.3 for $P(\to \tau) = P_0 + \rho_0 g h = P_h$ ➢ {0.1 for $P(\to \tau) = P_0$} ➢ 0.3 for $v(\to \tau) = -v_0 + gh/c$ ➢ {0.1 for $v(\to \tau) = -v_0$}

Part D 5.0	(D.1) 3.0	*Recursion relations for* ΔP_n *and* v_n. $\frac{\Delta P_n}{\rho_0 c} = -(v_n - v_{n-1}) - \frac{\Delta P_{n-1}}{\rho_0 c}$ $(n = 1,2,3,4)$ $\frac{v_n}{c} = \frac{-1+\sqrt{1+2k_n\left(\frac{v_{n-1}}{c}-\frac{\Delta P_{n-1}}{\rho c^2}\right)}}{k_n}$ $(n = 1,2,3)$ ➢ 0.2 setting $h = 0$ to simplify equations. ➢ 0.2 use $\Delta P = \mp\rho_0 c\Delta v$ for waves moving in $\mp x$ direction. ➢ 0.2 sign change of ΔP upon reflection at reservoir end. ➢ 0.2 no sign change of Δv upon reflection at reservoir end. ➢ 0.2 no sign change of ΔP upon reflection at valve end. ➢ 0.2 sign change of Δv upon reflection at valve end. ➢ 1.0 correct recursion formula for ΔP_n, n = 1,2,3,4. ➢ 0.4 use $\Delta P_n = \frac{1}{2}k_n\rho_0 v_n^2$ to eliminate ΔP_n in recursion formula ➢ 0.2 take positive root when solving for $\frac{v_n}{c}, n = 1,2,3$
	(D.2) 2.0	ΔP_n *and* $\rho_0 c v_n$ *by graphical method.* $\frac{\Delta P_n}{\rho_0 c} = -(v_n - v_{n-1}) - \frac{\Delta P_{n-1}}{\rho_0 c}$ ➢ 0.4 (0.1 each) ΔP_n vs. $\rho_0 c v_n$ line $(n = 1,2,3,4)$ passing through $(\rho_0 c v_{n-1}, -\Delta P_{n-1})$ with slope $= -1$ $(n = 1,2,3,4)$. ➢ 0.3 (0.1 each) parabola for ΔP_n vs. v_n curve $(n = 1,2,3)$. ➢ 0.1 Start with $(\rho_0 c v_0 = 6.0\ \text{MPa}, \Delta P_0 = 0)$ ➢ 0.1 End with $v_4 = 0$ ➢ 0.4 (0.1 each) each label n at $(\rho_0 c v_n, \Delta P_n)$ $(n = 1,2,3,4)$ ➢ 0.4 (0.1 each) estimate of ΔP_n $(n = 1,2,3,4)$. ➢ 0.3 (0.1 each) each estimate of $\rho_0 c v_n$ $(n = 1,2,3)$ Refer to plot and table on next page for values of $(\rho_0 c v_n, \Delta P_n)$.

Partial outcomes obtained for later problems which are incorrect solely because of errors being carried forward but are otherwise reasonable will not be further penalized. However, this rule does not apply to incorrect final outcomes.



Excess Pressures and particle velocities at the valve for slow closing							
n	r_n/R	C_n	k_n	$v_n/(\mathrm{m/s})$	$\rho_0 c v_n/\mathrm{MPa}$	$\Delta P_n/(\mathrm{MPa})$	$\Delta P_n/(\rho_0 c v_0)$
0	1.00	1.00	0.0	4.0	6.0	0.0	0.0
1	0.40	0.631	97.1	3.6	5.8	0.62	10 %
2	0.30	0.622	318.	2.5	3.8	1.0	17 %
3	0.20	0.616	1646.	1.1	1.7	1.1	18 %
4	0.00			0.0	0.0	0.64	11 %

$\rho_0 c = 1.50 \times 10^6\ \mathrm{kg\ m^{-2}\ s^{-1}}$ $v_0 = 4.0\ \mathrm{m/s}$

Theoretical Question 2: Ray tracing and generation of entangled light

Part A. Light propagation in isotropic dielectric media

A.1 0.4 pt

Ans: $\frac{1}{\sqrt{\mu_0 \epsilon}}$

Solution:

From $\vec{k}\times\vec{E} = \omega\vec{B} = \omega\mu_0\vec{H}$ and $\vec{k}\times\vec{H} = -\omega\vec{D}$, one obtains $\vec{k}\times(\vec{k}\times\vec{E}) = -\omega^2\mu_0\vec{D}$. By using the given identity $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$, one finds $\vec{k}\times(\vec{k}\times\vec{E}) = \vec{k}(\vec{k}\cdot\vec{E}) - k^2\vec{E}$. Since $\vec{D}\cdot\vec{k} = 0$ and $\vec{D} = \epsilon\vec{E}$, we find $\vec{k}\times(\vec{k}\times\vec{E}) = -k^2\vec{E}$ and the relation $\vec{k}\times(\vec{k}\times\vec{E}) = -\omega^2\mu_0\vec{D}$ reduces to $-k^2\vec{E} = -\omega^2\mu_0\epsilon\vec{E}$.

Now the phase delocity is determined by $\frac{d(\vec{k}\cdot\vec{r}-\omega t)}{dt} = 0$, we find that the phase velocity $\vec{v}_p = \frac{d\vec{r}}{dt} = \frac{\omega}{k}\hat{k}$. Clearly, we have $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon}}$. Hence $v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$.

A.2 0.2 pt

Ans: $c\sqrt{\mu_0\epsilon}$

Solution:

From $v_p = \frac{1}{\sqrt{\mu_0\epsilon}} = \frac{c}{n}$, we find $n = c\sqrt{\mu_0\epsilon}$

A.3 0.4 pt

Ans: $\hat{k}$, $v_r = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$

Solution:

To find the speed of the ray, we first note that the direction of the energy flow, given by the Poynting vector $\vec{S} = \vec{E}\times\vec{H}$, is in the same direction of $\vec{k}$. The electromagnetic energy density $u = u_e + u_m$ with $u_e = \frac{1}{2}\vec{E}\cdot\vec{D}$ and $u_m = \frac{1}{2}\vec{B}\cdot\vec{H}$.

Now, from $\vec{k}\times\vec{H} = -\omega\vec{D}$, one has $\vec{D} = -\frac{1}{v_p}\hat{k}\times\vec{H}$. Hence $u_e = -\frac{1}{2v_p}\vec{E}\cdot\hat{k}\times\vec{H} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Similarly, from $\vec{k}\times\vec{E} = \omega\vec{B}$, we find $u_m = \frac{1}{2v_p}\vec{B}\cdot\hat{k}\times\vec{E} = \frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Hence $u = \frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{B}$. We find $v_r = S/u = v_p = \frac{1}{\sqrt{\mu_0\epsilon}}$.

Part B. Light propagation in in uniaxial dielectric media

B.1 1.5pt

Ans: $n = n_o$, $\hat{B} = \pm\hat{k}\times\hat{y} = \pm(-\cos\theta, 0, \sin\theta)$, $\hat{D} = \pm\hat{y}$ or $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$, $\hat{B} = \pm\hat{y}$, $\hat{D} = \pm\hat{y}\times\hat{k} = \pm(\cos\theta, 0, -\sin\theta)$. For $\theta = 0$, there is only one permitted value for the refractive index

Solution:

From $\vec{k}\times\vec{E} = \omega\mu_0\vec{H}$ and $\vec{k}\times\vec{H} = -\omega\vec{D}$, one obtains $\vec{k}\times(\vec{k}\times\vec{E}) = -\omega^2\mu_0\vec{D}$. Writing out components and using $\omega = \frac{c}{n}k$, we find

$$-\cos^2\theta E_x + \cos\theta\sin\theta E_z = -\frac{n_o^2}{n^2}E_x,$$
$$-\cos^2\theta E_y - \sin^2\theta E_y = -\frac{n_o^2}{n^2}E_y,$$
$$-\sin^2\theta E_z + \cos\theta\sin\theta E_x = -\frac{n_e^2}{n^2}E_z.$$

After a bit rearrangement, we obtain

$$\left(1 - \frac{n_o^2}{n^2}\right)E_y = 0$$
$$\left(\frac{n_o^2}{n^2} - \cos^2\theta\right)E_x + \cos\theta\sin\theta E_z = 0$$
$$\cos\theta\sin\theta E_x + \left(\frac{n_o^2}{n^2} - \sin^2\theta\right)E_z = 0.$$

The vanishing of the determinant yields

$$\left(1 - \frac{n_o^2}{n^2}\right)\left[(\frac{n_o^2}{n^2} - \cos^2\theta)(\frac{n_e^2}{n^2} - \sin^2\theta) - \sin^2\theta\cos^2\theta\right] = 0. \tag{1}$$

Clearly, for a general θ, we have two solutions for n:

(1) $n = n_o$

In this case, $E_x = E_z = 0$. $\vec{E}$ is parallel to the y axis. From $\vec{k}\times\vec{E} = \omega\vec{B}$ and $\vec{k}\times(\mu_0\vec{B}) = -\omega\vec{D}$, we obtain the directions of $\vec{B}$ and $\vec{D}$ as $\hat{B} = \pm\hat{k}\times\hat{y} = \pm(-\cos\theta, 0, \sin\theta)$ and $\hat{D} = -\hat{k}\times\hat{B} = \pm(0,1,0) = \pm\hat{y}$.

(2) $(\frac{n_o^2}{n^2} - \cos^2\theta)(\frac{n_e^2}{n^2} - \sin^2\theta) - \sin^2\theta\cos^2\theta = 0$.

After rearrangement, we find $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$. Clearly, at $\theta = 0$, $n = n_o$, there is only one refractive index. This is the direction of the optic axis.

In this case, $E_y = 0$. Hence $\vec{E}$ lies in the xz plane. Hence the relation $\vec{k}\times\vec{E} = \omega\vec{B}$ implies $\hat{B} = \pm\hat{y}$. The relation $\vec{k}\times(\mu_0\vec{B}) = -\omega\vec{D}$ implies $\hat{D} = \pm\hat{y}\times\hat{k}$.

B.2 0.8 pt

Ans: (1) when $n = n_o$, $\hat{E} = \pm\hat{y}$ and this is an ordinary ray. $\tan\alpha = 0$.

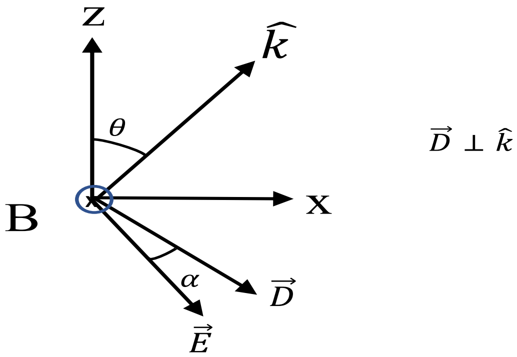
(2) when $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$, $\hat{E} = \pm\frac{1}{\sqrt{n_e^4\cos^2\theta + n_o^4\sin^2\theta}}(-n_e^2\cos\theta, 0, n_o^2\sin\theta)$ and this is an extraordinary ray. $\tan\alpha = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2\tan^2\theta}$.

Solution:

(1) For $n = n_o$, both $\vec{E}$ and $\vec{D}$ are parallel to the y axis. This is an ordinary ray with $\tan\alpha = 0$.

(2) For $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$, $n \neq n_o$, $E_y = 0$. By substituting n back into the equations of E_x and E_z, we find that $\frac{n_o^2}{n_e^2}\sin\theta E_x + \cos\theta E_z = 0$. Hence the electric field lies in xz plane with $\hat{E} = \pm\frac{1}{\sqrt{n_e^4\cos^2\theta + n_o^4\sin^2\theta}}(-n_e^2\cos\theta, 0, n_o^2\sin\theta)$ ($\vec{B}$ points in $\mp y$ direction.). Therefore, $\vec{E}$ is not perpendicular to $\vec{k}$ and lies in the xz plan in together with $\vec{D}$ and $\vec{k}$. This is the extraordinary ray.

Since $\vec{k} \times \vec{H} = -\omega\vec{D}$, $\vec{D}$ is perpendicular to $\hat{k}$. Hence $\hat{D} = \pm(-\cos\theta, 0, \sin\theta)$. Let $\vec{B} = \hat{y}$, the relative orientation of $\vec{E}$ and $\vec{D}$ for a given θ are shown in the following figure for the case when $n_e < n_o$.



Let the angle relative to x axis be θ_1 and θ_2 for $\vec{E}$ and $\vec{D}$. We have $\tan\theta_2 = -\tan\theta$ and $\tan\theta_1 = -\frac{n_o^2}{n_e^2}\tan\theta$. Hence $\tan\alpha = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_1\tan\theta_2} = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2\tan^2\theta}$. The same result remains when $n_e > n_o$ except that $\tan\alpha < 0$, indicating that the relative orientation of $\vec{E}$ and $\vec{D}$ is reversed.

B.3 0.6 pt

Ans: $n = n_o$, $\vec{E} = \pm\hat{k} \times \hat{z}/\sin\theta$ and this is an ordinary ray.
when $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$, $\hat{E} = \pm\frac{1}{\sqrt{n_e^4\cos^2\theta + n_o^4\sin^2\theta}}\frac{-n_e^2\cos\theta\hat{k} + (n_o^2\sin^2\theta - n_e^2\cos^2\theta)\hat{z}}{\sin\theta}$ and this is an extraordinary ray.

Solution: The problem has an axial symmetry so that in the plane formed by the z axis and $\hat{k}$, one can write $\vec{k} = k_z\hat{z} + k_\perp\hat{k}_\perp$ and $\vec{E} = E_z\hat{z} + E_\perp\hat{k}_\perp$, where $\hat{k}_\perp$ is perpendicular to $\hat{z}$. Clearly, we $k_z = k\cos\theta$, $k_\perp = k\sin\theta$, $E_z = E\cos\theta$, and $E_\perp = E\sin\theta$. Writing out the components for the equation: $\vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2\mu_0\vec{D}$, we get exactly the same equations except that E_x is replaced by $E_\perp$. Hence all of the solutions are the same except $\hat{x}$ is replaced by $\hat{k}_\perp$. Since $\hat{k}_\perp\sin\theta = \hat{k} - \cos\theta\hat{z}$, we obtain that when $n = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$, $\hat{E} = \pm\frac{1}{\sqrt{n_e^4\cos^2\theta + n_o^4\sin^2\theta}}[-n_e^2\cos\theta\frac{(\hat{k} - \cos\theta\hat{z})}{\sin\theta} + n_o^2\sin\theta\hat{z}] =$

$\pm\frac{1}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}\frac{-n_e^2\cos\theta\hat{k}+(n_o^2\sin^2\theta-n_e^2\cos^2\theta)\hat{z}}{\sin\theta}$.

B.4 0.8 pt

Ans: (1) $n=n_o$, $\tan\alpha_r=0$, $v_r=\frac{c}{n_o}$, $\hat{S}=(\sin\theta,0,\cos\theta)$

(2) $n=\frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta+n_e^2\cos^2\theta}}$, $\tan\alpha_r=\frac{(n_o^2-n_e^2)\tan\theta}{n_e^2+n_o^2\tan^2\theta}$, $v_r=\frac{c}{n_o n_e}\sqrt{\frac{n_e^4\cos^2\theta+n_o^4\sin^2\theta}{n_e^2\cos^2\theta+n_o^2\sin^2\theta}}$.

$\hat{S}=\frac{1}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}(n_o^2\sin\theta,0,n_e^2\cos\theta)$

(3) $n_s=\sqrt{(\hat{S}\cdot\hat{x})^2n_e^2+(\hat{S}\cdot\hat{z})^2n_o^2}$

Solution:

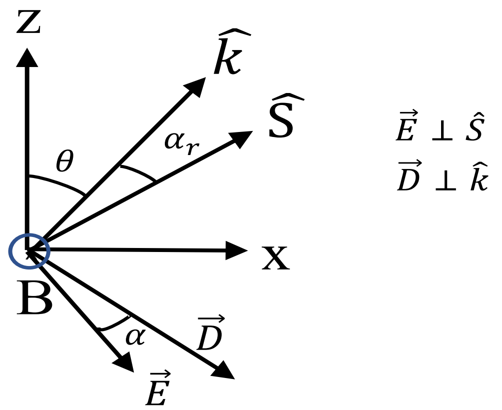
The direction of the energy flow is given by the Poynting vector, $\vec{S}=\vec{E}\times\vec{H}$. Let the energy density of EM wave be u and the ray velocity be v_r. Then $v_r=\frac{S}{u}$. Here $u=u_e+u_m$ with $u_e=\frac{1}{2}\vec{E}\cdot\vec{D}$ and $u_m=\frac{1}{2}\vec{B}\cdot\vec{H}$. There are two cases:

(i)$n=n_o$, $\vec{E}=(0,E,0)$, $\vec{D}=\epsilon\vec{E}$, $\vec{k}\times\vec{E}=\omega\mu_0\vec{H}$, $\vec{k}\times\vec{H}=-\omega\vec{D}$.

$\hat{k}$, $\vec{E}$ and $\vec{H}$ are mutually perpendicular to each other. Hence $\vec{S}$ is parallel to $\hat{k}$, i.e., $\hat{S}=(\sin\theta,0,\cos\theta)$ and $\tan\alpha_r=0$.

Now from $\vec{k}\times\vec{H}=-\omega\vec{D}$, one has $\vec{D}=-\frac{1}{v_p}\hat{k}\times\vec{H}$. Hence $u_e=-\frac{1}{2v_p}\vec{E}\cdot\hat{k}\times\vec{H}=\frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Similarly, we find $u_m=\frac{1}{2v_p}\vec{H}\cdot\hat{k}\times\vec{E}=\frac{1}{2v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Hence $u=\frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{H}$. Since $\hat{S}=\hat{k}$, we find $u=\frac{S}{v_p}$. Hence $v_r=\frac{S}{u}=v_p=\frac{\omega}{k}=\frac{c}{n_o}$.

(ii) $n=\frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta+n_e^2\cos^2\theta}}$. In this case, we can tak $\vec{B}=(0,B,0)$ (negative y direction works as well). $\vec{D}$, $\vec{E}$ and $\hat{k}$ are in the xz plane and $\vec{D}$ is perpendicular to $\hat{k}$. Therefore, the angle between $\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$ and $\hat{k}$ is equal to the angle between $\vec{D}$ and $\vec{E}$, i.e., $\alpha_r=\alpha$. This is shown in the following figure when $n_e<n_o$ (for $n_e>n_o$, both α and α_r are negative, the relative orientation of $\vec{E}$ and $\vec{D}$ is reversed and ordering of $\hat{S}$ and $\hat{k}$ are switched).



Therefore, from problem (d) (ii), we get $\tan\alpha_r = \tan\alpha = \frac{(n_o^2-n_e^2)\tan\theta}{n_e^2+n_o^2\tan^2\theta}$. Now, because $u = \frac{1}{v_p}\hat{k}\cdot\vec{E}\times\vec{H} = \frac{1}{v_p}|\vec{E}\times\vec{H}|\cos\alpha$, we obtain $v_r = \frac{S}{u} = \frac{v_p}{\cos\alpha}$. Hence the phase speed v_p and the ray speed are related by $v_p = v_r\cos\alpha$. From $\tan\alpha$, one finds $\cos\alpha = \frac{n_e^2\cos^2\theta+n_o^2\sin^2\theta}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}$. Hence $v_r = \frac{c}{n\cos\alpha} = \frac{c}{n_o n_e}\sqrt{\frac{n_e^4\cos^2\theta+n_o^4\sin^2\theta}{n_e^2\cos^2\theta+n_o^2\sin^2\theta}}$.
Clearly, $\hat{S} = (\sin(\theta+\alpha),\cos(\theta+\alpha))$. Since $\sin\alpha = \frac{(n_o^2-n_e^2)\sin\theta\cos\theta}{\sqrt{n_e^4\cos^2\theta+n_o^4\sin^2\theta}}$ and $\cos\alpha = \frac{n_e^2 c\cos^2\theta+n_o^2\sin^2\theta}{\sqrt{n_e^4 c\cos^2\theta+n_o^4\sin^2\theta}}$, we find $\hat{S} = \frac{1}{\sqrt{n_e^4 c\cos^2\theta+n_o^4\sin^2\theta}}(n_o^2\sin\theta, 0, n_e^2\cos\theta)$.
From $n_s^2 = \left(\frac{c}{v_r}\right)^2 = n_o^2 n_e^2\frac{n_e^2\cos^2\theta+n_o^2\sin^2\theta}{n_e^4\cos^2\theta+n_o^4\sin^2\theta} = \frac{(n_o^2\sin\theta)^2 n_e^2+(n_e^2\cos\theta)n_o^2}{n_e^4\cos^2\theta+n_o^4\sin^2\theta}$, we find $n_s = (\hat{S}\cdot\hat{x})^2 n_e^2 + (\hat{S}\cdot\hat{z})^2 n_o^2$.

B.5 1.1 pt

Ans: $\bar{A} = P_1(n^2\sin^2\theta_1 - P_1)$, $\bar{B} = -2P_3(n^2\sin^2\theta_1 - P_1)$, $\bar{C} = P_2 n^2\sin^2\theta_1 - P_3^2$.
$\phi = 0$, $\tan\theta_2 = \frac{n n_e\sin\theta_1}{n_o\sqrt{n_o^2-n^2\sin^2\theta_1}}$.
$\phi = \pi/2$, $\tan\theta_2 = \frac{n n_o\sin\theta_1}{n_e\sqrt{n_e^2-n^2\sin^2\theta_1}}$.

Solution:

Let the distance along z axis between A and B be d and the point of the interface that the ray passes be the origin O. The coordinates of B and A points can be expressed as $(h_2, 0, z)$ and $(h_1, 0, d-z)$. The distances are then given by $\overline{AO} \equiv d_1 = \sqrt{h_1^2+(d-z)^2}$ and $\overline{OB} \equiv d_2 = \sqrt{h_2^2+z^2}$. The propagation time from A to B is determined by the ray speed v_r as $(d_1 n_{s1} + d_2 n_{s2})/c$, where n_{si} are ray indices for medium i. According to the Fermat's principle, we need to minimize the optical path length defined by $\Delta \equiv d_1 n_{s1} + d_2 n_{s2}$. According to problem (e), we have $n_{s2}^2 = (\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{x}_2)^2 n_e^2 + (\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{z}_2)^2 n_o^2$. For an isotropic medium, the ray index is simply the refractive index, i.e., $n_{s1} = n$. Using the following relations

$$\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{x}_2 = \cos(\phi-\theta_2) = \frac{h_2}{d_2}\cos\phi + \frac{z}{d_2}\sin\phi,$$
$$\frac{\overrightarrow{OB}}{\overline{OB}}\cdot\hat{z}_2 = \cos(\frac{\pi}{2}+\phi-\theta_2) = \sin(\theta_2-\phi) = \frac{z}{d_2}\cos\phi - \frac{h_2}{d_2}\sin\phi,$$

we find

$$\Delta = n\sqrt{h_1^2+(d-z)^2} + \sqrt{(h_2\cos\phi+z\sin\phi)^2 n_e^2 + (-h_2\sin\phi+z\cos\phi)^2 n_o^2}.$$

The minimum occurs when $\frac{d\Delta}{dz} = 0$. We obtain

$$n\frac{z-d}{\sqrt{h_1^2+(d-z)^2}} + \frac{(h_2\sin\phi\cos\phi(n_e^2-n_o^2) + z(n_e^2\sin^2\phi+n_o^2\cos^2\phi)}{\sqrt{(h_2\cos\phi+z\sin\phi)^2 n_e^2 + (-h_2\sin\phi+z\cos\phi)^2 n_o^2}} = 0.$$

Recognizing $\frac{d-z}{\sqrt{h_1^2+(d-z)^2}} = \sin\theta_1$, moving the second term to the left and taking square of the equation, we obtain

$$n^2 \sin^2\theta_1 = \frac{(P_3 - P_1 \tan\theta_2)^2}{P_1 \tan^2\theta_2 - 2P_3 \tan\theta_2 + P_2},$$

where $P_1 = n_o^2 \cos^2\phi + n_e^2 \sin^2\phi$, $P_2 = n_o^2 \sin^2\phi + n_e^2 \cos^2\phi$, and $P_3 = (n_o^2 - n_e^2)\sin\phi\cos\phi$. By expanding the above equation out, we find

$$P_1(n^2 \sin^2\theta_1 - P_1)\tan^2\theta_2 - 2P_3(n^2 \sin^2\theta_1 - P_1)\tan\theta_1 + P_2 n^2 \sin^2\theta_1 - P_3^2 = 0.$$

Hence $\bar{A} = P_1(n^2 \sin^2\theta_1 - P_1)$, $\bar{B} = -2P_3(n^2 \sin^2\theta_1 - P_1)$, and $\bar{C} = P_2 n^2 \sin^2\theta_1 - P_3^2$.
For $\phi = 0$, we have $P_3 = 0$, $P_1 = n_o^2$, and $P_2 = n_e^2$. We find $n_o^2(n^2\sin^2\theta_1 - n_o^2)\tan^2\theta_2 + n_e^2 n^2 \sin^2\theta_1 = 0$. Hence $\tan\theta_2 = \frac{n n_e \sin\theta_1}{n_o\sqrt{n_o^2 - n^2\sin^2\theta_1}}$.
For $\phi = \pi/2$, we have $P_3 = 0$, $P_1 = n_e^2$, and $P_2 = n_o^2$. We find $n_e^2(n^2\sin^2\theta_1 - n_e^2)\tan^2\theta_2 + n_o^2 n^2 \sin^2\theta_1 = 0$. Hence $\tan\theta_2 = \frac{n n_o \sin\theta_1}{n_e\sqrt{n_e^2 - n^2\sin^2\theta_1}}$.

Part C. Entanglement of light

C.1 0.8 pt

Ans:(1) $\omega = \omega_1 \pm \omega_2$, $\vec{k} = \vec{k}_1 \pm \vec{k}_2$
(2) $\hbar\omega = \hbar\omega_1 \pm \hbar\omega_2$, $\hbar\vec{k} = \hbar\vec{k}_1 \pm \hbar\vec{k}_2$ represents the energy conservation and momentum conservation of photons.
(3) Splitting of photon: Energy conservation $\omega = \omega_1 + \omega_2$, momentum conservation: $\vec{k} = \vec{k}_1 + \vec{k}_2$.

Solution:

For a light wave with frequency ω and $\vec{k}$, the corresponding polarization density and the electric field are in the form of $\vec{A}\cos(\omega t - \vec{k}\cdot\vec{r})$, which can be rewritten as $\frac{\vec{A}}{2}(e^{i(\omega t - \vec{k}\cdot\vec{r})} + e^{-i(\omega t - \vec{k}\cdot\vec{r})})$. By substituting the above form into the equation $P_i^{NL} = \sum_j \sum_k \chi_{ijk}^{(2)} E_j E_k$ and equating the relevant exponents, we find all possible relations are

$$\omega = \omega_1 + \omega_2, \vec{k} = \vec{k}_1 + \vec{k}_2.$$
$$\text{or } \omega = \omega_1 - \omega_2, \vec{k} = \vec{k}_1 - \vec{k}_2,$$

where we have made use of the fact that the frequency is positive. The meaning for the these relations is clear if one recall that the energy and momentum of a photon is given by $\hbar\omega$ and $\hbar\vec{k}$. The relation of $\hbar\omega = \hbar\omega_1 + \hbar\omega_2, \hbar\vec{k} = \hbar\vec{k}_1 + \hbar\vec{k}_2$ represents the energy and momentum conservations when a photon with (ω, $\vec{k}$) is annihilated and split into two photons with (ω_1, $\vec{k}_1$) and (ω_2, $\vec{k}_2$), while the relation of $\hbar\omega = \hbar\omega_1 - \hbar\omega_2, \hbar\vec{k} = \hbar\vec{k}_1 - \hbar\vec{k}_2$ represents the energy and momentum conservations when a photon with (ω_1, $\vec{k}_1$) is annihilated and split into two photons with (ω, $\vec{k}$) and (ω_2, $\vec{k}_2$).

C.2 0.8 pt

Ans: $\mathbf{o} \rightarrow \mathbf{o} + \mathbf{o}$, $\mathbf{e} \rightarrow \mathbf{e} + \mathbf{e}$

Solution:

For the collinear case, the phase matching conditions become $\omega = \omega_1 + \omega_2$, $\frac{n_i(\omega)\omega}{c} = \frac{n_j(\omega_1)\omega_1}{c} + \frac{n_k(\omega_2)\omega_2}{c}$, where i, j, and k are indices of either **o** or **e**. Assuming that $\omega_1 \geq \omega_2$, one can solve ω_1 as $\omega_1 = \omega - \omega_2$. We obtain

$$n_i(\omega) - n_j(\omega_1) = \frac{\omega_2}{\omega}\left[n_k(\omega_2) - n_j(\omega_1)\right]. \tag{2}$$

Clearly, because $\omega > \omega_1 \geq \omega_2$, if $i = j = k$, $n_i(\omega) - n_j(\omega_1) > 0$ and $n_k(\omega_2) - n_j(\omega_1) \leq 0$, the above equation cannot be satisfied. For other cases, because there is no relation between n_o and n_e, the phase matching conditions can be satisfied. Hence only $\mathbf{o} \rightarrow \mathbf{o}+\mathbf{o}$ and $\mathbf{e} \rightarrow \mathbf{e}+\mathbf{e}$ are not possible.

C.3 1.5 pt

Ans: (1) $M = \frac{K_o[1 - N_e(\Omega_e, \theta)\cot\theta] + K_e}{2K_e K_o}$, $E = -N_e/2M$ and $F = -(\Omega - \Omega_e)(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M}$

(2) the angle between the axis of the cone and z' is $N/K_o = -\frac{2K_e N_e}{K_o[1 - N_e(\Omega_e, \theta)\cot\theta] + K_e}$

(3) the angle of cone is about $\frac{\sqrt{L/M}}{K_o} = -\frac{(\Omega - \Omega_e)}{MK_o}(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M^2 K_o}$.

Solution:

To satisfy the phase matching condition, we expand the angular frequencies ω_1 and ω_2 into $\omega_1 = \Omega_e + \nu$ and $\omega_2 = \Omega_o + \nu'$. Clearly, because $\Omega_e + \Omega_o = \Omega_p$, to satisfy $\omega_1 + \omega_2 = \omega$, $\nu' = -\nu$. Similarly, the conditions for the wavevectors, $\vec{k} = \vec{k}_1 + \vec{k}_2$, can be written as $k_z = k = K_p = k_{1z} + k_{2z}$ and $\vec{k}_{2\perp} = -\vec{k}_{1\perp} \equiv \vec{q}_\perp$. For the **o** light ray, we have $k_{2\perp}^2 + k_{2z}^2 = k_2^2$ with $k_2 = \frac{n_o(\omega_2)\omega_2}{c}$. One finds that $k_{2z} = \sqrt{k_2^2 - k_{2\perp}^2} = k_2 - \frac{k_{2\perp}^2}{2k_2}$. Expanding the dependence of ω_2 in k_2 to ν, we obtain

$$k_2 = \frac{n_o(\omega_2)\omega_2}{c} = \frac{n_o(\Omega_o)\Omega_o}{c} + \frac{dk_2}{d\omega_2}(\omega_2 - \Omega_o) = K_o - \frac{\nu}{u_o},$$

where u_o is the group velocity for the ordinary ray. Hence to the second order of corrections, we get

$$k_{2z} = K_o - \frac{\nu}{u_o} - \frac{q_\perp^2}{2K_o}.$$

Similarly, for the **e** light ray, we have $k_{1\perp}^2 + k_{1z}^2 = k_1^2$ with $k_1 = \frac{n_e(\omega_1,\theta_p)\omega_1}{c}$. One finds that $k_{1z} = \sqrt{k_1^2 - k_{1\perp}^2} = k_1 - \frac{k_{1\perp}^2}{2k_1}$. The expansion of k_1 is different from that for k_2 due to its angle dependence. Let the spherical angles for $\vec{k}_1$ be θ_1 and ϕ_1. We have

$$k_1 = \frac{n_e(\omega_1,\theta_1)\omega_1}{c} = \frac{n_e(\Omega_e,\theta)\Omega_e}{c} + \frac{dk_1(\Omega_e,\theta)}{d\Omega_e}(\omega_1 - \Omega_e) + \frac{\Omega_e}{c}\frac{dn_e(\Omega_e,\theta)}{d\theta}(\theta_1 - \theta) + \cdots$$

Here $\frac{n_e(\Omega_e,\theta)\Omega_e}{c} = K_e$, $\frac{dk_1(\Omega_e,\theta)}{d\Omega_e}$ is $1/u_e$ with u_e being the group velocity for the extraordinary ray and is given by

$$\frac{dk_1(\Omega_e,\theta)}{d\Omega_e} = \frac{n_e(\Omega_e,\theta)}{c} + \frac{\Omega_e}{c}\frac{dn_e(\Omega_e,\theta)}{d\Omega_e}.$$

Because $\frac{dn_e(\Omega_e,\theta)}{d\theta} = \frac{n_o n_e(n_e^2-n_o^2)\sin\theta\cos\theta}{(n_o^2\sin^2\theta+n_e^2\cos^2\theta)^{3/2}} = n_e(\Omega_e,\theta)N_e(\Omega_e,\theta)$, we find $N_e(\Omega_e,\theta) = \frac{(n_e^2-n_o^2)\sin\theta\cos\theta}{n_o^2\sin^2\theta+n_e^2\cos^2\theta}$. Note that for $n_e < n_o$, $N_e(\Omega_e,\theta) < 0$. To find $\delta\theta = \theta_1 - \theta$, we note that for any $\vec{k}_\alpha$, one has (cf. Fig. 2(a))

$$\hat{k}_\alpha \cdot \widehat{OA} = \cos\theta_\alpha = \cos\theta\cos\psi_\alpha + \sin\theta\sin\psi_\alpha\cos\phi_\alpha.$$

Since $\sin\psi_1 = |\vec{k}_{\perp,1}|/|\vec{k}_1| = q_\perp/k_1 \ll 1$ and $\cos\psi_1 = \sqrt{1-\sin^2\psi_1} = 1 - 1/2\sin^2\psi_1 + \cdots$, to the second order, we can replace k_1 by K_e and obtain

$$\hat{k}_1 \cdot \widehat{OA} = \cos\theta_1 = \cos\theta\left[1 - \frac{1}{2}\frac{q_\perp^2}{K_e^2} + \cdots\right] + \sin\theta\left[\frac{q_\perp}{K_e} + \cdots\right]\cos\phi_1.$$

On the other hand, $\cos\theta_1 = \cos\theta + \frac{d\cos\theta}{d\theta}(\theta_1-\theta) + \cdots = \cos\theta - \sin\theta(\theta_1-\theta) + \cdots$. Comparing this equation to the equaton for $\hat{k}_1 \cdot \widehat{OA}$, we obtain

$$\theta_1 - \theta = \frac{1}{2}\frac{q_\perp^2}{K_e^2}\cot\theta - \frac{q_\perp}{K_e}\cos\phi_1 \cdots = \frac{1}{2}\frac{q_\perp^2}{K_e^2}\cot\theta + \frac{q_{x'}}{K_e} + \cdots$$

Putting all together, we find

$$k_{1z} = K_e + \frac{1}{u_e}(\Omega - \Omega_e) + N_e(\Omega_e,\theta)q_{x'} + \frac{q_\perp^2}{2K_e}\left[N_e(\Omega_e,\theta)\cot\theta - 1\right] + \cdots.$$

The above equation when combined with the equation of k_{1z} and the relation $K_p = k_{1z} + k_{2z}$, we find

$$(\Omega - \Omega_e)(\frac{1}{u_e} - \frac{1}{u_o}) + N_e(\Omega_e,\theta)q_{x'} + q_\perp^2\left\{\frac{K_o\left[N_e(\Omega_e,\theta)\cot\theta - 1\right] - K_e}{2K_eK_o}\right\} = 0.$$

Because $n_e < n_o$, $N_e(\Omega_e, \theta) < 0$. The above equation can be rewritten in the form

$$M\left[q_{x'} - \frac{N_e}{2D}\right]^2 + Mq_{y'}^2 = -(\Omega - \Omega_e)(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M}.$$

Here $D = \frac{K_o[1-N_e(\Omega_e,\theta)\cot\theta]+K_e}{2K_eK_o} > 0$. Hence $E = -N_e/2M > 0$ $(N_e < 0)$ and $L = -(\Omega - \Omega_e)(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M}$. Clearly, the cone axis formed by $\vec{k}_2$ is characterized by $\vec{q}_\perp$. We find that the angle between the axis of the cone and z' is $\tan^{-1}(N/k_{1z})$, which is about $N/k_{1z} \approx N/K_o = -\frac{2K_eN_e}{K_o[1-N_e(\Omega_e,\theta)\cot\theta]+K_e}$. The angle of the cone is given by $\sin^{-1}\frac{\sqrt{L/M}}{k_2} \approx \frac{\sqrt{L/M}}{K_o} = -\frac{(\Omega-\Omega_e)}{MK_o}(\frac{1}{u_o} - \frac{1}{u_e}) + \frac{N_e^2}{4M^2K_o}$.

C.4 0.8pt

Ans: $P(\alpha,\beta) = \frac{1}{2}\sin^2(\alpha+\beta)$, $P(\alpha,\beta_\perp) = \frac{1}{2}\cos^2(\alpha+\beta)$, $P(\alpha_\perp,\beta) = \frac{1}{2}\cos^2(\alpha+\beta)$, $P(\alpha_\perp,\beta_\perp) = \frac{1}{2}\sin^2(\alpha+\beta)$

Solution:

For a-photon, let the electric field along the polarizer and perpendicular to the polarization represented by $|\alpha_x\rangle$ and $|\alpha_y\rangle$. Here α_x and α_x are essentially the electric field amplitudes in appropriate units. The electric fields (the states) along $\hat{x}'$ and $\hat{y}'$ can be written as

$$|\hat{x}'_a\rangle = \cos\alpha|\alpha_x\rangle - \sin\alpha|\alpha_y\rangle,$$
$$|\hat{y}'_a\rangle = \sin\alpha|\alpha_x\rangle + \cos\alpha|\alpha_y\rangle.$$

Similarly, for b-photon, we have

$$|\hat{x}'_b\rangle = \cos\beta|\beta_x\rangle - \sin\beta|\beta_y\rangle,$$
$$|\hat{y}'_b\rangle = \sin\beta|\beta_x\rangle + \cos\beta|\beta_y\rangle.$$

Hence we obtain

$$|\hat{x}'_a\rangle|\hat{y}'_b\rangle = (\cos\alpha|\alpha_x\rangle - \sin\alpha|\alpha_y\rangle)(\sin\beta|\beta_x\rangle + \cos\beta|\beta_y\rangle),$$
$$|\hat{y}'_a\rangle|\hat{x}'_b\rangle = (\sin\alpha|\alpha_x\rangle + \cos\alpha|\alpha_y\rangle)(\cos\beta|\beta_x\rangle - \sin\beta|\beta_y\rangle).$$

The state of the entangled photon pair can be written as

$$\begin{aligned}
& \frac{1}{\sqrt{2}}(|\hat{x}'_a\rangle|\hat{y}'_b\rangle + |\hat{y}'_a\rangle|\hat{x}'_b\rangle) \\
= & \frac{1}{\sqrt{2}}[(\cos\alpha\sin\beta + \sin\alpha\cos\beta)(|\alpha_x\rangle|\beta_x\rangle - |\alpha_y\rangle|\beta_y\rangle) \\
+ & (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(|\alpha_x\rangle|\beta_y\rangle - |\alpha_y\rangle|\beta_x\rangle)] \\
= & \frac{1}{\sqrt{2}}[\sin(\alpha+\beta)(|\alpha_x\rangle|\beta_x\rangle - |\alpha_y\rangle|\beta_y\rangle) + \cos(\alpha+\beta)(|\alpha_x\rangle|\beta_y\rangle - |\alpha_y\rangle|\beta_x\rangle)]
\end{aligned}$$

From the above equation, we obtain

$$P(\alpha, \beta) = \frac{1}{2}\sin^2(\alpha+\beta),$$
$$P(\alpha_\perp, \beta_\perp) = \frac{1}{2}\sin^2(\alpha+\beta),$$
$$P(\alpha, \beta_\perp) = \frac{1}{2}\cos^2(\alpha+\beta),$$
$$P(\alpha_\perp, \beta) = \frac{1}{2}\cos^2(\alpha+\beta).$$

C.5 0.5pt

Ans: $S = |\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')| + |\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')|$

$S = 2\sqrt{2}$. $S > 2$ indicates that it is not consistent with classical theories.

Solution:

One first realizes that $E(\alpha,\beta) = \frac{P(\alpha,\beta)+P(\alpha_\perp,\beta_\perp)-P(\alpha,\beta_\perp)-P(\alpha_\perp,\beta)}{P(\alpha,\beta)+P(\alpha_\perp,\beta_\perp)+P(\alpha,\beta_\perp)+P(\alpha_\perp,\beta)}$. Using expressions for P, we find

$$\begin{aligned} E(\alpha,\beta) &= \sin^2(\alpha+\beta) - \cos^2(\alpha+\beta) \\ &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)^2 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)^2 \\ &= -(\cos^2\alpha - \sin^2\alpha)(\cos^2\beta - \sin^2\beta) + 4\sin\alpha\sin\beta\cos\alpha\cos\beta \\ &= \sin(2\alpha)\sin(2\beta) - \cos(2\alpha)\cos(2\beta) = -\cos 2(\alpha-\beta). \end{aligned}$$

Hence $S = |\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')| + |\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')|$. For $\alpha = \frac{\pi}{4}$, $\alpha' = 0$, $\beta = -\frac{\pi}{8}$, $\beta' = \frac{\pi}{8}$, we find $S = |-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}| + |\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}| = 2\sqrt{2} > 2$. Hence classical theories do not apply.

Ray tracing and generation of entangled light – Marking Scheme

Remark: When student's solutions are correct and also show how solutions were obtained, it gets full credit. Only when student's solutions are incorrect or partially correct, the followings apply.

Part A. Light propagation in isotropic dielectric media

A.1 0.4 pt Solution: $\frac{1}{\sqrt{\mu_0 \epsilon}}$	Realize that the phase velocity is given by $\frac{\omega}{k}$	0.2 pt
	Correct expression for $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon}}$	0.2 pt
A.2 0.2 pt Solution: $n = c\sqrt{\mu_0 \epsilon}$	Correct relation $\frac{\omega}{k} = \frac{c}{n}$	0.1 pt
	Correct expression $n = c\sqrt{\mu_0 \epsilon}$	0.1 pt
A.3 0.4 pt Solution: $\hat{S} = \hat{k}$ $v_r = v_p = \frac{1}{\sqrt{\mu_0 \epsilon}}$	Correct expression for direction of $\hat{S}$	0.2 pt
	Correct result of computing the ratio S/u	0.1 pt
	Correct expression for $v_r = v_p = \frac{1}{\sqrt{\mu_0 \epsilon}}$	0.1 pt

Part B. Light propagation in uniaxial dielectric media

B.1 1.5pt Solution: $n = n_o$ $\hat{B} = \pm\hat{k} \times \hat{y} =$ $\pm(-\cos\theta, 0 \sin\theta)$ $\hat{D} = \pm\hat{y}$ $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$ $\hat{B} = \pm\hat{y}$ $\hat{D} = \pm\hat{y} \times \hat{k} =$ $\pm(\cos\theta, 0, -\sin\theta)$ $\theta = 0$ *or* π only one refractive index is allowed.	Realize that the determinant associated with equations for electric field has to vanish and correctly write out the form of the determinant.	0.2 pt
	Correct equation for n	0.1 pt
	Correct expressions for n n_o: 0.1pt, $\frac{n_o n_e}{\sqrt{n_e^2 \sin^2\theta + n_o^2 \cos^2\theta}}$: 0.2pt	0.3 pt
	Correct expressions for $\hat{B}$ (each direction is 0.2 pt, both + and – are given the fill credit)	0.4 pt

	Correct expressions for $\widehat{D}$ (each direction is 0.2 pt, both $+$ and $-$ are given the fill credit)	0.4 pt
	Correct value for the angle with only one refractive index	0.1 pt
B.2 0.8pt Solution: $n = n_o$ $\hat{E} = \pm\hat{y}$ ordinary light ray $\tan\alpha = 0$ $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$ $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 \cos^2\theta + n_o^4 \sin^2\theta}}$ $(-n_e^2\cos\theta, 0, n_o^2\sin\theta)$ Extraordinary light ray $\tan\alpha = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2\tan^2\theta}$	Correct ratio of $E_z{:}\,E_x$ for the case of $n = \frac{n_o n_e}{\sqrt{n_e^2 \sin^2\theta + n_o^2 \cos^2\theta}}$: 0.1 pt Correct expression for the polarization of the corresponding refractive index: $\hat{E} = \pm\hat{y}$: 0.1 pt $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 \cos^2\theta + n_o^4 \sin^2\theta}}$ $(-n_e^2\cos\theta, 0, n_o^2\sin\theta)$: 0.1pt	0.3 pt
	Correct expressions for the angle of $\vec{E}$ and $\vec{D}$ relative to x axis: 0.1 pt Correct expression for $\tan\alpha$: $\tan\alpha = 0$: 0.1 pt $\tan\alpha = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2\tan^2\theta}$: 0.1 pt	0.3 pt
	Correctly indicate types of light rays: Ordinary light ray 0.1 pt Extraordinary light ray 0.1pt	0.2 pt
B.3 0.6pt Solution: $n = n_o$ $\hat{E} = \pm\hat{z}\times\hat{k}\,/\sin\theta$ ordinary light ray $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$ $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 \cos^2\theta + n_o^4 \sin^2\theta}}$ $\frac{-n_e^2\cos^2\theta\hat{k} + (n_e^2\sin^2\theta + n_o^2\cos^2\theta)\hat{z}}{\sin\theta}$	Realize that the axial symmetry and replace $\hat{x}$ by $\hat{k}_\perp$	0.2pt
	Correct expressions for the polarization of the corresponding refractive index: $\hat{E} = \pm\hat{y}$: 0.1 pt $\hat{E} = \pm\frac{1}{\sqrt{n_e^4 \cos^2\theta + n_o^4 \sin^2\theta}}\times$ $\frac{-n_e^2\cos^2\theta\hat{k} + (n_e^2\sin^2\theta + n_o^2\cos^2\theta)\hat{z}}{\sin\theta}$: 0.1pt	0.2pt

extraordinary light ray	Correct expressions for n (0.1pt) and indications for type of light rays (0.1pt)	0.2pt
B.4 0.8 pt Solution: $n = n_o$: $\tan \alpha_r = 0$ $v_r = \frac{c}{n_o}$ $\hat{S} = (\sin\theta, 0, \cos\theta)$ $n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$: $\tan \alpha_r = \frac{(n_o^2 - n_e^2)\tan\theta}{n_e^2 + n_o^2 \tan^2\theta} = \tan\alpha$ $v_r = \frac{c}{n_o n_e} \frac{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}}{\sqrt{n_e^2 cos^2\theta + n_o^2 \sin^2\theta}}$ $\hat{S} = \frac{1}{\sqrt{n_e^4 cos^2\theta + n_o^4 \sin^2\theta}} \times$ $(n_o^2 \sin\theta, 0, n_e^2 \cos\theta)$ $n_s = \sqrt{(\hat{S}\cdot\hat{x})^2 n_e^2 + (\hat{S}\cdot\hat{z})^2 n_o^2}$	Correct expressions for $\tan \alpha_r$ (each expression 0.1pt for different n)	0.2 pt
	Correct expressions for v_r (each expression 0.1pt for different n)	0.2 pt
	Correct expressions for $\hat{S}$ (each expression 0.1pt for different n)	0.2 pt
	Correct expression for n_s	0.2 pt
B.5 1.1 pt Solution: $\bar{A} = P_1(n^2 \sin^2\theta_1 - P_1)$ $\bar{B} = -2P_3(n^2 \sin^2\theta_1 - P_1)$ $\bar{C} = P_2 n^2 \sin^2\theta_1 - P_3^2$ $\phi = 0$: $\tan\theta_2 = \frac{n n_e \sin\theta_1}{n_o \sqrt{n_o^2 - n^2 \sin^2\theta_1}}$ $\phi = \pi/2$: $\tan\theta_2 = \frac{n n_o \sin\theta_1}{n_o \sqrt{n_o^2 - n^2 \sin^2\theta_1}}$	Indicate that the path is determined by the optical path length $d_1 n_{s_1} + d_2 n_{s_2}$ where d_1 and d_2 are distances ,connecting A to O and O to B (0.1 pt). n_{s_1} and n_{s_2} are the corresponding refractive indices of the path d_1 and d_2 (0.2 pt)	0.3 pt
	Correct expression for the optical path length in terms of geometric factors (such as θ_1, ϕ, θ_2 and coordinates of points A and B) Each minor error in expression: -0.1 pt	0.3 pt
	Correct expression for $\bar{A}$	0.1 pt
	Correct expression for $\bar{B}$	0.1 pt

	Correct expression for $\bar{C}$	0.1 pt
	Correct expression for $\tan\theta_2$ when $\phi = 0$	0.1 pt
	Correct expression for $\tan\theta_2$ when $\phi = \pi/2$	0.1 pt

Part C. Entanglement of light

C.1 0.8 pt Solution: Relations: $\omega = \omega_1 \pm \omega_2$ $\vec{k} = \vec{k}_1 \pm \vec{k}_2$ $\vec{k} = \vec{k}_1 \pm \vec{k}_2$: momentum conservation $\omega = \omega_1 \pm \omega_2$: energy conservation Splitting: $\omega = \omega_1 + \omega_2$ $\vec{k} = \vec{k}_1 + \vec{k}_2$	Correct expressions for $\omega = \omega_1 \pm \omega_2$ (+: 0.1pt, −: 0.1pt)	0.2 pt
	Correct expressions for $\vec{k} = \vec{k}_1 \pm \vec{k}_2$ (+: 0.1pt, −: 0.1pt)	0.2 pt
	Adding $\hbar$ and interpretate $\hbar\vec{k} = \hbar\vec{k}_1 \pm \hbar\vec{k}_2$ as momentum conservation	0.1 pt
	Adding $\hbar$ and interpretate $\hbar\omega = \hbar\omega_1 \pm \hbar\omega_2$ as energy conservation	0.1 pt
	Correct expressions for splitting of $\omega = \omega_1 + \omega_2$ (0.1pt) and $\vec{k} = \vec{k}_1 + \vec{k}_2$ (0.1pt)	0.2 pt
C.2 0.8 pt Solution: $o \to o + o$ $e \to e + e$	Indicating that there is a confliction for splitting into the same type of the light ray due to that the refractive indices n_o and n_e are both increasing functions of ω.	0.4 pt
	Correctly listing $o \to o + o$	0.2 pt
	Correctly listing e$\to e + e$	0.2 pt
	Extra listing of splitting: -0.2 pt for each listing	

<table>
<tr>
<td rowspan="9">C.3 1.3 pt
Solution:
$M = \frac{K_o(1-N_e(\Omega_e,\theta)\cot\theta)+K_e}{2K_oK_e}$
$N = -\frac{N_e}{2M}$
$L = -(\Omega-\Omega_e)\left(\frac{1}{u_e}-\frac{1}{u_o}\right)+\frac{N_e^2}{4M}$
Angle between the axis of the cone and z' is $\frac{N}{K_o}$
$(= -\frac{K_eN_e}{K_o(1-N_e(\Omega_e,\theta)\cot\theta)+K_e})$

Angle of the cone is $\frac{\sqrt{L/M}}{K_o}$

$(= -\frac{(\Omega-\Omega_e)}{MK_o}\left(\frac{1}{u_e}-\frac{1}{u_o}\right)+\frac{N_e^2}{4M^2K_o})$</td>
<td>Realize the conservation of momentum along z direction: $K_p = k_{1z}+k_{2z}$</td>
<td>0.1 pt</td>
</tr>
<tr><td>Correct expansion of k_{2z}
Minor errors for numerical factors: -0.1 pt</td><td>0.3 pt</td></tr>
<tr><td>Correct expansion of k_{1z} in frequency
Minor errors for numerical factors: -0.1 pt</td><td>0.2 pt</td></tr>
<tr><td>Correct expansion of k_{1z} in momentum
Minor errors for numerical factors: -0.1 pt</td><td>0.2 pt</td></tr>
<tr><td>Correct expression for M</td><td>0.1 pt</td></tr>
<tr><td>Correct expression for N</td><td>0.1 pt</td></tr>
<tr><td>Correct expression for L</td><td>0.1 pt</td></tr>
<tr><td>Correct expression for the angle between the axis of the cone and z' (using N and K_o)</td><td>0.1 pt</td></tr>
<tr><td>Correct expression for the angle of the cone (using L, M and K_o)</td><td>0.1pt</td></tr>
<tr>
<td rowspan="2">C.4 0.9 pt
Solution:
$P(\alpha,\beta) = \frac{1}{2}\sin^2(\alpha+\beta)$
$P(\alpha,\beta_\perp) = \frac{1}{2}\cos^2(\alpha+\beta)$
$P(\alpha_\perp,\beta) = \frac{1}{2}\sin^2(\alpha+\beta)$
$P(\alpha_\perp,\beta_\perp) = \frac{1}{2}\cos^2(\alpha+\beta)$</td>
<td>Correctly expressing the electric fields along $\hat{x}'$ and $\hat{y}'$ direction in terms of the electric fields along the direction of the polarizer and perpendicular to the direction of polarizer for individual a-photon (0.1pt) and b-photon (0.1pt)</td>
<td>0.2 pt</td>
</tr>
<tr><td>Correctly expressing the entangled photon pair state $\frac{1}{\sqrt{2}}(|\hat{x}'_a\rangle|\hat{y}'_b\rangle + |\hat{y}'_a\rangle|\hat{x}'_b\rangle)$ in terms of combination of states using directions of</td><td>0.3 pt</td></tr>
</table>

	the polarizer: $\|\alpha_x\rangle\|\beta_x\rangle - \|\alpha_y\rangle\|\beta_y\rangle$, $\|\alpha_x\rangle\|\beta_y\rangle - \|\alpha_y\rangle\|\beta_x\rangle$	
	Correct expression of $P(\alpha,\beta)$	0.1 pt
	Correct expression of $P(\alpha,\beta_\perp)$	0.1 pt
	Correct expression of $P(\alpha_\perp,\beta)$	0.1 pt
	Correct expression of $P(\alpha_\perp,\beta_\perp)$	0.1 pt
C.5 0.5 pt Solution: $S = \|\cos 2(\alpha-\beta) - \cos 2(\alpha-\beta')\| + \|\cos 2(\alpha'-\beta) + \cos 2(\alpha'-\beta')\|$ Value of $S = 2\sqrt{2} > 2$ Inconsistent with classical theories	Correct expression of $E(\alpha,\beta)$ in terms of $P(\alpha,\beta_\perp)$, $P(\alpha_\perp,\beta)$, $P(\alpha_\perp,\beta)$ and $P(\alpha_\perp,\beta_\perp)$	0.3 pt
	Correct expression of $E(\alpha,\beta)$ in terms of α and β	0.1 pt
	Correct value of S and consistency with classical theories.	0.1 pt

Theory 3 Magnetic Levitation: Solution

Part A. Sudden appearance of a magnetic monopole: initial response and subsequent time evolution of the response in the thin film

Initial response

A.1 In the $z \geq 0$ region, excluding the point occupied by the monopole, the magnetic field $\vec{B} = \vec{B}' + \vec{B}_{\mathrm{mp}}$ at $t = t_0 = 0$ is given by

$$\vec{B}_{\mathrm{mp}} = \frac{\mu_0 q_{\mathrm{m}}}{4\pi} \frac{(z-h)\hat{z}+\vec{\rho}}{[(z-h)^2+\rho^2]^{3/2}}, \tag{A-1}$$

$$\vec{B}' = \frac{\mu_0 q_{\mathrm{m}}}{4\pi} \frac{(z+h)\hat{z}+\vec{\rho}}{[(z+h)^2+\rho^2]^{3/2}}, \tag{A-2}$$

$$\boxed{\vec{B} = \frac{\mu_0 q_{\mathrm{m}}}{4\pi}\left[\frac{(z-h)\hat{z}+\vec{\rho}}{[(z-h)^2+\rho^2]^{3/2}} + \frac{(z+h)\hat{z}+\vec{\rho}}{[(z+h)^2+\rho^2]^{3/2}}\right].} \tag{A-3}$$

A.2 In the $z \leq -d$ region, the magnetic field $\vec{B} = \vec{B}' + \vec{B}_{\mathrm{mp}}$ at $t = t_0 = 0$ is given by

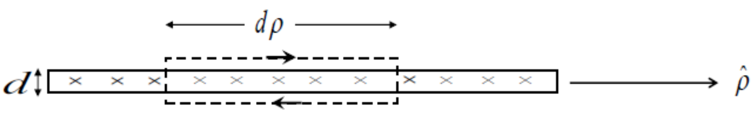
$$\boxed{\vec{B} = 0.} \tag{A-4}$$

A.3 From Eq. (A-3), $B_z' = 0$ at $z = 0$ for all ρ.
Therefore, the $\boxed{\text{total magnetic flux } \Phi_{\mathrm{B}} = 0 \text{ at } z = 0.}$ (A-5)
From Eq. (A-4), $B_z' = 0$ at $z = -d$.
Therefore, the $\boxed{\text{total magnetic flux } \Phi_{\mathrm{B}} = 0 \text{ at } z = -d.}$ (A-6)

A.4 Applying Ampere's law along the path shown in the figure below, and using the approximation $d \ll h$, we have

$$B_\rho(\rho,\ z = 0) d\rho = \mu_0\, j(\rho)\, d\rho \cdot d, \tag{A-7}$$

where the contributions from the $B_z d$ terms are smaller by a factor d/h and neglected.



The induced current density is given by

$$\boxed{\vec{j}(\vec{\rho}) = \frac{1}{\mu_0 d}\hat{z} \times \vec{B}(\vec{\rho}, z = 0) = \frac{q_{\mathrm{m}}}{2\pi d} \frac{\hat{z}\times\vec{\rho}}{(h^2+\rho^2)^{3/2}}.} \tag{A-8}$$

Subsequent response

A.5 Consider the form of an integral of Eq.(2), in the Question sheet, over the film thickness, we get, for $z \approx 0$ inside the film (that is $z < 0$ and $|z| \ll d$), that

$$\left.\frac{\partial B_z'}{\partial z}\right|_z - \left.\frac{\partial B_z'}{\partial z}\right|_{-d-z} = \mu_0 \sigma (d + 2z) \frac{\partial B_z'}{\partial t} \approx \mu_0 \sigma d \frac{\partial B_z'}{\partial t}. \tag{A-9}$$

Since B_z' is an even function of $z' = z + d/2$, therefore we have $\left.\frac{\partial B_z'}{\partial z}\right|_z = -\left.\frac{\partial B_z'}{\partial z}\right|_{-d-z}$ so that the left-hand side of Eq.(A-9) becomes $2\frac{\partial}{\partial z}B_z'(\rho, z; t)$. The right-hand side is approximated by the z-independent term of B_z' inside the film thickness. On the other hand, the z-dependent term of B_z' is even in z' and is of order $\sim z'^2 d/h$ so that it can be neglected based on the $h \gg d$ condition. As such the right-hand side is represented by $B_z'(\rho, z; t)$. Putting these results together, we get

$$2\frac{\partial}{\partial z}B_z'(\rho, z; t) = \mu_0 \sigma d \frac{\partial}{\partial t}B_z'(\rho, z; t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}B_z'(\rho, z; t) = v_0 \frac{\partial}{\partial z}B_z'(\rho, z; t).} \qquad \text{(A-10)}$$

Here $z \approx 0$, and $v_0 = 2/(\mu_0 \sigma d)$.

A.6 The equation in **A.5**, namely, Eq.(A-10) supports a solution of the form

$$\boxed{B'_z(\rho, z; t) = f(\rho, z + v_0 t),} \qquad \text{(A-11)}$$

and at $z \approx 0$.

A.7 At $t = 0$, $B_z'(\rho, z \geq 0) = \frac{\mu_0 q_m}{4\pi}\frac{(z+h)}{[(z+h)^2+\rho^2]^{3/2}}$, which is of the form

$$\boxed{B_z'(\rho, z \geq 0) = F(\rho, z + h).} \qquad \text{(A-12)}$$

For $t > 0$, we have according to Eq.(A-11), the replacement

$\boxed{z \to z + v_0 t,}$ to the $B_z'(\rho, z; t = 0)$. (A-13)

In other words, $B_z'(\rho, z \approx 0; t) = F(\rho, z + v_0 t + h)$.

This corresponds to a physical picture of a moving image monopole, with its position

$$\boxed{z_{\text{mp}} = -h - v_0 t.} \qquad \text{(A-14)}$$

Finally, $\boxed{v_0 = 2/(\mu_0 \sigma d).}$ (A-15)

Part B. Magnetic force acting on a point-like magnetic dipole moving at a constant h with a constant velocity

A moving monopole

B.1 The present locations of all the image magnetic monopoles of type q_m are at

$$\boxed{(x, z) = [-nv\tau, -h - nv_0\tau], \text{ for } n \geq 0.} \qquad \text{(B-1)}$$

The locations of all the image magnetic monopoles $-q_{\mathrm{m}}$ are at

$$\boxed{(x,z) = [-(n+1)v\tau, -h - nv_0\tau\,],\ \text{for}\ \ n \geq 0.} \qquad \text{(B-2)}$$

B.2 The magnetic potential $\Phi_+(x,z)$ due to all the image magnetic monopoles at $t=0$ is given by, in summation form

$$\Phi_+(x,z) = \frac{\mu_0 q_m}{4\pi}\sum_{n=0}^{\infty}\frac{1}{\sqrt{(x+nv\tau)^2+(z+h+nv_0\tau)^2}} - \frac{\mu_0 q_m}{4\pi}\sum_{n=0}^{\infty}\frac{1}{\sqrt{(x+(n+1)v\tau)^2+(z+h+nv_0\tau)^2}},$$

$$\Rightarrow \boxed{\Phi_+(x,z) = \frac{\mu_0 q_m}{4\pi}\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{(x+nv\tau)^2+(z+h+nv_0\tau)^2}} - \frac{1}{\sqrt{(x+(n+1)v\tau)^2+(z+h+nv_0\tau)^2}}\right].} \qquad \text{(B-3)}$$

In integral form

$$\Phi_+(x,z) = \frac{\mu_0 q_m}{4\pi\tau}\int_0^{\infty} dt'\left[\frac{1}{\sqrt{(x+vt')^2+(z+h+v_0t')^2}} - \frac{1}{\sqrt{(x+vt'+v\tau)^2+(z+h+v_0t')^2}}\right], \qquad \text{(B-4)}$$

$$= \frac{\mu_0 q_m}{4\pi\tau}\int_0^{\infty} dt'\,\frac{(x+vt')v\tau}{[(x+vt')^2+(z+h+v_0\tau)^2]^{3/2}}, \qquad \text{(B-5)}$$

$$\Rightarrow \boxed{\Phi_+(x,z) = \frac{\mu_0 q_m v}{4\pi}\,\frac{1}{(z+h)v - v_0 x}\left[\frac{z+h}{\sqrt{x^2+(z+h)^2}} - \frac{v_0}{\sqrt{v^2+v_0^2}}\right].} \qquad \text{(B-6)}$$

A moving dipole

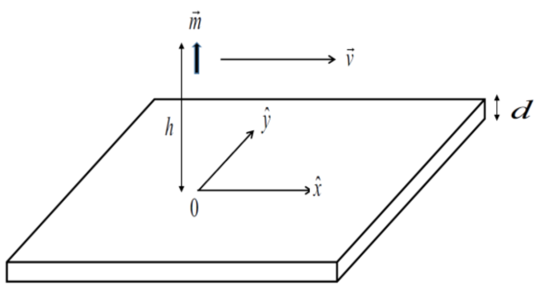
B.3

The total magnetic potential

$$\Phi_{\mathrm{T}}(x,z) = \Phi_+(x,z) + \Phi_-(x,z), \qquad \text{(B-7)}$$

where $\Phi_-(x,z) = -\Phi_+(x, z-\delta_{\mathrm{m}})$.

$$\Phi_{\mathrm{T}}(x,z) = \Phi_+(x,z) - \Phi_+(x,z-\delta_{\mathrm{m}})$$
$$= \delta_{\mathrm{m}} \times \partial\Phi_+(x,z)/\partial z. \qquad \text{(B-8)}$$



$$\Phi_{\mathrm{T}}(x,z) = -\frac{\mu_0 m v}{4\pi}\left[\frac{v}{[(z+h)v-v_0x]^2}\left(\frac{z+h}{\sqrt{x^2+(z+h)^2}} - \frac{v_0}{\sqrt{v^2+v_0^2}}\right) - \frac{x^2}{[(z+h)v-v_0x][x^2+(z+h)^2]^{3/2}}\right]. \quad \text{(B-9)}$$

Force acting on the point-like magnetic dipole:

$$F_z = -q_m \frac{d}{dz}\Phi_{\mathrm{T}}(0,z)\Big|_{z=h} + q_m \frac{d}{dz}\Phi_{\mathrm{T}}(0,z)\Big|_{z=h-\delta_{\mathrm{m}}}. \quad \text{(B-10)}$$

$$F_z = -\frac{\mu_0 m q_{\mathrm{m}}}{2\pi}\left(1 - \frac{v_0}{\sqrt{v^2+v_0^2}}\right)\left[\frac{1}{(2h)^3} - \frac{1}{(2h-\delta_{\mathrm{m}})^3}\right]. \quad \text{(B-11)}$$

$$\Rightarrow \quad \boxed{F_z = \frac{3\mu_0 m^2}{32\pi h^4}\left[1 - \frac{v_0}{\sqrt{v^2+v_0^2}}\right].} \quad \text{(B-12)}$$

$$F_x = -q_m \frac{d}{dx}\Phi_{\mathrm{T}}(x,h)\Big|_{x=0} + q_m \frac{d}{dx}\Phi_{\mathrm{T}}(x,h-\delta_{\mathrm{m}})\Big|_{x=0}, \quad \text{(B-13)}$$

$$\Rightarrow \quad \boxed{F_x = -\frac{3\mu_0 m^2}{32\pi h^4}\frac{v_0}{v}\left[1 - \frac{v_0}{\sqrt{v^2+v_0^2}}\right].} \quad \text{(B-14)}$$

Relation between v_0 and v and their relation

B.4 $\boxed{v_0 = \frac{2}{\mu_0 \sigma d} = \frac{2}{4\pi\times10^{-7}\times5.9\times10^{7}\times0.5\times10^{-2}} = 5.4\ \mathrm{m/s}\,.}$ (B-15)

B.5 In the small v regime, meaning that v is smaller than a certain typical velocity of the system (or a critical velocity v_{c} to be considered in the next task **B.**6) we have the characteristics basically akin to that of $v \approx 0$. For $v = 0$, the frequency ω is associated with v_0/h. Making use of the parameters given in **B.4**, the skin depth (Eq.(3) in the question sheet) δ is given by

$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \sqrt{\frac{2h}{v_0\mu_0\sigma}} = 1.58$ c.m., which is more than three times greater than d.

Thus we have, in the small v regime,

$\boxed{v_0(v) = v_0.}$ (B-16)

In the large v regime, we have the skin depth $\delta < d$ so that the effect thin film thickness

$$d_{\text{eff}} = \delta, \tag{B-17}$$

within which the field is more or less uniform (i.e. z independent).
In this case, $\omega = v/h$, (B-18)
so the

$$v_0(v) = \frac{2}{\mu_0 \sigma \delta} = \frac{2}{\mu_0 \sigma}\sqrt{\frac{\omega \mu_0 \sigma}{2}} = \sqrt{\frac{2}{\mu_0 \sigma}\frac{v}{h}} = \sqrt{\frac{d}{h} v\, v_0}, \quad \text{or}$$

$$\boxed{v_0(v) = v_0 \sqrt{\frac{d}{h}}\sqrt{\frac{v}{v_0}}} \tag{B-19}$$

B.6 The critical velocity v_{c} is determined from the condition $\delta = d$:

$$d = \sqrt{\frac{2}{\mu_0 \sigma v_{\text{c}}/h}} \;\Rightarrow\; \boxed{v_{\text{c}} = \frac{2h}{d^2 \mu_0 \sigma} = v_0 \frac{h}{d}} \; . \tag{B-20}$$

Part C Motion of the magnetic dipole when the conducting thin film is superconducting

When the electrical conductivity $\sigma \to \infty$, the receding velocity $v_0 \to 0$ so that there will not be a whole series of image magnetic monopoles. Instead, the image is simply one image magnetic dipole mirroring the instantaneous position of the magnetic dipole. In this case, the image magnetic dipole is $\vec{m} = m\hat{x}$ located at the location $(x, y, z) = (0,0,-h)$. It is then clear, from the symmetry of the image configuration, that the force on the magnetic dipole from the image aligns only along $\hat{z}$. For our convenience, we take the magnetic monopole $-q_{\text{m}}$ to locate at $x = 0$, and for the magnetic monopole q_{m} the location $x = \delta_{\text{m}}$.

C.1

The total magnetic potential $\Phi_{\text{T}}(x, z)$ from the image magnetic dipole is

$$\Phi_{\text{T}}(x, z) = -\frac{\mu_0 q_{\text{m}}}{4\pi}\frac{1}{\sqrt{x^2 + (z+h)^2}} + \frac{\mu_0 q_{\text{m}}}{4\pi}\frac{1}{\sqrt{(x-\delta_{\text{m}})^2 + (z+h)^2}}. \tag{C-1}$$

Approach 1:

The total vertical force F_z' acting on the magnetic dipole from the image magnetic dipole is given by

$$F_z' = (-q_{\text{m}})\left[-\frac{\partial}{\partial z}\Phi_{\text{T}}\right]\Bigg|_{\substack{x=0,\\ z=h}} + q_{\text{m}}\left[-\frac{\partial}{\partial z}\Phi_{\text{T}}\right]\Bigg|_{\substack{x=\delta,\\ z=h}} \tag{C-2}$$

$$F_z' = \frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\frac{z+h}{[x^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=0,\\ z=h}} - \frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\frac{z+h}{[(x-\delta_{\mathrm{m}})^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=0,\\ z=h}}$$

$$-\frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\frac{z+h}{[x^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=\delta_{\mathrm{m}},\\ z=h}} + \frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\frac{z+h}{[(x-\delta_{\mathrm{m}})^2+(z+h)^2]^{3/2}}\bigg|_{\substack{x=\delta_{\mathrm{m}},\\ z=h}},$$

$$F_z' = 2\frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\left(\frac{1}{2h}\right)^2\left[1-\frac{1}{\left(1+\left(\frac{\delta}{2h}\right)^2\right)^{3/2}}\right]. \quad \text{(C-3)}$$

$$F_z' = \frac{3\mu_0 m^2}{64\pi h^4}. \quad \text{(C-4)}$$

Equilibrium condition:

$$F_z' - M_0 g = 0, \quad \text{(C-5)}$$

$$\Rightarrow \quad \frac{3\mu_0 m^2}{64\pi h_0^4} = M_0 g,$$

$$\Rightarrow \quad \boxed{h_0 = \left[\frac{3\mu_0 m^2}{64\pi M_0 g}\right]^{\frac{1}{4}}.} \quad \text{(C-6)}$$

Approach 2:

We can use the direct force calculation.

$$F_z' = 2\frac{\mu_0 q_{\mathrm{m}}^2}{4\pi}\left[\left(\frac{1}{2h}\right)^2 - \frac{2h}{\left(\delta_m^2+(2h)^2\right)^{3/2}}\right] \quad \text{(C-7)}$$

$$= \frac{\mu_0 q_{\mathrm{m}}^2}{2\pi}\left(\frac{1}{2h}\right)^2\left[1-\frac{1}{\left(1+\left(\frac{\delta}{2h}\right)^2\right)^{3/2}}\right] \quad \text{(C-8)}$$

$$= \frac{3\mu_0 m^2}{64\pi h^4}.$$

The equilibrium condition $F_z' - M_0 g = 0$ gives the same equilibrium position h_0 as in Eq. (C-6),

$$\Rightarrow \quad \boxed{h_0 = \left[\frac{3\mu_0 m^2}{64\pi M_0 g}\right]^{\frac{1}{4}}.}$$

C.2

The oscillation frequency about the equilibrium is obtained from

$$F_z' \approx M_0 + \frac{dF_z'}{dz}\Delta z, \qquad \text{(C-9)}$$

where $\Delta z = z - h_{0.}$

And from $\frac{dF_z'}{dz} = -k = -M_0\Omega^2$ (C-10)

we have

$$k = -\frac{d}{dz}\frac{3\mu_0 m^2}{64\pi h^4} = \frac{3\mu_0 m^2}{16\pi h_0^5} = \frac{4}{h_0}\frac{3\mu_0 m^2}{64\pi h_0^4} = \frac{4M_0 g}{h_0} = M_0\Omega^2 \qquad \text{(C-11)}$$

The angular oscillation frequency

$$\boxed{\Omega = \sqrt{\frac{4g}{h_0}}\,.} \qquad \text{(C-12)}$$

C.3

$$h_0 = \left[\frac{3\mu_0\left(\frac{4}{3}\pi R^3 M\right)^2}{64\pi\left(\frac{4}{3}\pi R^3 \rho_0 g\right)}\right]^{1/4} = \left[\frac{R^3 M^2 \mu_0}{16\rho_0 g}\right]^{1/4} \qquad \text{(C-13)}$$

$$\boxed{h_0 = \left[\frac{10^{-18}\times 75^2\times 10^{-4}}{16\times 7400\times 9.8\times \mu_0}\right]^{1/4}\ \text{m} = 25.\,\mu\text{m}.} \qquad \text{(C-14)}$$

C.4 $$\boxed{\Omega = \sqrt{\frac{4g}{h_0}} = \sqrt{\frac{4\times 9.8}{30\times 10^{-6}}}\ \text{s}^{-1} = 1.3\ \text{kHz}.} \qquad \text{(C-15)}$$

Theory 3: Magnetic Levitation – Marking Scheme

Part A Sudden appearance of a magnetic monopole (3.0 points)

Initial response **(1.6 points)**

A.1	$$\vec{B}_{\rm mp} = \frac{\mu_0 q_{\rm m}}{4\pi}\frac{(z-h)\hat{z}+\vec{\rho}}{[(z-h)^2+\rho^2]^{3/2}}$$	**0.1**	**0.4**
	$$\vec{B}' = \frac{\mu_0 q_{\rm m}}{4\pi}\frac{(z+h)\hat{z}+\vec{\rho}}{[(z+h)^2+\rho^2]^{3/2}}$$	**0.2**	
	$$\vec{B} = \frac{\mu_0 q_{\rm m}}{4\pi}\left[\frac{(z-h)\hat{z}+\vec{\rho}}{[(z-h)^2+\rho^2]^{3/2}} + \frac{(z+h)\hat{z}+\vec{\rho}}{[(z+h)^2+\rho^2]^{3/2}}\right]$$	**0.1**	
A.2	$\vec{B} = 0$	**0.2**	**0.2**
A.3	$B_z' = 0$ at $z = 0$	**0.1**	**0.4**
	$\Phi_{\rm B} = 0$ at $z = 0$	**0.1**	
	$B_z' = 0$ at $z = -d$	**0.1**	
	$\Phi_{\rm B} = 0$ at $z = -d$	**0.1**	
A.4	$B_\rho(\rho, z=0)d\rho = \mu_0\, j(\rho)\, d\rho \cdot d$	**0.4**	**0.6**
	$$\vec{j}(\vec{\rho}) = \frac{1}{\mu_0 d}\hat{z}\times\vec{B}(\vec{\rho}, z=0) = \frac{q_{\rm m}}{2\pi d}\frac{\hat{z}\times\vec{\rho}}{(h^2+\rho^2)^{3/2}}$$	**0.2**	

Subsequent response **(1.4 points)**

A.5	$$\left.\frac{\partial B_z'}{\partial z}\right\|_z - \left.\frac{\partial B_z'}{\partial z}\right\|_{-d-z} = \mu_0\sigma(d+2z)\frac{\partial B_z'}{\partial t} \approx \mu_0\sigma d\frac{\partial B_z'}{\partial t}$$	**0.2**	**0.6**
	$$\left.\frac{\partial B_z'}{\partial z}\right\|_z = -\left.\frac{\partial B_z'}{\partial z}\right\|_{-d-z}$$	**0.2**	
	$$\frac{\partial}{\partial t}B_z'(\rho, z; t) = 2/(\mu_0\sigma d)\times\frac{\partial}{\partial z}B_z'(\rho, z; t)$$	**0.2**	
A.6	$B'_z(\rho, 0; t) = f(\rho, z+v_0 t)$ near $z \approx 0$	**0.4**	**0.4**
A.7	At $t=0$ $B_z'(\rho, z \geq 0)$ is of the form $F(\rho, z+h)$	**0.1**	**0.4**
	For $t > 0$ $z \longrightarrow z + v_0 t$	**0.1**	
	$v_0 = 2/(\mu_0\sigma d)$	**0.2**	

Part B Magnetic force acting on a point-like magnetic dipole moving at a constant *h* with a constant velocity (4.0 points)

A moving monopole **(1.5 points)**

B.1	Present positions of $q_{\rm m}$: $(x, z) = [-n v\tau, -h - n v_0\tau]$, for $n \geq 0$.	**0.4**	**0.8**
	Present positions of $-q_{\rm m}$: $(x, z) = [-(n+1)v\tau, -h - n v_0\tau]$, for $n \geq 0$.	**0.4**	

B.2	Magnetic potential : $\Phi_+(x,z)=\frac{\mu_0 q_m}{4\pi}\left[\sum_{n=0}^{\infty}\frac{1}{\sqrt{(x+nv\tau)^2+(z+h+nv_0\tau)^2}}-\sum_{n=0}^{\infty}\frac{1}{\sqrt{(x+(n+1)v\tau)^2+(z+h+nv_0\tau)^2}}\right]$	**0.3**	**0.7**
	$\Phi_+(x,z)=\frac{\mu_0 q_m}{4\pi\tau}\int_0^{\infty}dt'\left[\frac{1}{\sqrt{(x+vt')^2+(z+h+v_0t')^2}}-\frac{1}{\sqrt{(x+vt'+v\tau)^2+(z+h+v_0t')^2}}\right]$	**0.2**	
	$\Phi_+(x,z)=\frac{\mu_0 q_m v}{4\pi}\frac{1}{(z+h)v-v_0x}\left[\frac{z+h}{\sqrt{x^2+(z+h)^2}}-\frac{v_0}{\sqrt{v^2+v_0^2}}\right]$	**0.2**	

A moving dipole **(1.5 points)**

B.3	$\Phi_{\mathrm{T}}(x,z)=\Phi_+(x,z)+\Phi_-(x,z)$ where $\Phi_-(x,z)=-\Phi_+(x,z-\delta_{\mathrm{m}})$	**0.2**	**1.5**
	$\Phi_{\mathrm{T}}(x,z)=\Phi_+(x,z)-\Phi_+(x,z-\delta_{\mathrm{m}})=\delta_{\mathrm{m}}\times\partial\Phi_+(x,z)/\partial z$	**0.2**	
	$\Phi_{\mathrm{T}}(x,z)=-\frac{\mu_0 m v}{4\pi}\left[\frac{v}{[(z+h)v-v_0x]^2}\left(\frac{z+h}{\sqrt{x^2+(z+h)^2}}-\frac{v_0}{\sqrt{v^2+v_0^2}}\right)-\frac{x^2}{[(z+h)v-v_0x][x^2+(z+h)^2]^{3/2}}\right]$	**0.3**	
	$F_z=-q_m\frac{d}{dz}\Phi_{\mathrm{T}}(0,z)\Big\|_{z=h}+q_m\frac{d}{dz}\Phi_{\mathrm{T}}(0,z)\Big\|_{z=h-\delta_{\mathrm{m}}}$	**0.2**	
	$F_z=\frac{3\mu_0 m^2}{32\pi h^4}\left[1-\frac{v_0}{\sqrt{v^2+v_0^2}}\right]$	**0.2**	
	$F_x=-q_m\frac{d}{dx}\Phi_{\mathrm{T}}(x,h)\Big\|_{x=0}+q_m\frac{d}{dx}\Phi_{\mathrm{T}}(x,h-\delta_{\mathrm{m}})\Big\|_{x=0}$	**0.2**	
	$F_x=-\frac{3\mu_0 m^2}{32\pi h^4}\frac{v_0}{v}\left[1-\frac{v_0}{\sqrt{v^2+v_0^2}}\right]$	**0.2**	

Relation between v_0 and v **(1.0 points)**

B.4	$v_0=\frac{2}{\mu_0\sigma d}=\frac{2}{4\pi\times10^{-7}\times5.9\times10^{7}\times0.5\times10^{-2}}=5.4\ \mathrm{m/s}$	**0.3**	**0.3**
B.5	In the $v<v_{\mathrm{c}}$ regime: $v_0(v)=v_0$	**0.1**	**0.4**
	In the $v>v_{\mathrm{c}}$ regime: $v_0(v)=\frac{2}{\mu_0\sigma\delta}=\frac{2}{\mu_0\sigma}\sqrt{\frac{\omega\mu_0\sigma}{2}}$	**0.1**	
	$\omega=v/h$	**0.1**	
	$v_0(v)=v_0\sqrt{\frac{d}{h}}\sqrt{\frac{v}{v_0}}$	**0.1**	
B.6	$\delta=d$	**0.1**	**0.3**
	$v_{\mathrm{c}}=\frac{2h}{d^2\mu_0\sigma}=v_0\frac{h}{d}$	**0.2**	

Part C Motion of the magnetic dipole when the conducting thin film is superconducting (3.0 points)

C.1	**Approach 1:** Start from the total magnetic potential		**1.2**
	$\Phi_T(x,z) = -\frac{\mu_0 q_m}{4\pi}\frac{1}{\sqrt{x^2+(z+h)^2}} + \frac{\mu_0 q_m}{4\pi}\frac{1}{\sqrt{(x-\delta_m)^2+(z+h)^2}}$	**0.3**	
	$F_z' = (-q_m)\left[-\frac{\partial}{\partial z}\Phi_T\right]\Big\|_{x=0,\ z=h} + q_m\left[-\frac{\partial}{\partial z}\Phi_T\right]\Big\|_{x=\delta_m,\ z=h}$	**0.3**	
	$F_z' = \frac{3\mu_0 m^2}{64\pi h^4}$	**0.4**	
	$h_0 = \left[\frac{3\mu_0 m^2}{64\pi M_0 g}\right]^{\frac{1}{4}}$	**0.2**	
	Approach 2: Start from the force		
	$F_z' = 2\frac{\mu_0 q_m^2}{4\pi}\left[\left(\frac{1}{2h}\right)^2 - \frac{2h}{(\delta_m^2+(2h)^2)^{3/2}}\right]$	**0.6**	
	$F_z' = \frac{3\mu_0 m^2}{64\pi h^4}$	**0.4**	
	$h_0 = \left[\frac{3\mu_0 m^2}{64\pi M_0 g}\right]^{\frac{1}{4}}$	**0.2**	
C.2	$\frac{dF_z'}{dz} = -k = -M_0\Omega^2$	**0.5**	**0.8**
	$\Omega = \sqrt{\frac{4g}{h_0}}$	**0.3**	

C.3	$h_0 = \left[\frac{3\mu_0\left(\frac{4}{3}\pi R^3 M\right)^2}{64\pi\left(\frac{4}{3}\pi R^3\rho_0 g\right)}\right]^{1/4} = \left[\frac{R^3M^2\mu_0}{16\rho_0 g}\right]^{1/4}$	**0.3**	**0.7**
	$h_0 = \left[\frac{10^{-18}\times 75^2\times 10^{-4}}{16\times 7400\times 9.8\times\mu_0}\right]^{1/4}$ m	**0.2**	
	$h_0 = 25\ \mu\text{m}$	**0.2**	
C.4	$\Omega = \sqrt{\frac{4g}{h_0}} = \sqrt{\frac{4\times 9.8}{30\times 10^{-6}}}\ \text{s}^{-1} = 1.3\ \text{kHz}$	**0.3**	**0.3**