Theory Q1: Solutions
RF Reflectometry

Version 1.32.

A. LUMPED ELEMENT MODEL OF A CO-AXIAL TRANSMISSION LINE

A.1 The speed of wave propagation in free space ($c_0 = 299\,792\,458\,\mathrm{m/s}$) is $c_0 = 1/\sqrt{\varepsilon_0\,\mu_0}$. The speed in the dielectric & diamagnetic medium is

$$\boxed{v = \frac{c_0}{\sqrt{\varepsilon_\mathrm{r}\,\mu_\mathrm{r}}}} \tag{A.1}$$

A.2 Gauss law for the flux through a cylindrical surface with radius r co-axial with the the core, $a < r < b$:

$$\Delta x\, 2\pi r\, E(r) = \frac{\Delta q}{\varepsilon_\mathrm{r}\varepsilon_0} \Rightarrow E(r) = \frac{\Delta q}{\Delta x}\frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0 r} \tag{A.2}$$

A.3 The capacitance

$$C_x\,\Delta x = \frac{\Delta q}{\varphi} \tag{A.3}$$

where the potential φ of the core with respect to the shield is

$$0 - \varphi = -\int_a^b E(r)\,dr \Rightarrow \varphi = \frac{\Delta q}{\Delta x}\frac{1}{2\pi\varepsilon_\mathrm{r}\varepsilon_0}\ln\frac{b}{a} \tag{A.4}$$

$$\boxed{C_x = \frac{2\pi\varepsilon_\mathrm{r}\varepsilon_0}{\ln\frac{b}{a}}} \tag{A.5}$$

A.4 The magnetic flux through a rectangular contour paralel to the axis equal inductance times the current:

$$\Delta x \int_a^b B(r)\,dr = L_x\,\Delta x\, I \tag{A.6}$$

Biot-Savart law $B(r) = \frac{\mu_\mathrm{r}\mu_0}{2\pi}\frac{I}{r}$ gives

$$\boxed{L_x = \frac{\mu_\mathrm{r}\mu_0}{2\pi}\ln\frac{b}{a}} \tag{A.7}$$

A.5 i. Adding δx length of the cable should not change its impedance. Hence the impedance Z of the following circuit must be equal to Z_0:

$$\frac{1}{Z} = \frac{1}{Z_0 + j\omega\delta L} + \frac{1}{\frac{1}{j\omega\delta C}} = \frac{1}{Z_0} \tag{A.8}$$

$$Z_0^2 + j\,\omega\,\delta L\,Z_0 - \delta L/\delta C = 0 \tag{A.9}$$

(here engineering notation for $j^2 = -1$ is used.) $\delta L/\delta C = L_x/C_x$ and $\delta L \to 0$ for $\delta x \to 0$, hence

$$\boxed{Z_0 = \sqrt{L_x/C_x}} \tag{A.10}$$

ii.

$$Z_0 = \sqrt{L_x/C_x} = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_\mathrm{r}\mu_0}{\varepsilon_\mathrm{r}\varepsilon_0}} = \ln(b/a)\sqrt{\frac{\mu_\mathrm{r}}{\varepsilon_\mathrm{r}}} \times 59.96\,\Omega \tag{A.11}$$

For $Z_0 = 50\,\Omega$, $\varepsilon_\mathrm{r} = 4.0$ and $\mu_\mathrm{r} = 1.0$ this gives $\boxed{b = 5.30\,a}$.

B. HYPOTHETICAL TRANSMISSION LINE WITH RETURN ALONG A GROUNDED PLANE

B.1 The high-conductance ground plate can be replaced by an image of the wire with opposite direction of the current at distance $2d$ from the real wire. The magnetic fields from the real and the imaginary wires add up and need to be integrated to get the magnetic flux between the wire and the plate:

$$L_x \, \Delta x \, I = \frac{\mu\mu_0}{2\pi} I \int_a^d \left(\frac{1}{r} + \frac{1}{2d - r} \right) dr \, \Delta x \tag{B.1}$$

$$L_x = \frac{\mu\mu_0}{2\pi} \ln \left(\frac{2d}{a} - 1 \right) \approx \frac{\mu\mu_0}{2\pi} \ln \frac{2d}{a} \tag{B.2}$$

The potential difference between the wire and the plate can be obtained similarly by integrating the combined field for the wire and its image:

$$\varphi = \frac{\Delta q}{\Delta x} \frac{1}{2\pi\varepsilon_{\mathrm{r}}\varepsilon_0} \int_a^d \left(\frac{1}{r} + \frac{1}{2d - r} \right) dr = \frac{\Delta q}{\Delta x} \frac{\ln(2d/a)}{2\pi\varepsilon_{\mathrm{r}}\varepsilon_0} \tag{B.3}$$

$$C_x = \frac{\Delta q}{\Delta x} \frac{1}{\varphi} \approx \frac{2\pi\varepsilon_{\mathrm{r}}\varepsilon_0}{\ln(2d/a)} \tag{B.4}$$

Hence the characterstic impedance $Z_0 = \sqrt{L_x / C_x}$ of the wire-plate system is

$$\boxed{Z_0 = \frac{\ln(2d/a)}{2\pi} \sqrt{\frac{\mu_{\mathrm{r}}\mu_0}{\varepsilon_{\mathrm{r}}\varepsilon_0}}} \tag{B.5}$$

C. BASICS OF RF REFLECTOMETRY

C.1 At the interface, values of the voltage on both transmission lines have to coincide:

$$V_{\mathrm{i}} + V_{\mathrm{r}} = V_{\mathrm{t}} \tag{C.1}$$

The current has to be conserved at the interface, however, the incident and the reflected waves carry the current in opposite directions:

$$\frac{V_{\mathrm{i}}}{Z_0} - \frac{V_{\mathrm{r}}}{Z_0} = \frac{V_{\mathrm{t}}}{Z_1} \tag{C.2}$$

It is clear from the equation above that $V_{\mathrm{t}} \neq 0$ if $Z_0 \neq Z_1$ – impedance mismatch has to cause reflection. Solving the voltage and the current equations for $\Gamma = V_{\mathrm{r}}/V_{\mathrm{i}}$ gives

$$\boxed{\Gamma = \frac{Z_1 - Z_0}{Z_1 + Z_0}} \tag{C.3}$$

C.2 A π-shift implies opposite signs of V_{i} and V_{r} and hence requires $\Gamma < 0$. This implies $\boxed{Z_1 < Z_0}$.

D. THE SINGLE ELECTRON TRANSISTOR

D.1 i. Since any capacitance beyond C_g is neglected in our model, the quantum dot can be thought as a capacitor plate with the gate being the other plate of the same capacitor with capacitance C_g. The fixed number n of electrons trapped on the quantum dot sets a fixed-charge ($q = -ne$) boundary condition for the capacitor C_g on the QD, while the gate side is kept at a constant potential V_g. (We denote the elementary charge by $e > 0$). The implies that an excess charge of opposite sign, $-q = ne$ will accumulate on the gate, to keep electric field confined between the QD and the gate. The potential jump across the capacitor from the gate to the QD will be equal to the capacitor $q/C_g = -ne/C_g$. Hence the potential on the QD is

$$\boxed{\varphi_n = V_g + \frac{-ne}{C_{\mathrm{g}}}} \tag{D.1}$$

ii. Bringing an infinitesimal charge δq from potential 0 to potential $\varphi(q)$ requires energy $\delta E = \varphi(q)\delta q$, and the dependence of potential $\varphi(q)$ on the accumulated charge q is linear. For the single-electron transfer, the additional charge of the electron, $-e$, changes the potential from φ_n to $\varphi_{n+1} = \varphi_n - e/C_g$. Hence the work necessary to accumulate an extra e on the QD is the integral of δE

$$\Delta E_n = -e\frac{\varphi_n + \varphi_{n+1}}{2} \tag{D.2}$$

$$\boxed{\Delta E_n = \frac{e^2}{C_g}\left(n + \frac{1}{2}\right) - eV_g} \tag{D.3}$$

Alternatively, ΔE_n can be obtained from energy conservation, by computing the change of the energy of the capacitor the dork the work done against the electromotive force of the battery (=− "work done by the battery') for a charge $+e$ to be brought from the ground potential via the battery to the gate-side plate of the capacitor:

$$\Delta E_n = \frac{e^2(n+1)^2}{2C_g} - \frac{e^2n^2}{2C_g} - eV_g \tag{D.4}$$

Note that without $C_t \ll C_g$ approximation, the answer is $\Delta E_n = \frac{e^2}{C_g+2\,C_t}\left(n + \frac{1}{2}\right) - eV_gC_g/(2\,C_t + C_g)$ (not required to receive full marks).

D.2 $\mathcal{N}$ is a minimal integer n for which $\Delta E_n \geq 0$. Consider the marginal case of $\Delta E_{\mathcal{N}} = 0$ which is achieved at some $V_g = V_0$,

$$\Delta E_{\mathcal{N}}(V_0) = 0 = \frac{e^2}{C_g}\left(\mathcal{N} + \frac{1}{2}\right) - eV_0 \tag{D.5}$$

If V_g would go slightly larger than V_0, then ΔE_n would go negative and then minimal n that makes a positive ΔE_n would jump from $\mathcal{N}$ to $\mathcal{N}+1$. Hence $E_c = \Delta E_{\mathcal{N}+1}(V_0)$. This gives

$$\Delta E_{\mathcal{N}+1}(V_0) = E_c = \frac{e^2}{C_g}\left(\mathcal{N} + 1 + \frac{1}{2}\right) - eV_0 = \boxed{\frac{e^2}{C_g}} \tag{D.6}$$

D.3 In a metal, only electrons in an energy range $\pm \approx k_BT$ around the Fermi level take part in the thermal motion. (Here k_{B} is the Boltzmann constant.) Typical energy of these electrons is $k_{\mathrm{B}}T$ per particle and it may not exceed characteristic single-electron addition energy E_c, $\boxed{k_{\mathrm{B}}T < E_c}$.

D.4 i. $\boxed{\tau = R_t\,C_t}$

ii. Quantum uncertainty of energy (life-time broadening) h/τ must be less than the energy difference between the states with n and $n+1$ electrons,

$$h/\tau < E_c \Rightarrow \frac{h}{R_tC_t} < \frac{e^2}{C_g} \tag{D.7}$$

$$\boxed{R_t > \frac{h}{e^2}\frac{C_g}{C_t}} > \frac{h}{e^2} \tag{D.8}$$

E. RF REFLECTOMETRY TO READ OUT SET STATE

E.1

$$\Gamma = \frac{Z_{\mathrm{SET}} - Z_0}{Z_{\mathrm{SET}} + Z_0} \tag{E.1}$$

$$\Gamma_{\mathrm{ON}} = \frac{10^5 - 50}{10^5 + 50} \approx 1 - 2\frac{50}{10^5} \tag{E.2}$$

$$\Gamma_{\mathrm{OFF}} = \lim_{Z_1\to\infty}\frac{Z_1 - Z_0}{Z_1 + Z_0} = 1 \tag{E.3}$$

$$\Delta\Gamma = |\Gamma_{\mathrm{ON}} - \Gamma_{\mathrm{OFF}}|\boxed{\approx 1.0 \cdot 10^{-3}} \tag{E.4}$$

E.2 Large change in reflectance requires the impedance Z_1 of the circuit to switch between $Z_1 < Z_0$ to $Z_1 > Z_0$ as the SET between ON ($Z_{\mathrm{SET}} = 100\mathrm{k}\Omega$) and OFF ($Z_{\mathrm{SET}} = \infty$).

In the OFF state of the SET, the circuit is an disspationless LC contour with resonance frequency $\omega_0 = 1/\sqrt{L_0 C_0}$ and its impedance is 0. If we choose

$$\boxed{L_0 = \frac{1}{\omega_{\mathrm{rf}}^2 C_0}} \tag{E.5}$$

then the imedance of the $\omega_0 = \omega_{\mathrm{rf}}$.

Since Z_{tot} (the total impedance of the circuit) in the OFF state of the SET equals to 0, the reflectance i $\Gamma_{\mathrm{OFF}} = -1$. As we switch to the ON state with $Z_{\mathrm{SET}} = R_{\mathrm{SET}} = 10^5\Omega$, the change in reflectance will be large if $|Z_{\mathrm{tot}}|$ in this ON state is on the order of Z_0 or larger, which is indeed the case.

For the ON state and $\omega_0 = \omega_{\mathrm{rf}}$

$$Z_{\mathrm{tot}} = \left(\frac{1}{\frac{1}{j\,\omega\,C_0}} + \frac{1}{R_{\mathrm{SET}}} \right)^{-1} + j\,\omega L_0 = \frac{R_{\mathrm{SET}}}{1 + j\,\omega C_0\,R_{\mathrm{SET}}} + j\,\omega\,L_0 = \frac{R_{\mathrm{SET}} + j\,\sqrt{L_0/C_0}}{1 + R_{\mathrm{SET}}^2 C_0/L_0} \tag{E.6}$$

For $C_0 = 0.4 \cdot 10^{-12}\,\mathrm{F}$, $Z_0 = 50\,\Omega$ and $\omega_{\mathrm{rf}} = 2\pi \cdot 10^8\,\mathrm{Hz}$, we have $\boxed{L_0 = 6.33\,\mu\mathrm{H}}$, $Z_{\mathrm{tot}} = (158 + 6.3\,j)\Omega$, $\Gamma_{\mathrm{ON}} = 0.5198 + 0.0145\,j$, and $\boxed{\Delta\Gamma = 1.52}$.

F. CHARGE SENSING WITH A SINGLE LEAD QUANTUM DOT

F.1 The SLQD readout circuit contains only reactive elements, so $|\Gamma| = 1$ will always be one. The OFF state of the SLQD corresponds to an inductor L_0 and a capacitor C_0 connected in parallel. We again choose

$$\boxed{\omega_{\mathrm{rf}} = 1/\sqrt{L_0 C_0}} \tag{F.1}$$

so that Z_{tot} is the OFF state is infinite and $\Gamma_{\mathrm{OFF}} = 1$.

The ON state corresponds to $Z_{\mathrm{SET}} = -j\frac{1}{\omega_{\mathrm{rf}} C_q}$ and Z_{tot} at $\omega_{\mathrm{rf}} = \omega_0$ is just the impedance of the SLQD

$$Z_{\mathrm{tot}} = \frac{1}{(j\omega_{\mathrm{rf}} L_0)^{-1} + j\omega_{\mathrm{rf}}(C_0 + C_q)} = -j\frac{1}{\omega_0 C_q} = -j\frac{C_0}{C_q} Z_C \tag{F.2}$$

For the complex phase of $\Gamma_{\mathrm{ON}} = (Z_{\mathrm{tot}} - Z_0)/(Z_{\mathrm{tot}} + Z_0)$ to be significantly different from zero, we need $|Z_{\mathrm{tot}}| \sim Z_0$ since Z_{tot} is purely imaginary. Hence

$$\boxed{Z_C \sim \frac{C_q}{C_0} Z_0} \tag{F.3}$$

F.2 If L_0 is fixed, we can still operate the circuit at the frequency

$$\boxed{\omega_{\mathrm{rf}} = 1/\sqrt{L_0 C_0}} \tag{F.4}$$

that gives $\Gamma_{\mathrm{OFF}} = 1$. However, we need to deduce a way to increase $|Z_{\mathrm{tot}}|$ even if $Z_C \ll C_q Z_0/C_0$ is not sufficient. One of the ways to do that is to add an additional capacitance C_m is series with rest of the circuit. This will give (at $\omega_{\mathrm{rf}} = \omega_0$)

$$Z_{\mathrm{tot}} = -j\left(\frac{C_0}{C_q} Z_C + \frac{1}{\omega_0 C_m} \right) = -j\omega_0^{-1}\left(C_q^{-1} + C_m^{-1} \right) \tag{F.5}$$

We can satisfy the condition $|Z_{\mathrm{tot}}| = Z_0$ (and hence $\Gamma_{\mathrm{ON}} = j$ and $\Delta\Gamma = \sqrt{2} \sim 1$) with

$$C_m = \frac{C_q}{Z_0 C_q \omega_{\mathrm{rf}} - 1} = \frac{C_q\sqrt{L_0 C_0}}{Z_0 C_q - \sqrt{L_0 C_0}} \tag{F.6}$$

$$\boxed{C_m = \frac{C_q Z_C}{Z_0 C_q/C_0 - Z_C}} \overset{Z_C \ll Z_0 C_q/C_0}{\approx} \frac{1}{Z_0\,\omega_{\mathrm{rf}}} \tag{F.7}$$

Theory
Question 2: X-ray jets from active galactic nuclei
Solutions

Asian Physics Olympiad Adelaide 2019

Part A: 1d fluid model of a jet

A1

If you consider a prism of plasma in the jet frame, it contains a number of particles N, has length l in the direction of motion, and cross sectional area A. The total number of particles in the volume is invariant on transformation into the AGN frame, however the volume occupied by the plasma changes as lengths are contracted in the direction of motion, while perpendicular lengths are unchanged. Hence, $A' = A$, and $l' = l/\gamma$.

This gives us two relationships:

$$N = n(s)Al \tag{1}$$

and

$$N = n'(s)Al/\gamma \tag{2}$$

Equating these gives

$$n(s)Al = n'(s)Al/\gamma \quad ,$$

which leads to

$$n'(s) = \gamma n(s) \quad . \tag{3}$$

A2

The particles in the jet have a bulk flow speed of $v(s)$, so in a time Δt a volume $V = A(s)v(s)\Delta t$ crosses the cross section of the jet. Using the number density in the AGN frame,

$$F_{\mathrm{p}}(s) = n'(s)A(s)v(s) \tag{4}$$

$$= \gamma(s)n(s)A(s)v(s) \tag{5}$$

A3

As the plasma travels along the jet there are no particles passing through the side boundary of the jet. Hence, the total flux through the curved edges of the jet is zero, and the total flux into the jet is the flux in through the cross section at s_1 is $F_{\mathrm{p}}(s_1)$ and the total flux out of the jet is $F_{\mathrm{p}}(s_2)$. There is an additional term in the continuity equation due to the mass injection. There are $\alpha V/\mu_{\mathrm{pp}}$ particles injected.

This gives

$$\gamma(s_2)v(s_2)n(s_2)A(s_2) - \gamma(s_1)v(s_1)n(s_1)A(s_1) = \alpha V/\mu_{\mathrm{pp}} \tag{6}$$

A4

Similarly, in the AGN frame the energy flux

$$F_{\mathrm{E}}(s) = n'(s)A'(s)v(s)\epsilon'_{\mathrm{av}}(s) \quad . \tag{7}$$

We use previous results for all quantities except average energy per particle.

Consider the total energy in a volume ΔV of the plasma, $E_{\mathrm{tot}} = \epsilon_{\mathrm{av}}N$ in the jet frame. As this is the proper frame v(s)=0.

Transforming to the AGN frame, $E'_{\mathrm{tot}} = \gamma(s)\epsilon_{\mathrm{av}}N$, and $\epsilon'_{\mathrm{av}} = \gamma\epsilon_{\mathrm{av}}$.

Hence,

$$F_{\mathrm{E}}(s) = (\gamma(s))^2\, n(s)A's)v(s)\epsilon_{\mathrm{av}}(s) \quad . \tag{8}$$

Energy conservation requires that the total energy flux out of the jet is equal to the energy added through injection of mass, so

$$(\gamma(s_2))^2\, v(s_2)n(s_2)A(s_2)\epsilon_{\mathrm{av}}(s_2) - (\gamma(s_1))^2\, v(s_1)n(s_1)A(s_1)\epsilon_{\mathrm{av}}(s_1) = \alpha V c^2 \tag{9}$$

A5

From the defintion of jet power and also (8),

$$P_j(s) = (\gamma(s))^2\, n(s)A's)v(s)\epsilon_{\rm av}(s) - \dot{M}c^2 \quad . \tag{10}$$

Here $\dot{M}$ is the flux of mass flux across the surface, so $\dot{M} = F_{\rm p}(s)\mu_{\rm pp}$ and

$$P_j(s) = (\gamma(s))^2\, n(s)A's)v(s)\epsilon_{\rm av}(s) - F_{\rm p}(s)\mu_{\rm pp}c^2 \quad . \tag{11}$$

In order to find how jet power varies along the jet, we consider jet power at two points along the jet.

$$P_{\rm j}(s_2) - P_{\rm j}(s_1) = (\gamma(s_2))^2\, n(s_2)A'(s_2)v(s_2)\epsilon_{\rm av}(s_2) - F_{\rm p}(s_2)\mu_{\rm pp}c^2 \tag{12}$$

$$- \left((\gamma(s_2))^2\, n(s_1)A'(s_1)v(s_1)\epsilon_{\rm av}(s_1) - F_{\rm p}(s_1)\mu_{\rm pp}c^2\right) \quad . \tag{13}$$

We can identify the two terms with $\epsilon_{\rm av}$ to be those from the left hand side of (8), and the two terms with $\mu_{\rm pp}$ are $\mu_{\rm pp}c^2$ times the left hand side of (6). Making these substitutions,

$$P_{\rm j}(s_2) - P_{\rm j}(s_1) = \alpha V c^2 - \alpha V c^2 = 0 \quad . \tag{14}$$

This argument applies to arbitary s_1 and s_2, so the jet power is constant along the jet and $\frac{dP_{\rm j}}{ds} = 0$.

A6

We start from (10) and substitute $\epsilon_{\rm av} = \mu_{\rm pp}c^2 + \frac{13}{4}\frac{P}{n}$, to arrive at

$$P_{\rm j}(s) = (\gamma(s))^2\, n(s)A(s)v(s)(\mu_{\rm pp}c^2 + \frac{13}{4}\frac{P}{n(s)}) - \gamma(s)n(s)A(s)v(s)\mu_{\rm pp}c^2 \tag{15}$$

$$= (\gamma(s)-1)\gamma(s)n(s)A(s)v(s)\mu_{\rm pp}c^2 + (\gamma(s))^2\, A(s)v(s)\frac{13}{4}P \tag{16}$$

$$= (\gamma(s)-1)\dot{M}c^2 + (\gamma(s))^2\, A(s)v(s)\frac{13}{4}P \tag{17}$$

Rearranging to find $\dot{M}$ gives

$$\dot{M} = \frac{P_{\rm j} - \gamma(s)^2 A(s)v(s)\frac{13}{4}P}{(\gamma(s)-1)c^2} \tag{18}$$

Using the relationship $P(s) = 5.7 \times 10^{-12}\left(\frac{s}{s_0}\right)^{-1.5}$ and substituting values for s_1 and s_2 respectively into (18), give $\dot{M}_1 = 2.8 \times 10^{19}\ \mathrm{kg\,s^{-1}}$ and $\dot{M}_2 = 5.2 \times 10^{19}\ \mathrm{kg\,s^{-1}}$.

Note: some of the input values are given to one significant figure only. Hence, answers which are correct to this degree of precision and are given to one or two significant figures are accepted as correct.

A7

From lorentz transforming $\epsilon_{\rm av}$ from the jet frame where $v = 0$ to the AGN frame, the average momentum per particle is $p_{\rm av} = \gamma(s)\frac{v(s)}{c^2}\epsilon_{\rm av}$. As the momentum is directly proportional to the total energy, the flux argument is the same, and

$$\Pi(s) = \frac{F_{\rm E}}{c}\frac{v(s)}{c} \quad . \tag{19}$$

This can be related to the jet power and $\dot{M}$,

$$\Pi(s) = \left(\frac{P_{\rm j}}{c} + \dot{M}c\right)\frac{v(s)}{c} \quad . \tag{20}$$

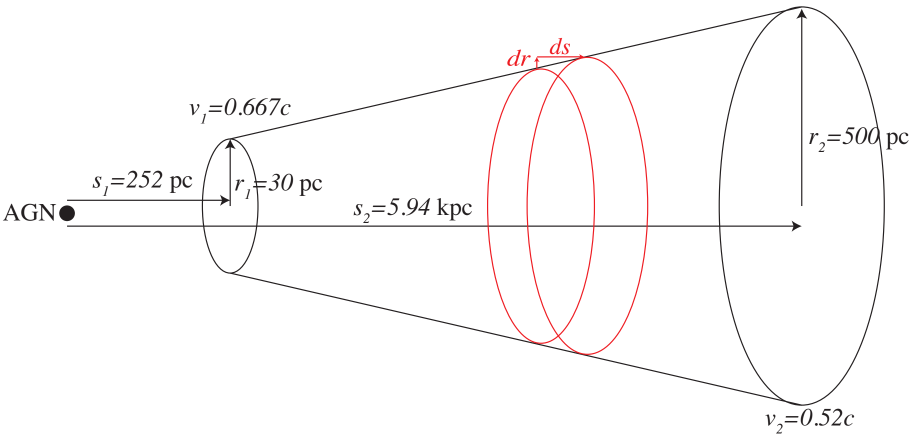
Again, there is no particle flux, and hence no momentum flux through the sides of the jet, so the total momentum flux out of the jet is

$$\Pi = \Pi(s_2) - \Pi(s_1) \quad . \tag{21}$$

Substituting values for the jet at s_2 and s_1 gives $\Pi = 1.9 \times 10^{27}\ \mathrm{kg\,m\,s^{-2}}$.

A8

The total force on the jet due to external pressure has contributions from the cross section at s_1, $F_1 = P(s_1)A(s_1)$, at s_s, $F_2 = P(s_2)A(s_2)$, and from the pressure on the curved surface. We have a linear relationship $s(r) = s_1 + \frac{s_2 - s_1}{r_2 - r_1}(r - r_1)$.



The nett pressure force on the surface is only the component in the s direction. As the force is perpendicular to the surface, this results in a factor of $\frac{dr}{ds}$. Consequently

$$dF = 2\pi r P(s) dr \quad , \tag{22}$$

where $P(s) = 5.7 \times 10^{-12} \left(\frac{s}{s_0}\right)^{-1.5}$.

The total force due to the external pressure,

$$F_{\rm Pr} = F_1 - F_2 + \int_{r_1}^{r_2} dF \quad . \tag{23}$$

Evaluating the integral gives $\int_{r_1}^{r_2} dF = 9.8 \times 10^{26}$ N, so $F_{\rm Pr} = 8.2 \times 10^{26}$ N.

A9

As there are no other forces on the jet, it is expected that $\Pi = F_{\rm Pr}$.

The % deviation is $|(\Pi - F_{\rm Pr})/F_{\rm Pr}| \approx 40\%$

Gas of ultrarelativistic electrons

B1

The total energy per volume is

$$\int_0^\infty \epsilon f(\epsilon) d\epsilon$$

B2

Consider the particles colliding with a surface ΔA,with the normal to the surface in the z-direction, in time Δt. As the electrons are ultrarelativistic, theirs speeds are all approximately c. We assume that the collisions with wall are elastic, and electrons depart with their parallel mometnum unchanged and $p_{z,\text{ final}} = -p_z$. Hence, $\Delta p_z = 2p_z$, where $p_z = \frac{\epsilon}{c}\cos\theta$, since the electrons are ultrarelativistic and $E \approx pc$.

The distribution is isotropic so electrons are equally likely to be travelling in any direction.

All electrons within a parallelepiped of length $c\Delta t$ which approach the surface at an angle θ will hit it in the time Δt. The volume of the paralleleiped is $c\Delta t \Delta A \cos\theta$. From here, the total change in momentum is

$$\begin{aligned}
\Delta p_z &= \int_0^\infty \int_0^{\pi/2} \int_0^{2\pi} 2 f(\epsilon) p_z c \Delta t \Delta A \cos\theta \frac{\sin\theta}{4\pi} d\phi d\theta d\epsilon && (24)\\
&= \frac{2\Delta t \Delta A}{4\pi} \int_0^{\pi/2} \sin\theta \cos^2\theta d\theta \int_0^{2\pi} d\phi \int_0^\infty \epsilon f(\epsilon) d\epsilon && (25)\\
&= \frac{2\Delta t \Delta A}{4\pi} \times \frac{1}{3} \times 2\pi \int_0^\infty \epsilon f(\epsilon) d\epsilon && (26)
\end{aligned}$$

B3

As the remaining integral in the expression above was identified as the energy per volume in B1, $\Delta p_z = \Delta t \Delta A \frac{1}{3}\frac{E}{V}$. The pressure is the force per area normal to the wall, so $P = \frac{\Delta p_z}{\Delta t}\frac{1}{\Delta A}$. Combining these gives $P = \frac{E}{3V}$, or $E = 3PV$, which is the equation of state.

B4

For an adiabatic process $dQ = 0$ so $dE = dW = -PdV$. $dE = d(3PV) = 3PdV + 3VdP$, so equating these expressions gives

$$3PdV + 3VdP = -pdV \tag{27}$$

$$4PdV = -3VdP \tag{28}$$

$$4\frac{dV}{V} = -3\frac{dP}{P} \tag{29}$$

$$4\int_{V_0}^{V}\frac{dV'}{V'} = -3\int_{P_0}^{P}\frac{dP'}{P} \tag{30}$$

$$4\ln\left(\frac{V}{V_0}\right) = -3\ln\left(\frac{P}{P_0}\right) \tag{31}$$

$$\frac{PV^{4/3}}{P_0V_0^{4/3}} = 1 \tag{32}$$

Synchrotron emission

C1

An electron in a magnetic field has a component of its velocity, $v\cos\phi$ along the magnetic field, and $v\sin\phi$ perpendicular to the field. The parallel component of the velocity remains constant, but in the perpendicular direction the electron experiences a force in a direction perpendicular to its motion, so it undergoes simple harmonic motion. The perpendicular component of its velocity is Ωr where Ω is its angular frequency and r the radius of the circular motion. The force on the electron is $\mathbf{F}_\mathrm{B} = q\mathbf{v}\times\mathbf{B} = e\Omega rB\sin\phi$. The acceleration of the electron is perpendicular to the direction of motion, so $F_B = \gamma ma$, where a is the acceleration and m the mass of the electron. For uniform circular motion, $a = -\Omega^2 r$, so

$$F_\mathrm{B} = \gamma m\Omega^2 r \tag{33}$$

$$e\Omega rB\sin\phi = \gamma m\Omega^2 r \tag{34}$$

$$\Omega = \frac{eB\sin\phi}{\gamma m} \tag{35}$$

C2

The observer only sees the synchrotron emission when they are within the forward light cone. As the electron is gyrating around the magnetic field, this direction is changing. The observer is in this light cone for time $\Delta t = \frac{2\theta}{\Omega} = \frac{2m}{eB}$. However, the emitting electron is moving directly toward the observer over this time, so although the light emitted at the start of the pulse is ahead of the light at the end of the pulse, it is only ahead by $c\Delta t\left(1-\frac{v}{c}\right)$. The pulse then has an apparent duration of

$$\Delta t_a = \Delta t\left(1-\frac{v}{c}\right) \quad .$$

Since $\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right) = 1-\frac{v^2}{c^2} = \frac{1}{\gamma^2}$, we can write $\left(1-\frac{v}{c}\right) = \frac{1}{\gamma^2\left(1+\frac{v}{c}\right)}$. As the electrons are ultrarelativistic, $\left(1+\frac{v}{c}\right) = 2$, and

$$\Delta t_\mathrm{a} = \frac{m_e}{\gamma^2 eB} \quad .$$

C3

$$\nu_\mathrm{chr} \approx \frac{1}{\Delta t_\mathrm{a}} = \frac{\gamma^2 eB}{m_e}$$

C4

Making a linear approximation,

$$\tau \approx -\frac{E}{\left(\frac{dE}{dt}\right)} \tag{36}$$

$$= \frac{6\pi\varepsilon_0 m^4 c^5}{e^4 B^2 \sin^2\phi}\frac{1}{E} \tag{37}$$

Synchrotron emission from an AGN jet

D1

As the magnetic field is frozen in, and magnetic flux is constant, the magnetic field must decrease as the area increases in the expansion.

For a small area A, $B_0 A_0 = BA$. Since $A \propto V^{2/3}$, $B = B_0(A_0/A) = B_0\left(\frac{V}{V_0}\right)^{-2/3}$

D2

A volume of plasma V_0 with number density n_0 contains a total number of particles $N = n_0 V_0$. As the volume expands, the total number remains constant, so $n = N/V = (V/V_0)n_0$.

The internal energy of the plasma $E = 3PV$, and since $PV^{4/3} = P_0 V_0^{4/3}$, $EV^{1/3} = E_0 V_0^{1/3}$. The scaling for particle energy with volume is then $E = (V/V_0)^{-1/3}E_0$. This means that the particles initially with energies between ϵ_0 and $\epsilon + d\epsilon$, will have energies between $(V/V_0)^{-1/3}\epsilon_0$ and $(V/V_0)^{-1/3}(\epsilon + d\epsilon)$. As $((V/V_0)^{-1/3}\epsilon)^{-p} = (V/V_0)^{-p/3}\epsilon^{-p}$.

Hence, we can write

$$f(\epsilon) = \kappa\epsilon^{-p} \quad .$$

The value of κ is determined by the relationship

$$\int_0^\infty \kappa\epsilon^{-p}d\epsilon = N/V \quad .$$

Given

$$\int_0^\infty \kappa_0\epsilon^{-p}d\epsilon = N/V_0$$

$\kappa_0 V_0 = \kappa V$, and

$$f(\epsilon) = \left(\frac{V}{V_0}\right)^{-1}\kappa_0\epsilon^{-p}$$

D3

As the energy loss rate due to synchrotron emission increases as E^2, and the cooling time decreases as $1/E$, the more energetic electrons lose energy more rapidly. If we consider electrons with energies $\epsilon_1 < \epsilon_2$, both will move to lower energies in the distribution, but $df/dt \propto E^2$, so $\frac{df}{dt}|_{\epsilon_2} > \frac{df}{dt}|_{\epsilon_1}$. This will reduce the relative number of electrons with higher energies, and steepen the power law of the electron energy distribution.

D4

For the knots in Centaurus A there is no change in the x-ray spectrum, so this rules out synchrotron cooling as in that case the spectrum would steepen (Part D3). Hence adiabatic cooling is more likely for these two knots.

For the knots in M87, there is no change in brightness in other bands. Adiabatic expansion would reduce the number density at all energies (Part D2) and hence brightness at all wavelengths, so this is not likely. Hence, synchrotron cooling is more likely for these two knots.

Reference sheet for markers

Note: some results below were used for the previous version of part **A.10**, and are no longer needed.

Coordinate systems for convenience (note: use of matrices not needed) xyz from XYZ

$$\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{X}}\\ \hat{\mathbf{Y}}\\ \hat{\mathbf{Z}}\end{bmatrix}$$

123 from xyz

$$\begin{bmatrix}\hat{\mathbf{1}}\\ \hat{\mathbf{2}}\\ \hat{\mathbf{3}}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{y}}\\ \hat{\mathbf{z}}\end{bmatrix}$$

Position of point A from centre of mass, in xyz and 123 frames:

$$\begin{aligned}\mathbf{a} &= \alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}} \qquad (1)\\ &= \alpha R\sin\theta\hat{\mathbf{x}} + R(\alpha\cos\theta - 1)\hat{\mathbf{z}}\\ &= R\sin\theta\hat{\mathbf{1}} + R(\alpha - \cos\theta)\hat{\mathbf{3}}\end{aligned}$$

Useful products:

$$\hat{\mathbf{z}} \times \hat{\mathbf{3}} = \sin\theta\hat{\mathbf{y}} \qquad (2)$$

$$(3)$$

Note (given in question):

$$\left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\mathbf{K}} = \left(\frac{\partial \mathbf{A}}{\partial t}\right)_{\widetilde{\mathbf{K}}} + \boldsymbol{\omega} \times \mathbf{A} \qquad (4)$$

Time derivatives:

$$\dot{\hat{\mathbf{3}}} = \boldsymbol{\omega} \times \hat{\mathbf{3}} \qquad (5)$$

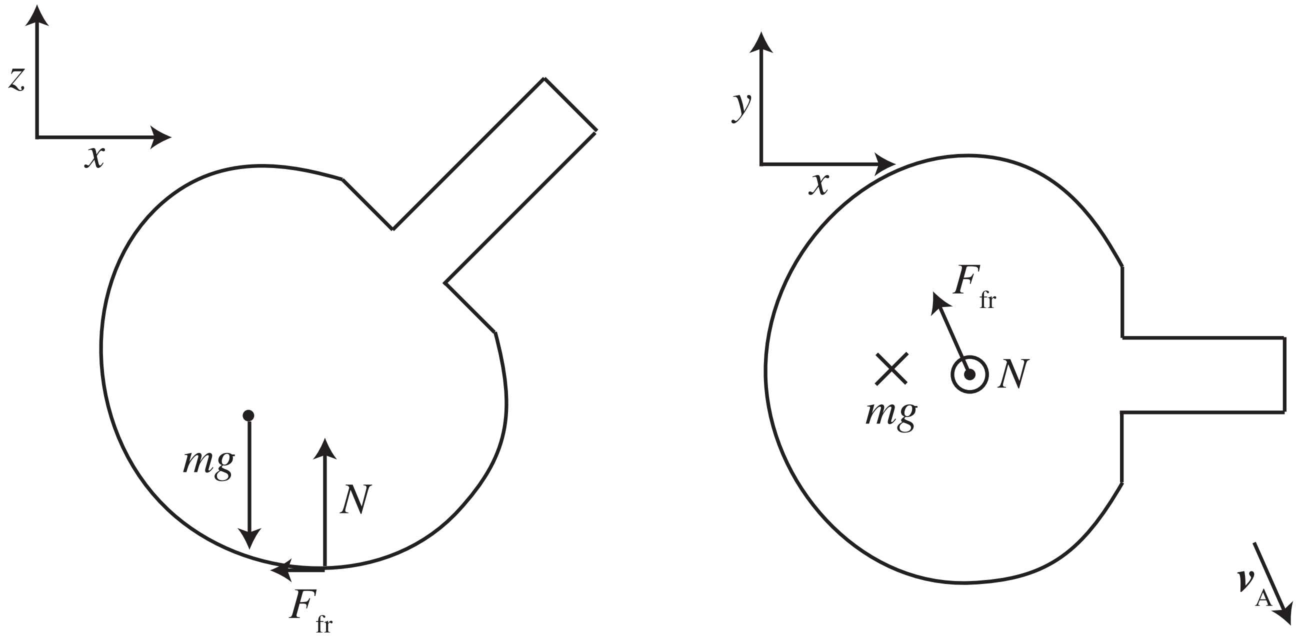
$$\dot{\hat{\mathbf{x}}} = \dot{\phi}\hat{\mathbf{y}} \qquad (6)$$

$$\dot{\hat{\mathbf{y}}} = -\dot{\phi}\hat{\mathbf{x}} \qquad (7)$$

Solutions: Tippe Top

1. (**1.0 marks**)

   Free body diagrams:



   Note: the direction of $\mathbf{F}_f$ must be opposite to the direction of $\mathbf{v}_A$, but is otherwise unimportant. Sum of forces:

$$\begin{aligned}\mathbf{F}_{\text{ext}} &= (N - mg)\hat{z} + \mathbf{F}_f \quad \text{(sufficient for full marks)} \\ &= (N - mg)\hat{z} - \frac{\mu_k N}{|v_A|}\mathbf{v_A}\end{aligned} \tag{8}$$

   Sketched $\mathbf{v_A}$ must be in opposite direction to $\mathbf{F}_f$ on xy diagram.

2. (**0.8 marks**)

   Sum of torques:

$$\begin{aligned}\boldsymbol{\tau}_{\text{ext}} &= \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \quad (9)\\ &= (\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \times (N\hat{\mathbf{z}} + F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) \\ &= \alpha RN\hat{\mathbf{3}} \times \hat{\mathbf{z}} + \alpha R(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}) \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) - R\hat{\mathbf{z}} \times (F_{f,x}\hat{\mathbf{x}} + F_{f,y}\hat{\mathbf{y}}) \\ &= -\alpha RN\sin\theta\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} + \alpha R\cos\theta F_{f,x}\hat{\mathbf{y}} - \alpha R\cos\theta F_{f,y}\hat{\mathbf{x}} - RF_{f,x}\hat{\mathbf{y}} + RF_{f,x}\hat{\mathbf{x}} \\ &= RF_{f,y}(1 - \alpha\cos\theta)\hat{\mathbf{x}} + [RF_{f,x}(\alpha\cos\theta - 1) - \alpha RN\sin\theta]\,\hat{\mathbf{y}} + \alpha R\sin\theta F_{f,y}\hat{\mathbf{z}} \quad (10)\end{aligned}$$

3. (**0.4 marks**)

   Motion at A satisfies

$$\mathbf{v_A} = \dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a} \tag{11}$$

   where $\boldsymbol{\omega}$ is the total angular velocity of the top in the centre of mass frame (this is deteremined in the next part). Want to show that $\mathbf{v_A} \cdot \hat{\mathbf{z}} = 0$.

   To show this, take time derivative of contact condition in XYZ or xyz frame (note: either is suitable, as we only need the $\hat{\mathbf{z}}$ component, and $\hat{\mathbf{z}} = \hat{\mathbf{Z}}$).

Contact condition:

$$\begin{aligned} (\mathbf{s}+\mathbf{a})\cdot\hat{\mathbf{z}} &= 0 \quad \text{at all times} \\ \Rightarrow \frac{d}{dt}(\mathbf{s}+\mathbf{a})\cdot\hat{\mathbf{z}} &= 0 \quad \text{at all times} \end{aligned} \tag{12}$$

Note we only care about the z-component, and $(\boldsymbol{\omega}\times\hat{\mathbf{z}})\cdot\hat{\mathbf{z}} = 0$. Then, using 11, 1, and 5,

$$\begin{aligned} \mathbf{v_A}\cdot\hat{\mathbf{z}} &= (\dot{\mathbf{s}} + \boldsymbol{\omega}\times\mathbf{a})\cdot\hat{\mathbf{z}} \\ &= \left(\dot{\mathbf{s}} + \alpha R\boldsymbol{\omega}\times\hat{\mathbf{3}}\right)\cdot\hat{\mathbf{z}} \\ &= \left(\dot{\mathbf{s}} + \alpha R\frac{d\hat{\mathbf{3}}}{dt}\right)\cdot\hat{\mathbf{z}} \\ &= (\dot{\mathbf{s}}+\dot{\mathbf{a}})\cdot\hat{\mathbf{z}} = 0 \end{aligned} \tag{13}$$

4. (**0.8 marks**)

Total angular velocity $\boldsymbol{\omega}$ of top is the sum of three distinct rotations:

$$\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}}$$

Use transformations shown in figure 3 or otherwise to transform into xyz or 123 frame:

$$\boldsymbol{\omega} = \dot{\psi}\sin\theta\hat{\mathbf{x}} + \dot{\theta}\hat{\mathbf{y}} + (\dot{\psi}\cos\theta + \dot{\phi})\hat{\mathbf{z}} \tag{14}$$

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}} + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \tag{15}$$

5. (**1.0 marks**)

Where $\mathbf{I}$ is the inertia tensor

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3, \end{bmatrix}$$

we have

$$\begin{aligned} E_T &= K_T + K_R + U_G \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\boldsymbol{\omega} + \frac{1}{2}m\dot{\mathbf{s}}^2 + mgR(1-\alpha\cos\theta) \end{aligned}$$

From 11,

$$\begin{aligned} \dot{\mathbf{s}} &= \mathbf{v_A} - \boldsymbol{\omega}\times\mathbf{a} \\ &= \mathbf{v_A} - (\dot{\theta}\hat{\mathbf{2}} + \dot{\phi}\hat{\mathbf{z}} + \dot{\psi}\hat{\mathbf{3}})\times(\alpha R\hat{\mathbf{3}} - R\hat{\mathbf{z}}) \\ &= v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} - \left(\dot{\theta}\alpha R\hat{\mathbf{1}} - \dot{\theta}R\hat{\mathbf{z}} + \dot{\phi}\alpha R\hat{\mathbf{z}}\times\hat{\mathbf{3}} - \dot{\psi}R\hat{\mathbf{3}}\times\hat{\mathbf{z}}\right) \\ &= \left(v_x + \dot{\theta}R(1-\alpha\cos\theta)\right)\hat{\mathbf{x}} + \left(v_y - R\sin\theta(\alpha\dot{\phi}+\dot{\psi})\right)\hat{\mathbf{y}} + \dot{\theta}\alpha R\sin\theta\hat{\mathbf{z}} \end{aligned}$$

using 2. Thus

$$\begin{aligned} E_T = &\frac{1}{2}\left[I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi}+\dot{\phi}\cos\theta)^2\right] \\ &+ \frac{m}{2}\left[\left(v_x + \dot{\theta}R(1-\alpha\cos\theta)\right)^2 + \left(v_y - R\sin\theta(\alpha\dot{\phi}+\dot{\psi})\right)^2 + \dot{\theta}^2\alpha^2R^2\sin^2\theta\right] + mgR(1-\alpha\cos\theta) \end{aligned}$$

6. (**0.4 marks**)

From 10,

$$\frac{d\mathbf{L}}{dt}\cdot\hat{\mathbf{z}} = \sum \boldsymbol{\tau}\cdot\hat{\mathbf{z}} = \alpha R\sin\theta F_{f,y} \tag{16}$$

7. (**1.4 marks**)

Changes in energy: $h = \mathbf{s}\cdot\hat{\mathbf{z}}$ increases, so $\dot{U}_G > 0$.

At start and end (phases **I** and **V**) there is little translation so $K_T \sim 0$ at **I** and **V**. Thus, energy transfer is from K_R to U_G.

Normal force does no work. Frictional force does work at point A. Direction is $-\mathbf{v_A}$:

$$W = \int \mathbf{F}_f \cdot \mathbf{v_A}\ dt < 0$$
$$\Rightarrow \frac{d}{dt}E_T = -\mu_k N|\mathbf{v_A}|$$

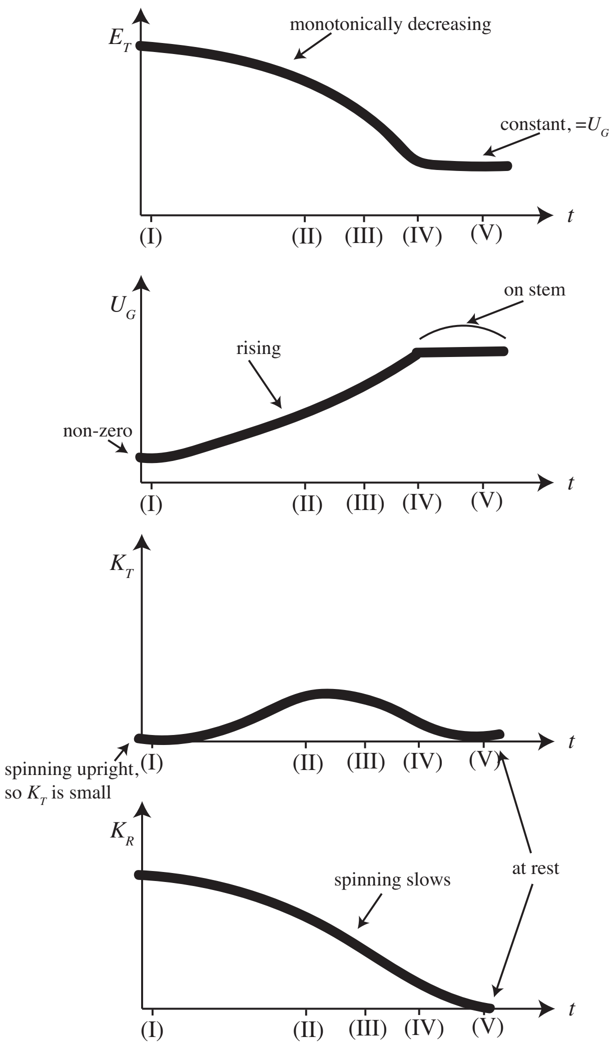
Thus $\mathbf{F}_f$ decreases the total energy monotonically.

16 implies only the $\mathbf{F}_f.\hat{\mathbf{y}}$ acts to decrease $\mathbf{L}\cdot\hat{\mathbf{z}}$. Energy transfer from K_R to U_G, caused by component of frictional force in $\hat{\mathbf{y}}$ direction, so component of resultant torque is in the $\mathbf{a}\times\hat{\mathbf{y}}$ direction.

8. (**2.0 marks**)

Expectation (see figure):

- E_T: monotonically decreasing
- K_R: monotonically decreasing; zero at V
- K_T: zero at I and V; higher between; close to zero at IV
- U_G: flat at start and finish; higher at end; increases from I to IV then flat; increase roughly at same time that K_{rot} decreases



9. (**0.5 marks**)

From 15,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = I_1\left(-\dot{\phi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{2}}\right) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{3}} \tag{17}$$

Taking cross product with $\hat{\mathbf{3}}$:

$$\begin{aligned} \mathbf{L} \times \hat{\mathbf{3}} &= I_1\left(\dot{\phi}\sin\theta\hat{\mathbf{2}} + \dot{\theta}\hat{\mathbf{1}}\right) \\ &= I_1(\boldsymbol{\omega} \times \hat{\mathbf{3}}) \end{aligned} \tag{18}$$

10. (**1.7 marks**)

About any axis through the centre of mass,

$$\frac{d\mathbf{L}}{dt} \neq 0 \Leftrightarrow \tau_{\text{ext}} \neq 0$$

External torque given by 9,

$$\begin{aligned} \boldsymbol{\tau}_{\text{ext}} &= \mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F}_f) \\ &\Rightarrow \tau_{\text{ext}} \cdot \mathbf{a} = 0 \\ \frac{d\mathbf{L}}{dt} &\cdot \mathbf{a} = 0 \end{aligned}$$

Thus, angular momentum in the direction of $\mathbf{a}$ must be constant, so $\mathbf{v} = \mathbf{a}$.

To demonstrate this mathematically, 5, 10, 18 allow

$$\begin{aligned} -\dot{\lambda} &= \frac{d\mathbf{L}}{dt} \cdot \mathbf{a} + \alpha R \mathbf{L} \cdot \frac{d\hat{3}}{dt} \\ &= (\mathbf{a} \times (N\hat{\mathbf{z}} + \mathbf{F_f})) \cdot \mathbf{a} + \frac{\alpha R}{I_1}\mathbf{L} \cdot (\boldsymbol{\omega} \times \mathbf{L}) \\ &= 0 \end{aligned}$$