Theory Q1
Optical trap of neutral atoms (12 points),
Solution and Marking Scheme

1.1 **0.75pt**	At the instance when the seperation between charge centers is $\vec{x}$, the external field $\vec{E}$ exerts on them opposite forces $\vec{F} = \pm e\vec{E}$.	**0.15**
	After a time interval dt, the seperation is changed to $\vec{x} + d\vec{x}$, work done by the external field on the charges is thus $dW = \vec{F}d\vec{x} = \vec{F} = ed\vec{x} \cdot \vec{E} = d\vec{p} \cdot \vec{E}$	**0.3**
	The power received by the atomic dipole $$P_{abs} = \frac{dW}{dt} = \frac{d\vec{p}}{dt} \cdot \vec{E} = \dot{\vec{p}} \cdot \vec{E}$$	**0.3**
1.2 **0.75pt**	Total work can be obtained by integration $$W = \int_0^{\vec{E}_0} d\vec{p} \cdot \vec{E} = \int_0^{\vec{E}_0} \alpha d\vec{E} \cdot \vec{E} = \frac{1}{2}\alpha \vec{E}_0^2 = \frac{1}{2}\vec{p}_0\vec{E}_0$$	**0. 5**
	Potential energy of the dipole is $$U_{dip} = -W = -\frac{1}{2}\vec{p}_0\vec{E}_0$$	**0.25**
	If the sign of U_{dip} is incorrect or the factor 1/2 is missing, students get 0pt.	

2.1 **1.0pt**	The time average of any time dependent function is denoted by $\langle f(t) \rangle = \frac{\omega}{2\pi}\int\limits_0^{2\pi/\omega} f(t)\,dt$	
	$U_{dip}(\vec{r}) = -\frac{1}{4}\alpha(\omega)\cos\varphi.E_0^2(\vec{r})$ (1)	**0.5**
	$U_{dip}(\vec{r}) = -\frac{\alpha(\omega)\cos\varphi.I(\vec{r})}{2\varepsilon_0 c}$ (2)	**0.5**
	If student gets directly to eq. (2) – full mark (1.0pt) *If the answer is still correct but expressed in any quantity other than those requested – 0.5 pt.*	

3.1 **1.0pt**	The power absorbed by the oscillator from the driving field (and re-emitted as dipole radiation) is given by	
	$\langle P_{abs}(\vec{r}) \rangle = \langle \dot{\vec{p}}\vec{E} \rangle = -\frac{\sin\varphi.\alpha(\omega)\omega}{2}E_0^2(r)$	**0.5**
	$\langle P_{abs}(\vec{r}) \rangle = -\frac{\sin\varphi.\alpha(\omega)\omega}{\varepsilon_0 c}I(\vec{r})$ (3)	**0.25**
	The corresponding *scattering rate* is $\Gamma_{sc}(\vec{r}) = \frac{\langle P_{abs} \rangle}{\hbar\omega} = -\frac{\alpha(\omega)\sin\varphi}{\hbar\varepsilon_0 c}I(\vec{r})$. (4)	**0.25**

4.1 **2.0pt**	In one dimensional Lorentz's model, we replace $\vec{E}(\vec{r},t) \to E(x,t)$. One can find the solution of the form $x = x_0\cos(\omega t + \varphi)$ thus from the equation of motion,	
	$\ddot{x} + \gamma_\omega\dot{x} + \omega_0^2 x = -eE_0\cos\omega t / m_e$ => $x_0(\omega_0^2 - \omega^2)\cos(\omega t + \varphi) - x_0\omega\gamma_\omega\sin(\omega t + \varphi) = -eE_0\cos\omega t / m_e$	**0.25**

$$x_0\left\{\left[\left(\omega_0^2-\omega^2\right)\cos\varphi-\omega\gamma_\omega\sin\varphi\right]\cos\omega t-\left[\left(\omega_0^2-\omega^2\right)\sin\varphi+\omega\gamma_\omega\cos\varphi\right]\sin\omega t\right\}=$$

$$=-eE_0\cos\omega t/m_e$$ **0.25**

Comparing coefficients before $\cos\omega t$ and $\sin\omega t$ on both sides, one has

$$\left(\omega_0^2-\omega^2\right)\cos\varphi-\omega\gamma_\omega\sin\varphi=-\frac{eE_0}{m_e x_0}$$ **0.5**

$$\left(\omega_0^2-\omega^2\right)\sin\varphi+\omega\gamma_\omega\cos\varphi=0$$

$$x_0=\frac{eE_0/m_e}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}};$$ **0.25**

$$\sin\varphi=\frac{\omega\gamma_\omega}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}} \qquad (4)$$

0.25

$$\cos\varphi=-\frac{(\omega_0^2-\omega^2)}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}} \qquad (5)$$

$$p=-ex=-ex_0\cos(\omega t+\varphi)=\alpha E_0\cos(\omega t+\varphi) \qquad (6)$$ **0.25**

$$\alpha\left(\omega\right)=-\frac{e^2}{m_e\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma_\omega^2\omega^2}} \qquad (7)$$ **0.25**

Note: students can obtain φ via any of sin, cos, tan functions: full mark (0.25 pt)

5.1 1.0pt

The power radiated due to the damping force, thus

$$-m_e\gamma_\omega v.v=-\frac{1}{6\pi\varepsilon_0}\frac{e^2a^2}{c^3}$$ **0.25**

$$\Rightarrow -m_e\gamma_\omega(\omega r)^2=-\frac{1}{6\pi\varepsilon_0}\frac{e^2(\omega^2 r)^2}{c^3},$$ **0.25**

$$\gamma_\omega=\frac{1}{6\pi\varepsilon_0}\frac{e^2\omega^2}{m_e c^3}. \qquad (8)$$ **0.5**

6.1 0.5pt

Substituting $e^2/m_e=6\pi\varepsilon_0 c^3\gamma_\omega/\omega^2$ the on-resonance damping rate $\gamma\equiv\gamma_{\omega_0}=\left(\omega_0/\omega\right)^2\gamma_\omega$.

Using Eq. (1), (4), (5) and (6) one has

$$\frac{U_{dip}\left(\vec{r}\right)}{\hbar\Gamma_{sc}\left(\vec{r}\right)}=\frac{-\frac{1}{2\varepsilon_0 c}\alpha\left(\omega\right)\cos\varphi}{-\hbar\frac{\alpha\left(\omega\right)\sin\varphi}{\hbar\varepsilon_0 c}}=\frac{1}{2}\frac{1}{\tan\varphi}=-\frac{1}{2}\frac{\omega_0^2-\omega^2}{\left(\omega^3/\omega_0^2\right)\gamma},$$ **0.5**

7.1 0.5pt

From (1), (5) and (6) one has

	$U_{depth}=\left\|U_0\right\|=\left\|\frac{\alpha(\omega)\cos(\varphi)I(0,0)}{2\varepsilon_0 c}\right\|=\left\|\frac{\alpha(\omega)\cos(\varphi)}{2\varepsilon_0 c}\frac{2P}{\pi D_0^2}\right\|=\left\|6c^2\frac{(\omega_0^2-\omega^2)\gamma/\omega_0^2}{\left[(\omega_0^2-\omega^2)^2+\gamma^2\frac{\omega^6}{\omega_0^4}\right]}\frac{P}{D_0^2}\right\|$	**0.5**
7.2 **1.0pt**	Trap depth when $P=4mW$, laser wavelength $\lambda=985nm$, and $D_0=6\mu m.$ For sodium $\lambda_0=589nm$. One has: $\omega=\frac{2\pi c}{\lambda};\omega_0=\frac{2\pi c}{\lambda_0}$; And $\gamma=\frac{1}{6\pi\varepsilon_0}\frac{e^2\omega_0^2}{m_e c^3}=\frac{2\pi e^2}{3\varepsilon_0 m_e c\lambda_0^2}=6.4\times10^7 s^{-1}$ $U_{depth}=f\,k_B T_0$ *(factors f = 3/2, 1/2,1 are all accepted)* $\Rightarrow f.T_0=4.13\mu K$	 **0.5** **0.25** **0.25**

8.1 **0.5pt**	Using linear expansion, we have $\Omega_\rho=\sqrt{\frac{4k_B fT_0}{mD_0^2}}$ and $\Omega_z=\sqrt{\frac{2k_B fT_0}{mz_R^2}}$	**0.25** **0.25**

9.1 **0.5pt**	Mean potential enery $U(z_0)=const+\frac{1}{2}m\Omega_z^2 z_0^2.$ To estimate the particle momentum, we assume $p\sim\Delta p,\Delta z\sim z_0.$ The uncertainty principle is written now $p\sim\frac{\hbar}{z_0}.$ Kinetic energy $K=\frac{p^2}{2m}=\frac{\hbar^2}{2mz_0^2}.$ Total energy of the particle $E=\frac{1}{2}m\Omega_z^2 z_0^2+\frac{\hbar^2}{2mz_0^2}+const$ Minimal energy corresponds to the energy balance $\frac{1}{2}m\Omega_z^2 z_0^2=\frac{\hbar^2}{2mz_0^2}\Rightarrow z_0=\sqrt{\frac{\hbar}{m\Omega_z}}.$ *If the student followed a correct analysis any obtained correct answer upto some multiplication factor: full mark* *If the student obtained correct answer using dimensional analysis: only 0.1 pt is granted*	 **0.2** **0.1** **0.1** **0.1**
9.2 **0.25pt**	Insert the expression of the cloud size $z_0=\sqrt{\frac{\hbar}{m\Omega_z}}$ to the energy expression $E_{\min}=\frac{1}{2}m\Omega_z^2 z_0^2+\frac{\hbar^2}{2mz_0^2}+const$ one obtains $E_{\min}=\hbar\Omega_z+const.$ *If the student obtained the answer* $E_{min}=\frac{\hbar\Omega_z}{2}$ *by using* $E_n=\hbar\Omega_z\left(n+\frac{1}{2}\right)$ *: full mark*	 **0.25**
9.3	From the uncertainty principle, the particle velocity therefore is estimated to be	**0.25**

0.25pt	$$mv_z = \frac{\hbar}{z_0} = \sqrt{m\hbar\Omega_z} \Rightarrow v_z = \sqrt{\frac{\hbar\Omega_z}{m}}.$$ Alternative estimation is constructed from kinetic energy: $\frac{1}{2}mv_z^2 = K = \frac{1}{2}\hbar\Omega_z \Rightarrow v_z = \sqrt{\frac{\hbar\Omega_z}{m}}$	

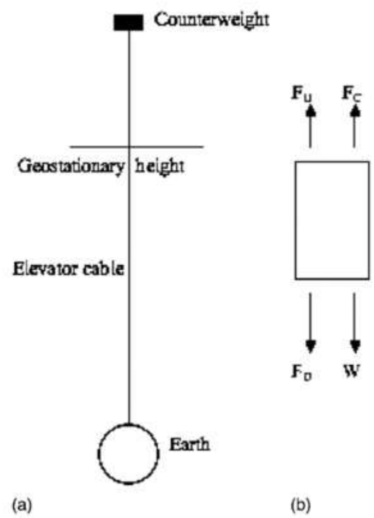
10.1 0.5pt	For the three dimendional trap, one has: $z_0 = \sqrt{\frac{\hbar}{m\Omega_z}}$. Similarly for x, y coordinates $x_0 = y_0 = \sqrt{\frac{\hbar}{m\Omega_\rho}}$ and thus $\rho_0 = \sqrt{x_0^2 + y_0^2} = \sqrt{\frac{2\hbar}{m\Omega_\rho}}$. The condensate aspect ratio: $\frac{z_0}{\rho_0} = \sqrt{\frac{\Omega_\rho}{2\Omega_z}}$. *Student may use either x_0, y_0 or ρ_0 in estimating the radial size of the cloud. Correct answers upto multiplication factor: full mark*	**0.2** **0.2** **0.1**
10.2 0.5pt	$$v_z = \sqrt{\frac{\hbar\Omega_z}{m}},$$ $$v_x = v_y = \sqrt{\frac{\hbar\Omega_\rho}{m}}, \Rightarrow v_\rho = \sqrt{v_x^2 + v_y^2} \sim \sqrt{\frac{2\hbar\Omega_\rho}{m}},$$ $$\frac{v_\rho}{v_z} \sim \sqrt{\frac{2\Omega_\rho}{\Omega_z}}.$$ *Student may use either v_x, v_y or v_ρ in estimating expansion velocity in the radial direction. Correct answers upto some multiplication factor: full mark*	**0.25** **0.25**
10.3 0.5pt	After the time t, the sizes of the condensate cloud are: $z_L = z_0 + v_z t \approx v_z t \qquad \rho_L = \rho_0 + v_\rho t \approx v_\rho t$. The cloud aspect ratio after the time t, $\frac{z_L}{\rho_L} \approx \frac{v_z}{v_\rho} \sim \sqrt{\frac{\Omega_z}{2\Omega_\rho}} \ll 1$. *Correct final answers upto some multiplication factor: full mark*	**0. 25** **0.25**
10.4 0.5pt	Due to isotropic nature of thermal cloud, described by the Maxwell distribution: $$v_{T,z} = v_{T,\rho} \Rightarrow \frac{v_{T,\rho}}{v_{T,z}} \approx 1.$$ one can easily find $z_{T,L} = z_0 + v_z t \approx v_z t$, $\rho_{T,L} = \rho_0 + v_\rho t \approx v_\rho t$. After a very long time, the aspect ratio of the thermal cloud therefore: $\rho_{T,L} : z_{T,L} \sim 1$ *Note: students use different velocities (arithmetrical, rms, projection….etc.) to estimate the expansion of the cloud, as long as they give the correct ratio $\rho_L : z_T \sim 1$, full mark of this sub question is granted. In this question, the correct multiplication factor is requested. For incorrect multiplication factor: zero mark*	**0.2** **0.2** **0.1**

Theory Q2

Space elevator (8 points)

Solution and Marking Scheme

1 Cylindrical Space Elevator with Uniform Cross Section

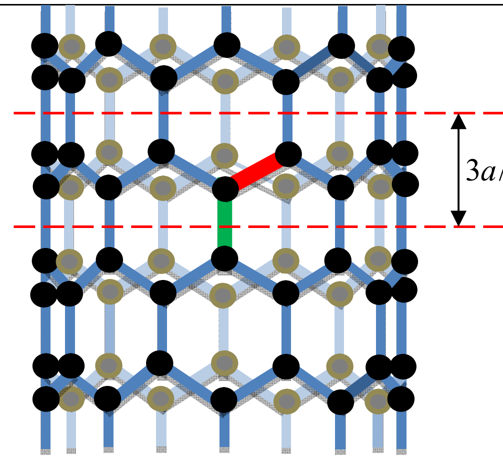
1.1 **0.5pt**	Consider a small element of the cylinder of thickness dr at position r, there are four forces acting on that element: gravitational $\vec{W}(r)$, centrifugal $\vec{F}_C(r)$, cable tension $\vec{F}_D = \vec{T}(r)$ at position r, tension $\vec{F}_U = \vec{T}(r+dr)$ at position $r+dr$. Positive direction is chosen from the Earth center outward. The net force must be zero, therefore: $-W + F_C + T(r+dr) - T(r) = 0$ $\Leftrightarrow -W + F_C + A.\sigma(r+dr) - A.\sigma(r) = 0$, Hence $Ad\sigma = \frac{GM(A\,dr\,\rho)}{r^2} - (A\,dr\,\rho)\omega^2 r$ $\Rightarrow \frac{d\sigma}{dr} = GM\rho\left(\frac{1}{r^2} - \frac{r}{R_G^3}\right)$  Note that, the tensions at the ends of the cylinder are zero. Integrating the above equation from R to R_G, one obtains the stress at R_G $\sigma(R_G) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3}\right]$, Similarly, integrating from R_G to H (the distance from the Earth center to the upper end of the cylinder), one obtains the same stress at R_G $\sigma(R_G) = GM\rho\left[\frac{1}{H} - \frac{3}{2R_G} + \frac{H^2}{2R_G^3}\right]$ Equating the two above expressions, one arrives to the equation: $RH^2 + R^2H - 2R_g^3 = 0$, from where H is determined: $H = \frac{R}{2}\left[\sqrt{1+8\left(\frac{R_G}{R}\right)^3} - 1\right] = 1.51\times10^5$ km. The height of the cylinder $L = H - R = \frac{R}{2}\left[\sqrt{1+8\left(\frac{R_G}{R}\right)^3} - 3\right] = 1.45\times10^5$ km. *Note: Students can just equalize the net gravitational force and the net centrifugal force acting on the cylinder to obtain H correctly: full mark.*	**0.1** **0.1** **0.1** **0.1** **0.1**
1.2 **0.5pt**	The maximal stress is determined from the requirement	

	$$\frac{d\sigma}{dr} = GM\rho\left(\frac{1}{r^2} - \frac{r}{R_G^3}\right) = 0$$ which yields $r = R_G$	**0.25** **0.25**
1.3 **0.5pt**	Maximal stress is expressed by $\sigma(R_G) = GM\rho\left[\frac{1}{R} - \frac{3}{2R_G} + \frac{R^2}{2R_G^3}\right]$ (1) $\sigma(R_G) = \rho g\left[R - \frac{3R^2}{2R_G} + \frac{R^4}{2R_G^3}\right]$ (2) Numerical calculation with $\rho = 7900 kg/m^3$ one obtains the ratio: $\frac{\sigma(R_G)}{5.0GA} = \frac{383\ GPa}{5.0\ \ GPa} = 76.5$, This ratio is much larger than 1, therefore steel is not suitable to build this kind of elevator. *If eq. (2) is not obtained and other correct equation like eq. (1) is derived - 0.1pt from full mark (get only 0.15pt for maximal stress).*	**0.25** **0.25**

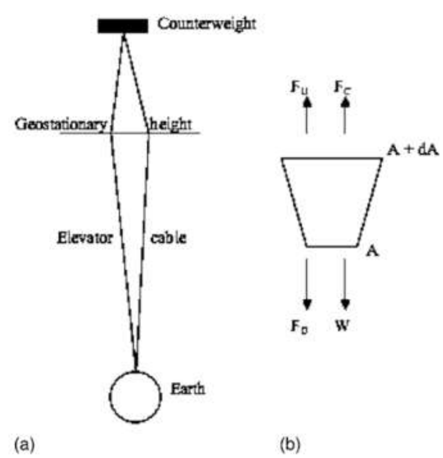
2 Carbon Nanotubes

2.1 **0.25pt**	Expand exponential function in series, and limit to the lowest power of x, one has $V = V_0\left(-1 + \frac{4x^2}{a^2}\right)$ and gets $P = -V_0$ and $Q = \frac{4V_0}{a^2}$.	**0.1** **0.15**
2.2 **0.25pt**	$F = -\frac{dV}{dx} = -\frac{8V_0}{a^2}x$ then $k = \frac{8V_0}{a^2} = 313 Nm^{-1}$.	**0.1** **0.15**
2.3 **0.5pt**	Young's modulus of the carbon nanotube. Denote d the diameter of the carbon nanotube, one has $d = 27b/\pi$. $E_1 = \frac{stress\ \sigma}{strain\ \varepsilon} = \frac{F/A}{x/a} = \frac{kx/A}{x/a} = \frac{ka}{A} = \frac{32V_0}{a\pi d^2}$ $E\ =\ NE_1 = 342\,\text{GPa}$	**0.25** **0.25**

2.4 **0.5pt**	$V_0 = \frac{1}{2}kx_{\max}^2 \Rightarrow x_{\max} = \sqrt{\frac{2V_0}{k}} = \frac{1}{2}a$ $= 0.071\,\text{nm}$	**0.25** **0.25**
2.5 **0.5pt**	Tensile strength of the carbon nanotube, $\sigma_0 = E\frac{x_{\max}}{a} = E/2 = 171\,\text{GPa}$.	**0.5**

2.6 **0.5pt**	Volume $\frac{\pi d^2}{4}\times\frac{3a}{2}$ contains 18 carbon atoms, therefore the density of the carbon nanotube, $\rho=\frac{2\times 27\times 12\times 10^{-3}}{N_A\times\frac{\pi d^2}{4}\times\frac{3a}{2}}=1440\,\text{kg/m}^3$. 	**0.25** **0.25**

3 Tapered Space Elevator with Uniform Stress

3.1 **0.5pt**	The solution to this section is analogous to that given in the previous section, however, now one has to take into account the fact that the stress σ is constant, but the cross section area A varies along the tower. $\sigma dA=\frac{GM\left(A\,dr\,\rho\right)}{r^2}-\left(A\,dr\,\rho\right)\omega^2 r$ $\Rightarrow\frac{dA}{A}=\frac{\rho gR^2}{\sigma}\left(\frac{1}{r^2}-\frac{r}{R_G^3}\right)dr$ where $g=GM/R^2$ is gravitational acceleration at the Earth surface. By integration one can obtain the tower cross section as:  $A(h)=A_S\exp\left[\frac{\rho gR^2}{\sigma}\left(\frac{1}{R}+\frac{R^2}{2R_G^3}-\frac{1}{R+h}-\frac{(R+h)^2}{2R_G^3}\right)\right]$	**0.25** **0.25**
3.2 **0.5pt**	Using the condition A(H)=A(R)=A_S one arrives to the equation $R\,H^2+R^2H-2R_G^3=0$, which allows to determine $H=\frac{R}{2}\left[\sqrt{1+8\left(\frac{R_G}{R}\right)^3}-1\right]=151000\,\text{km}.$	**0.25** **0.25**
3.3 **0.5pt**	The ratio $\frac{A_G}{A_S}=\exp[\frac{R}{2L_C}\{\left(\frac{R}{R_G}\right)^3-3\left(\frac{R}{R_G}\right)+2\}]=1.623$ where $L_C=\frac{\sigma}{\rho g}$	**0.5**
3.4 **1.0pt**	Net force exerted on the counterweight must be zero $\frac{GMm_C}{\left[R_G+h_C\right]^2}+A\left(R_G+h_C\right).\sigma=m_C\omega^2\left[R_G+h_C\right]$, replacing $A\left(R_G+h\right)$ from the equation for cross section area, one can determine the counterweight mass.	**0.5**

	$$m_C = \frac{\rho A_S L_C \exp\left[\frac{R^2}{2L_C R_G^3}\left(\frac{2R_G^3 + R^3}{R} - \frac{2R_G^3 + \left(R_G + h_C\right)^3}{R_G + h_C}\right)\right]}{\frac{R^2\left(R_G + h_C\right)}{R_G^3}\left[1 - \left(\frac{R_G}{R_G + h_C}\right)^3\right]}.$$	**0.50**

4 Applications

4.1 0.5pt	An object can leave the Earth if its energy at the distance r satisfies $E = \frac{m(\omega r)^2}{2} - \frac{GMm}{r} \geq 0$ from which $r_C = \left(2GM / \omega^2\right)^{\frac{1}{3}} = 53200km$ In order to launch an object, the upper end of the tower must locate above the distance r_C.	**0.25** **0.25**

4.2 1.0pt	We denote the Earth orbital velocity as v_E, the spacecraft velocity when it's released from the tower top as $v_1 = \omega h_0$. The spacecraft can reach the furthest distance from the Sun if $\vec{v}_1$ is parallel to $\vec{v}_E$. The spacecaft velocity relative to the Sun is $v_E + v_1$. The Earth orbital radius R_E also is the smallest distance from the sun (if one neglects the tower length compared to the radius of the Earth's orbit). r_2 is the apogee distance of the spacecraft from the Sun, v_2 is its velocity at apogee. Angular momentum and energy convervation laws read $m\left(v_E + v_1\right) R_E = m v_2 r_2$ $\frac{1}{2} m\left(v_E + v_1\right)^2 - \frac{GM_S m}{R_E} = \frac{1}{2} m v_2^2 - \frac{GM_S m}{r_2}$ Here the energy term $-\frac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has $\left[\left(v_E + \omega h_0\right)^2 - \frac{2GM_S}{R_E}\right] r_2^2 + 2GM_S r_2 - \left(v_E + \omega h_0\right)^2 R_E^2 = 0$ from which $r_{Max} = r_2 = \frac{\left(v_E + \omega h_0\right)^2 R_E^2}{2GM_S - \left(v_E + \omega h_0\right)^2 R_E}$. Numerical calculation gives r_2=5.3AU, that covers Jupiter's orbit. Similarly, for the spacecraft to approach as close as possible to the Sun, the released velocity $\vec{v}_1$ must be antiparallel to $\vec{v}_E$. The spacecaft velocity relative to the Sun is $v_E - v_1$, r_2 is the perigee distance of the spacecraft from the Sun, v_2 is its velocity at perigee. The previous angular momentum and energy convervation laws still hold, $m\left(v_E - v_1\right) R_E = m v_2 r_2$	**0.1** **0.1** **0.1** **0.1** **0.1** **0.1**

	$\frac{1}{2}m\left(v_E - v_1\right)^2 - \frac{GM_S m}{R_E} = \frac{1}{2}mv_2^2 - \frac{GM_S m}{r_2}$ Here the energy term $-\frac{GMm}{h_0}$ due the earth's gravity is neglected. Eliminating v_2 one has	**0.1**
	$\left[\left(v_E - \omega h_0\right)^2 - \frac{2GM_S}{R_E}\right]r_2^2 + 2GM_S r_2 - \left(v_E - \omega h_0\right)^2 R_E^2 = 0$	**0.1**
	from which $r_{\min} = r_2 = \frac{\left(v_E - \omega h_0\right)^2 R_E^2}{2GM_S - \left(v_E - \omega h_0\right)^2 R_E}$.	**0.1**
	Numerical calculation gives $r_{\min} = 0.43\,\text{AU}$, meaning the Mercury's orbit is within our reach.	**0.1**

References

[1] Artsutanov, Y. Kosmos na elektrovoze. *Komsomolskaya Pravda* July 31 (1960); contents described in Lvov *Science* **158**, 946–947 (1967).

[2] Pearson, J. The Orbital Tower: a Spacecraft Launcher Using the Earth's Rotational Energy. *Acta Astronautica* **2**, 785 (1975)

[3] Aravind, P. K. The physics of the space elevator. *American Journal of Physics* **75**, 125 (2007).

[4] Bochníček, Z. A Carbon Nanotube Cable for a Space Elevator**.** *The Physics Teacher* **51**, 462 (2013).

Theory Q3

Thermoelectric effects and theirapplication in thermoelectric generator and refrigerator(10 points)

Solution and Marking scheme

A. Heat transfer and thermoelectric generator

A1. Heat transfer in a homogeneous conducting bar

A1.1 0.75 pt

Consider heat transfer in the segment dx of the bar in the steady state. Equation for the balance of the energy exchange through the cross-sectional area is written as

$$-kS\frac{dT(x)}{dx}+\rho\frac{dx}{S}I^2=-kS\frac{dT(x+dx)}{dx}=-kS\frac{dT(x)}{dx}-kS\frac{d^2T(x)}{dx^2}.dx$$

Hence

$$-kS\frac{d^2T(x)}{dx^2}=\frac{\rho I^2}{S} \qquad \text{(A1)}$$

0.25

Integration of (A1) gives

$$\frac{dT(x)}{dx}=-\frac{\rho I^2}{kS^2}x+C_1, \qquad \text{(A2)}$$

$$T(x)=-\frac{\rho I^2}{2kS^2}x^2+C_1x+C_2. \qquad \text{(A3)}$$

Constants $C_1,\ C_2$ are derived from the boundary conditions

$$x=0\Rightarrow T=T_1\Rightarrow C_2=T_1, \qquad \text{(A4)}$$

$$x=L\Rightarrow T=T_2\Rightarrow C_1=\frac{T_2-T_1}{L}+\frac{1}{2}\frac{\rho L}{S^2k}I^2. \qquad \text{(A5)}$$

0.25

Equation for the temperature distribution in the bar is

$$T(x)=T_1+\left(\frac{\rho LI^2}{2kS^2}-\frac{T_1-T_2}{L}\right)x-\frac{\rho I^2}{2kS^2}x^2. \qquad \text{(A6)}$$

0.25

A1.2 1.0 pt

Using (A2) –(A5) we obtain the equation for the heat current at x

$$q(x)=-kS\frac{dT(x)}{dx}=\frac{kS}{L}(T_1-T_2)+\frac{\rho I^2}{S}\left(x-\frac{L}{2}\right), \qquad \text{(A7)}$$

0.5

at $x=0,$ and $x=L$

$$q(x=0)=\frac{kS}{L}(T_1-T_2)-\frac{\rho LI^2}{2S}=K(T_1-T_2)-\frac{RI^2}{2}, \qquad \text{(A8)}$$

0.25

$$q(x=L)=\frac{kS}{L}(T_1-T_2)+\frac{\rho LI^2}{2S}=K(T_1-T_2)+\frac{RI^2}{2}. \qquad \text{(A9)}$$

0.25

Here $K=\frac{kS}{L},\ R=\frac{\rho L}{S}.$

A2. Relation between Peltier and Seebeck Coefficients

Thermocouple consists of two subsystems: a) the conducting electron gas that performs an ideal themodynamic cycle; b) Nuclei and bounded electrons of the bar crystal that oscillate around equillibrium positions at finite temperature and participate in heat conduction process. If the resistance of the thermocouple is neglected, these two subsystems may be considered as noninteracting, the electron gas exchanges heat only with the heat source at T_1 and the heat sink at T_2, performing the ideal Carnot cycle.

A2.1 **0.25 pt**	Electron gas receives heat from heat source due to the Peltier effect $q_1 = \pi_1 I$ (A10)	**0.25**
A2.2. **0.25 pt**	The heat amount transferred to the heat sink due to the Peltier effect $q_2 = \pi_2 I$ (A11)	**0.25**
A2.3. **0.5 pt**	Power delivered by the electron gas due to the Seebeck emf is $P = \varepsilon I = \alpha(T_1 - T_2)I$ (A12)	**0.5**
A2.4 **0.5 pt**	The efficiency of the ideal Carnot cycle applied to the thermocouple can be written as $\eta = \frac{P}{q_1}, \eta = \frac{T_1 - T_2}{T_1}$. (A13) Thus $\frac{T_1 - T_2}{T_1} = \frac{\alpha(T_1 - T_2)}{\pi_1}$ (A14) Comparing these equations, one has $\pi_1 = \alpha T_1$. This is the Peltier coefficient at the first junction contacting with the heat source. Generally, one has $\pi = \alpha T$.	**0.25** **0.25**

A3. Thermoelectric generator

A.3.1. **0.5 pt**	Power received by the thermocouple from the heat source (see also (A8)) is $q_1 = K(T_1 - T_2) + \alpha T_1 I - \frac{1}{2} I^2 R.$ (A15) Here α is the Seebeck coefficient of the thermocouple and $K = K_A + K_B = \frac{k_A S_A}{L} + \frac{k_B S_B}{L}$, (A16) $R = R_A + R_B = \frac{\rho_A L}{S_A} + \frac{\rho_B L}{S_B}$, (A17) are its thermal conductance and internal resistance. The heat sink receives a power (see also (A9)) $q_2 = K(T_1 - T_2) + \alpha T_2 I + \frac{1}{2} I^2 R.$ (A.18)	**0.25** **0.25**
A3.2. **0.75 pt**	The efficiency of the thermoelectric generator is $\eta = \frac{P_L}{q_1} = \frac{I^2 R_L}{K(T_1 - T_2) + \alpha T_1 I - I^2 R/2} = \frac{m}{\frac{K(T_1 - T_2)}{I^2 R} + \frac{\alpha T_1}{IR} - \frac{1}{2}}$. (A19) Here we use $R_L = mR$. The electrical current in the circuit is	**0.25**

	$I=\frac{\alpha\left(T_1-T_2\right)}{R_L+R}=\frac{\alpha\left(T_1-T_2\right)}{\left(1+m\right)R}.$ (A20) Substituting (A20) into (A19) we obtain the expession for the efficiency $\eta=\frac{m\left(T_1-T_2\right)}{\frac{KR\left(1+m\right)^2}{\alpha^2}+T_1\left(1+m\right)-\frac{T_1-T_2}{2}}.$ (A21)	0.25 0.25
A3.3. **0.25**	Replacing the figure of merit $Z=\frac{\alpha^2}{KR}$ (A22) and $\eta_c=\frac{T_1-T_2}{T_1}$ the efficiency of the ideal Carnot cycle in (A21), one has $\eta=\eta_c\frac{m}{\frac{\left(1+m\right)^2}{ZT_1}+\left(1+m\right)-\frac{1}{2}\eta_c}.$ (A23) From (A23) one sees that larger Z leads to the larger efficiency of the corresponding thermoelectric generator. The condition $ZT_1\geq 1$ can be used for material application in thermoelectric generators.	 0.25

A4. The maximum efficiency

A4.1 **0.25 pt**	When $R_L=R$ or *m=1,* the power consumed on the load is maximum. The efficiency in that case is $\eta_P=\frac{T_1-T_1}{\left[\frac{4}{Z}+\frac{3T_1+T_2}{2}\right]}.$ (A24)	 0.25
A4.2. **0.75 pt**	Equation (A23) may be rewritten as $\eta=\frac{m}{a\left(1+m\right)^2+b\left(1+m\right)-1/2},$ (A25) where $a=\frac{1}{Z\left(T_1-T_2\right)}$, $b=\frac{T_1}{T_1-T_2}$. Equation $\frac{d\eta}{dm}=0$ has the solution $M=\sqrt{1+\frac{2b-1}{2a}}$ or $M=\sqrt{1+Z\frac{\left(T_1+T_2\right)}{2}}.$ (A26)	0.25 0.25 0.25
A4.3. **0.25 pt**	Using (A25), (A26) we obtain the maximum efficiency of the thermoelectric generator $\eta_{\max}=\frac{T_1-T_2}{T_1}\frac{\left(M-1\right)}{\left(M+\frac{T_2}{T_1}\right)}$ (A27) (*Correct expression containing either* M, Z *or both is also accepted*)	0.25

A5. The maximum figure of merit

A5.1 **0.5**	According to (A22) Z takes the maximum value $Z = Z_m$ when $KR = y$ is smallest. Denoting $\left(k_A S_A + k_B S_B\right)\left(\frac{\rho_A}{S_A} + \frac{\rho_B}{S_B}\right) = y,\ x = \frac{S_A}{S_B}$ one has the equation $\left(k_A x + k_B\right)\left(\frac{\rho_A}{x} + \rho_B\right) = y$.	**0.25**
	It is easily to show the function y has the minimum at $x=x_m$, where $x_m = \sqrt{\frac{\rho_A k_B}{\rho_B k_A}}$ or $\frac{S_A}{S_B} = \left(\frac{\rho_A k_B}{\rho_B k_A}\right)^{1/2}$. (A28)	**0.25**
A5.2 **0.25 pt**	If the ratio of cross-sectional areas satisfies (A28) then $y_m = \left[\left(\rho_A k_A\right)^{1/2} + \left(\rho_B k_B\right)^{1/2}\right]^2$ and the maximum figure of merit of the thermocouple is $Z_m = \frac{\alpha^2}{\left[\left(\rho_A k_A\right)^{1/2} + \left(\rho_B k_B\right)^{1/2}\right]^2}$. (A.29)	**0.25**

A6. The optimal efficiency

A6.1. **0.5 pt**	The thermocouple with two bars made from material A and B has the following the figure of merit $Z_m = \frac{\alpha^2}{\left[\left(\rho_A k_A\right)^{1/2} + \left(\rho_B k_B\right)^{1/2}\right]^2} = \frac{\alpha^2}{4\rho_A k_A} = 3.15\times10^{-3}\,\mathrm{K}^{-1}$. (A.30)	**0.15**
	The optimal efficiency of the thermocouple AB when T_1= 423K, T_2 = 303K has the following value $\eta_{opt} = \frac{T_1 - T_2}{4Z_m^{-1} + \frac{3T_1 + T_2}{2}} = \frac{120}{4\frac{1}{3.2\times10^{-3}} + \frac{3\times423+303}{2}} = 5.84\%$. (A.31)	**0.25**
	The corresponding ideal Carnot efficiency for that case is $\eta_C = \frac{T_1 - T_2}{T_1} = \frac{120}{423} = 28.4\%$ (A32) $\eta_{opt}/\eta_C = 0.21$.	**0.1**
A6.2 **0.25 pt**	The maximum efficiency of the thermoelectric generator designed from AB materials is $M = \sqrt{1 + Z_m\frac{\left(T_1+T_2\right)}{2}} = \sqrt{1+3.2\times10^{-3}\times363} = 1.46$ $\eta_{\max} = \eta_C\frac{\left(M-1\right)}{\left(M + \frac{T_2}{T_1}\right)} = 6.0\%$. (A.33)	**0.25**

B. Thermoelectric refrigerator

B1. The cooling power and the maximum temperature difference

B1.1 **0.25pt**	For cooling purpose we choose the current direction so that heat is absorbed at upper junction (temperature T_1) due to Peltier effect and transferred to the A & B bars. Using (A.9) one gets cooling power taken out from heat source at T_1 $q_C = \alpha T_1 I + K(T_1 - T_2) - \frac{RI^2}{2}$ (B.1) where K, R are thermal conductance and internal resistance of thermocouple.	**0.25**
B1.2. **0.5**	Condition for the maximum cooling power q_{CM} is founded from $\frac{dq_C}{dI} = 0$, one has $I_q = \frac{\alpha T_1}{R}$, (B2) $q_{CM} = \frac{\alpha^2 T_1}{2R} - K(T_2 - T_1)$. (B3) The maximum temperature depression is derived from the condition $q_{CM} = 0$, which gives $\Delta T_{\max} = T_2 - T_{1\min} = \frac{\alpha^2 T_{1\min}^2}{2KR} = \frac{Z T_{1\min}^2}{2}$. (B4) Here $Z = \frac{\alpha^2}{KR}$ is the figure of merit of the thermocouple.	**0.25** **0.25**

B2. The working current

B2.1 **0.25pt**	Thermocouple AB with $Z_m = 3.15\times10^{-3}\,\mathrm{K}^{-1}$ is used for a refrigerator. The lowest cooling temperature T_{1min} is found from the same equation (B4) $0 = T_{1\min}^2 + \frac{2}{Z_m} T_{1\min} - \frac{2}{Z_m} T_2$ $T_{1\min} = \frac{1}{Z_m}\left(\sqrt{1 + 2 Z_m T_2} - 1\right)$. (B5) Putting $T_2 = 300\,\mathrm{K}$ and $Z_m = 3.15\times10^{-3}\,\mathrm{K}^{-1}$ in (B.5) we obtain $T_{1\min} = 2.22\times10^2\,\mathrm{K}$. (B.6)	**0.1** **0.15**
B2.2. **0.5**	Putting the value of the internal resistance $R = \frac{\rho_A L}{S_A} + \frac{\rho_B L}{S_B} = \frac{2\rho_B L}{S_B} = 4.0\times10^{-3}\,\Omega$ in (B2), one gets the working current $I_W = \frac{\alpha T_{1\min}}{R} = \frac{4.2\times10^{-4}\times 221.5}{4\times10^{-3}}\,\mathrm{A} = 23.3\,\mathrm{A}$ (B7)	**0.25** **0.25**

B3. The coefficient of performance

B3.1 0.5pt	According to the energy conservation law, the power supplied by the electrical source P equals to the Joule heat plus Peltier's heat taken away in thermocouple per unit of time: $P=\alpha(T_2-T_1)I+RI^2$. (B.8) The equation for Coefficient of Performance (COP) is $$\beta=\frac{q_C}{P}=\frac{\alpha T_1 I-K\left(T_2-T_1\right)-\frac{RI^2}{2}}{\alpha(T_2-T_1)I+RI^2}$$ (B9)	**0.25** **0.25**
B3.2. 0.25	Electrical current I_β corresponds to the maximum of the COP is found from the equation $\frac{d\beta}{dI}=0$. (B9) may be rewritten in convenience form $$\beta=-\frac{1}{2}+\frac{\alpha\left(T_1+T_2\right)I-2K\left(T_2-T_1\right)}{2\left[\alpha(T_2-T_1)+RI\right]I}\ .$$ (B10) The equation $\frac{d\beta}{dI}=0$ leads to $$-\alpha R\left(T_1+T_2\right)I^2+4K\left(T_2-T_1\right)RI+2K\alpha\left(T_2-T_1\right)^2=0,$$ $$I^2-\frac{2K\left(T_2-T_1\right)I}{\alpha T_M}-\frac{K}{RT_M}\left(T_2-T_1\right)^2=0,$$ (B.11) with $\mathrm{T}_M=\frac{\left(T_2+T_1\right)}{2}$. (B.12) Solution of (B.11) is $$I_\beta=\frac{K\left(T_2-T_1\right)}{\alpha T_M}\left\{\sqrt{1+Z\,\mathrm{T}_M}+1\right\}.$$ (B.13) (*Taking into account that* $Z=\frac{\alpha^2}{KR}$, *(B.13) can be written in other form* $$I_\beta=\frac{\alpha\left(T_2-T_1\right)}{R\left\{\sqrt{1+Z\,\mathrm{T}_M}-1\right\}}$$*)* (B.14)	**0.25**
B3.3. 0.25	Substituting (B.14) into (B.9) one has $$\beta_{\max}=\frac{T_1\left[\sqrt{1+ZT_M}-T_2/T_1\right]}{\left(T_2-T_1\right)\left[\sqrt{1+ZT_M}+1\right]}.$$ (B.15)	**0.25**