Vortices in superfluid

Introduction

Superfluidity is a property of flowing without friction. Everyday experience tells us that motion of an ordinary fluid (say, water at room temperature) is always accompanied by viscous dissipation of energy, so that the flow gradually becomes slower unless it is maintained by external forces. In contrast, superfluid exhibits no loss of kinetic energy: once excited the motion of superfluid can continue indefinitely. Superfluidity was originally discovered experimentally in liquid helium.

We study properties of superfluid helium at zero temperature. It will be treated as an incompressible fluid with density ρ. Flow continuity (the fact that the mass flowing into and the mass flowing out of a given infinitesimal volume are equal) implies that the flux of helium velocity $\vec{v}$ through a closed surface is always zero. Superfluid velocity in this aspect is analogous to the magnetic field intensity. By analogy with the magnetic field lines, "streamlines" are tangential to the fluid velocity at each point and their density is proportional to its magnitude.

True superflow has an important property of being irrotational: circulation of superfluid velocity $\vec{v}$ along any closed path within helium is zero

$$\int_L \vec{v} \cdot d\vec{l} = 0 \tag{1}$$

This statement must be amended if superfluidity is absent along a thin "vortex filament". The thickness of the filament itself is of approximately atomic dimensions a, but the vortex induces long range velocity field in surrounding superfluid: velocity circulation around such filament is the circulation quantum[1]

$$\left| \int_L \vec{v} \cdot d\vec{l} \right| = 2\pi\kappa, \tag{2}$$

and zero if the path can be contracted to a single point without crossing the vortex, see Fig. 1. This supports the analogy between superflow and the magnetic field created by wires carrying current: superposition of two valid velocity distributions is a valid velocity distribution and the velocity at any point is equal (up to a dimensional factor) to the magnetic field produced by the unit currents running through a system of wires representing vortex filaments.

[1] Circulation quantization is a macroscopic quantum effect and corresponds to the angular momentum quantization in Bohr model. The circulation quantum can be expressed as $\kappa = \hbar/m_{\mathrm{He}}$, where m_{He} is the mass of helium atom.

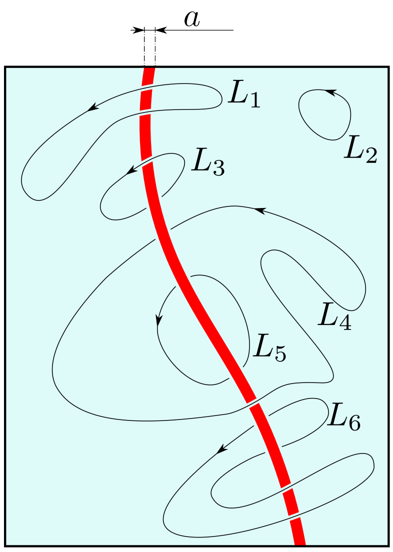


Fig. 1: Vortex filament (red) in superfluid (light blue). Velocity circulations along paths L_1, L_2, L_5, and L_6 are all zero, but those for L_3 and L_4 are equal to $\pm 2\pi\kappa$. Note that circulations along L_3 and L_4 have opposite signs.

Part A. Steady filament (0.75 points)

Consider a cylindrical beaker (radius $R_0 \gg a$) of superfluid helium and a straight vertical vortex filament in its center Fig. 2.

A.1	Plot the streamlines. Find out the velocity v at a point $\vec{r}$.	0.25pt

A.2	Work out the free surface shape (height as a function of coordinate $z(\vec{r})$) around the vortex. Free fall acceleration is g. Surface tension can be neglected.	0.5pt

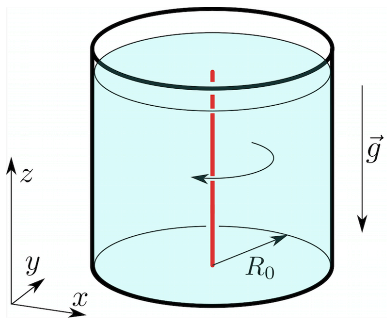


Fig. 2: Straight vortex along the axis of a beaker.

Part B. Vortex motion (1.4 points)

Free vortices move about in space with the flow[2]. In other words each element of the filament moves with the velocity $\vec{v}$ of the fluid at the position of that element.

[2]This is a consequence of momentum conservation, see next section.

As an example, consider a pair of counter-rotating straight vortices placed initially at distance r_0 from each other, see Fig. 3. Each vortex produces velocity $v_0 = \kappa/r_0$ at the axis of another. As a result, the vortex pair moves rectilinearly with constant speed $v_0 = \kappa/r_0$ so that the distance between them remains unchanged.

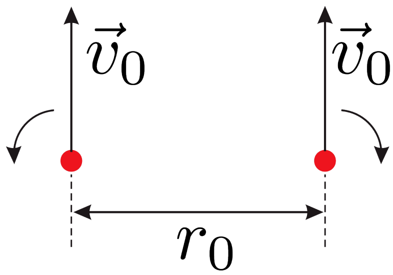


Fig. 3: Parallel vortex filaments with opposite circulations.

B.1	Consider two identical straight vortices initially placed at distance r_0 from each other as shown in Fig. 4. Find initial velocities of the vortices and draw their trajectories.	0.25pt

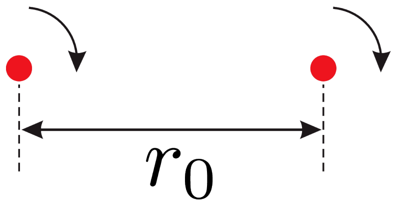


Fig. 4: Parallel vortex filaments with equal circulations.

A beaker of helium (see Part A) is filled with triangular lattice ($u \ll R_0$) of identical vertical vortices, see Fig. 5.

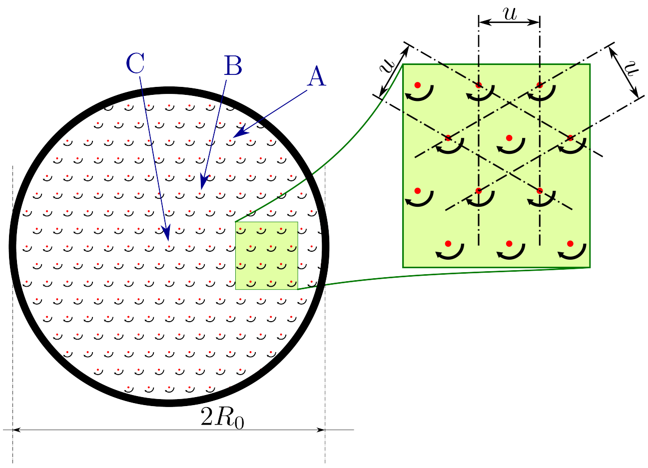


Fig. 5: Triangular lattice of vortices in a beaker. The view from above.

B.2	Draw the trajectories of vortices A, B, and C (located in the center).	0.15pt

B.3	Find velocity $v(\vec{r})$ of a vortex positioned at $\vec{r}$.	0.4pt

B.4	Find the distance AB(t) between the vortices A and B at time t. Treat AB(0) as given.	0.35pt

B.5	Work out the "smoothed out" (omitting the lattice structure) free helium surface shape $z(\vec{r})$.	0.25pt

Part C. Momentum and energy (1.75 points)

The long range velocity field is the major contribution to the energy of a system of vortices, it is insensitive to exact structure of the filament. The filament itself can not be properly described by the macroscopic theory and apparent singularities (infinities) are insignificant. Real physical quantities, such as the energy, of the region inside a thin tube of radius a around the filament should be neglected. Outside of this tube the density of superflow kinetic energy $\rho v^2/2$ (where $\rho =$ const) is analogous to the energy density of the magnetic field $B^2/(2\mu_0)$ — they are both quadratic in respective variables. This analogy together with the correspondence between magnetic field and superfluid velocity generated by vortices (currents) facilitates calculation of the flow energy for a given system. For instance, given the inductance of a circular wire loop $L \approx \mu_0 R \log(R/a)$, where R is the loop radius and a is wire radius, we get the superfluid vortex loop energy[3]

$$U \approx 2R\rho\pi^2\kappa^2 \log(R/a) \tag{3}$$

Total fluid momentum is also determined by the long range velocity distribution. It is obtained by integration of the momentum density $\rho\vec{v}$. Again, consider a flow generated by a circular vortex loop placed in xy plane. It is obvious from the symmetry considerations, that total momentum has only z component:

$$P = \int \rho v_z dV = \rho \int\int \underbrace{\left(\int v_z dz\right)}_{q(x,y)} dxdy \tag{4}$$

The innermost integration is in fact an integration along appropriate paths parallel to z-axis, see Fig. 6. From the circulation identity (2) it follows that

$$q(x,y) = \int\limits_{L(x,y)} \vec{v}\cdot d\vec{l} \tag{5}$$

is piecewise constant. Particularly, it is zero for paths passing outside the ring and $2\pi\kappa$ for paths inside it. Total momentum is therefore

$$P = \rho\cdot\pi R^2\cdot 2\pi\kappa = 2\pi^2\rho R^2\kappa \tag{6}$$

[3] This expression is also valid only if $\log R/a \gg 1$.

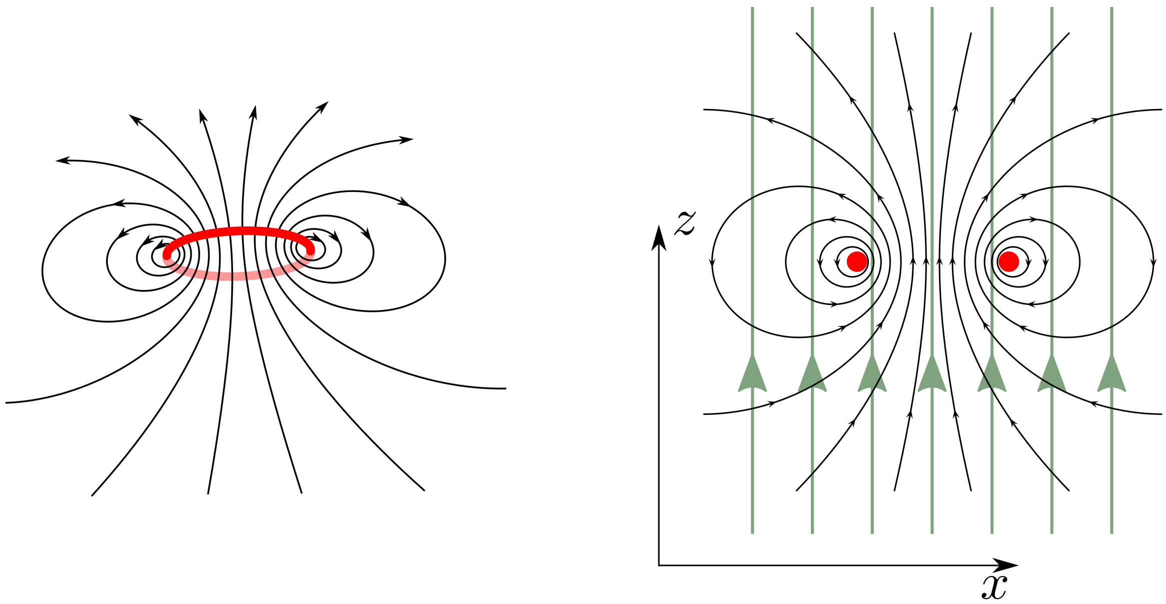


Fig. 6: Velocity field of a circular vortex loop and integration paths (green) for $q(x, y)$ calculation.

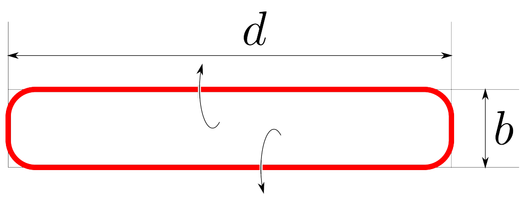


Fig. 7: A nearly rectangular vortex loop, $b \ll d$.

C.1	Consider a nearly rectangular vortex loop $b \times d$, $b \ll d$, Fig. 7. Indicate the direction of its momentum $\vec{P}$. Find out the momentum magnitude.	0.3pt

C.2	Calculate its energy U.	0.7pt

C.3	Suppose we shift a long straight vortex filament by a distance b in x direction, see Fig. 8. How much does the fluid momentum change? Indicate the momentum change direction. The filament length (constrained by the vessel walls) is d.	0.75pt

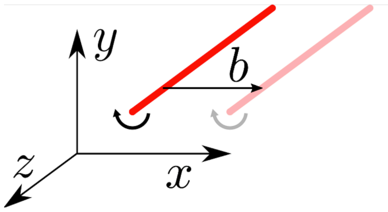


Fig. 8: Momentum changes whenever the vortex shifts with respect to the fluid.

Part D. Trapped charges (2.85 points)

Electrons, if injected in helium, get trapped in the vortex filaments. Here and below polarizability of helium can be neglected ($\epsilon = 1$).

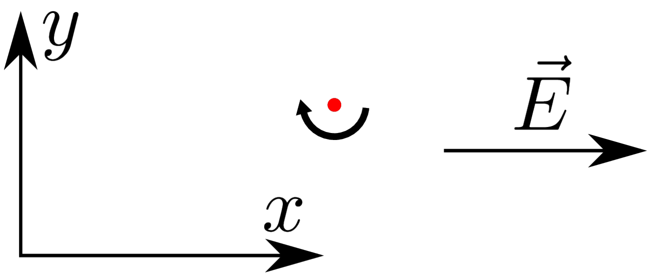


Fig. 9: Straight vortex in a uniform electric field.

D.1	Consider a straight vortex charged with uniform linear density $\lambda < 0$ in a uniform electric field $\vec{E}$. Draw the vortex trajectory. Find its velocity as a function of time.	0.5pt

A circular vortex loop of radius R_0 initially charged with uniform linear density $\lambda < 0$ is placed in a uniform electric field $\vec{E}$ perpendicular to its plane, opposite to its momentum $\vec{P}_0$.

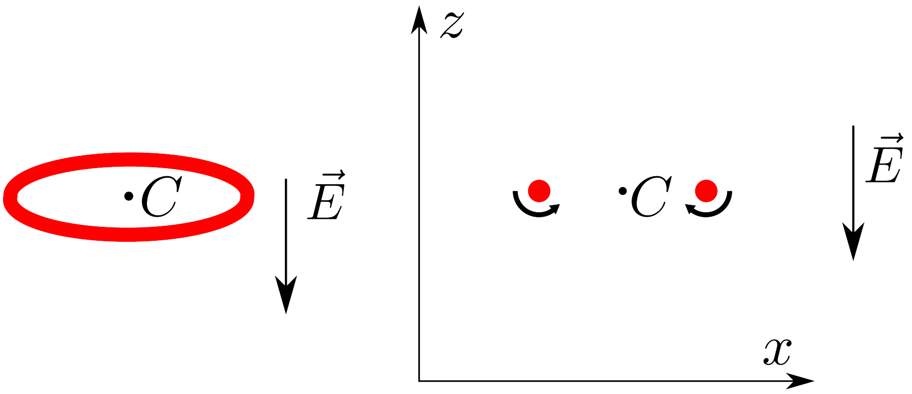


Figure 10: (left) Vortex ring in a uniform electric field. (right) Cross section of the ring.

D.2	Draw the trajectory of the loop center C. Find the radius of the loop as a function of time.	0.6pt

D.3	Find its velocity $v(t)$ as a function of time.	1.5pt

D.4	The field is switched off at a time t^* when the velocity reaches the value $v^* = v(t^*)$. Find the loop velocity $v(t)$ at a later time $t > t^*$.	0.25pt

Part E. Influence of the boundaries (3.25 points)

Solid walls alter the velocity field created by a vortex filament, because the fluid cannot flow through them. Mathematically this means that the wall-normal velocity component vanishes at the wall surface.

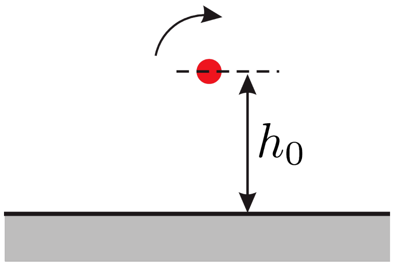


Fig. 11: Straight vortex filament near a flat wall.

E.1	Draw the trajectory of a straight vortex, initially placed at a distance h_0 from a flat wall. Find its velocity as a function of time.	0.5pt

Consider a straight vortex placed in a corner at a distance h_0 from both walls.

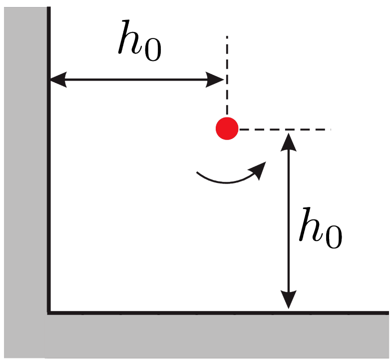


Fig. 12: Straight vortex filament in a corner.

E.2	What is the initial velocity v_0 of the vortex?	0.75pt

E.3	Draw the trajectory of the vortex.	0.5pt
E.4	What is the velocity of the vortex v_∞ after very long time?	1.5pt

Evolution of Supermassive Black Holes Binary

Introduction

The concept of gravitational waves is one of the most impressive predictions of Einstein's Theory of General Relativity. Gravitational waves are the space-time ripples propagating with the speed of light similarly to electromagnetic waves. Direct detection of gravitational waves is incredibly difficult, however, the first signal was detected on September 14, 2015, by LIGO and VIRGO collaborations.

Gravitational waves are emitted during the rapid motion of massive objects. The most powerful source of gravitational waves is the merging of two Supermassive Black Holes (SBH). Black holes predicted by Theory of General Relativity represent extremely compact objects which might have very large masses. Other specific properties of black holes will not be needed in the solution of this problem.

In the generally accepted theory of galaxies' evolution, it is supposed that there is SBH with the mass ranging from $10^5 - 10^9$ of Solar masses in the galaxy's center. Galaxies are huge stellar systems containing $10^{10} - 10^{11}$ stars. During their evolution, two galaxies can collide and merge into one. What happens to two SBHs initially located in their centers? The evolution of the SBH binary system can be divided into *three* main stages. At each stage SBHs approach each other, although the underlying physical phenomena differ. We will examine these phenomena separately in the first three parts of the problem. In the fourth part, we will use the obtained relations to calculate the total time of the SBH binary system evolution.

At the end of their evolution, two SBHs will eventually approach each other and merge into a single black hole. The merging process lasts about an hour and is accompanied by an intense burst of gravitational radiation. Future observatories like LISA will be able to detect this gravitational radiation. Still, the research on the SBH evolution is under way now, at the dawn of gravitational-wave astronomy.

General information

1. Express all your numerical answers in parsecs (pc) for the distances and giga-years (Gy) for the time intervals. We will use Solar mass (M_S) as a reference mass. You might need these values:

$1\ \mathrm{pc} = 3.1 \times 10^{16}\ \mathrm{m}$,

$M_s = 2.0 \times 10^{30}\ \mathrm{kg}$,

$t_H = 13.7\ \mathrm{Gy}$, age of the Universe,

$G = 6.67 \times 10^{-11}$ N $\times$ m^2/kg^2,

$c = 3.0 \times 10^8$ m/s.

2. When you encounter the word "**estimate**", you are not demanded the exact answer. It is sufficient to obtain a result that differs from the accurate one, not more than by a factor of 10. On the contrary, when you encounter the word "**find**", you are supposed to achieve the exact answer. The word "**calculate**" asks you to bring the numerical answer.

3. Throughout the problem, assume every star in the galaxy to have the same mass $m = M_S$.

4. Throughout the problem we will not take into account the effects of the Theory of General Relativity except gravitational waves emission. All stars and black holes are considered as point masses governed by Newton's gravitation law.

Part A. Dynamic Friction (1.6 points)

In this part we shall study the simplified model of the galaxy. You can ignore the velocities of the stars in the galaxy and assume the constant stellar concentration n. The characteristic size of the galaxy is R. The stellar concentration is small enough, so the stellar collisions are extemely rare and negligible. Let us consider a SBH with the mass $M \gg m$ moving with the velocity v through the galaxy. Surprisingly, the SBH experiences nonzero average force from the stars. This force slows the motion of the SBH and is called the force of dynamical friction for this reason. This part is devoted to the determination of this force.

A.1 Let us work in the SBH's reference frame and consider the transit of one star with impact parameter b (fig. 1). Assume that **0.75pt**

$$b \gg b_1 = \frac{GM}{v^2}. \qquad (1)$$

The angular deflection of the star $\alpha = kb_1/b$, where k is some coefficient. Find the value of k. If you cannot find k, assume $k = 1$ hereafter.

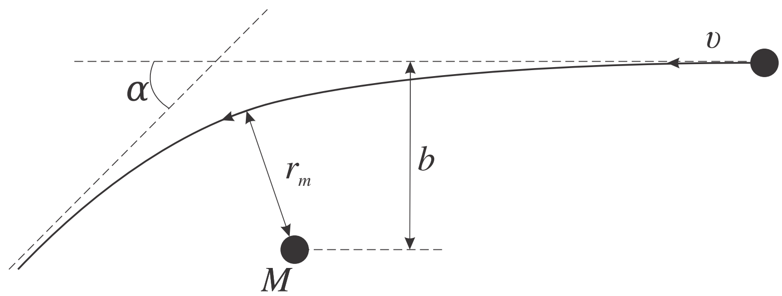


Fig. 1: The deflection of a star by the SBH with mass M. The impact parameter is b, the minimal distance between the star and the SBH is r_m.

A.2 Let Ox axis be directed along the SBH's velocity. Find the momentum component Δp_x transferred from the star to the SBH. **0.25pt**

A.3 Estimate the average force F_{DF} acting on the SBH by taking the average over impact parameter b . Neglect the contribution of the stars with impact parameters $b < b_1$. Assume the SBH to reside in the central part of a galaxy. Express F_{DF} in terms of M, v, R, G and stellar density $\rho = mn$. **0.4pt**

A.4 As you obtained in the previous task, the expression for F_{DF} includes the factor $\log R/b_1$, which we will denote further as $\log \Lambda$. Calculate the value of $\log \Lambda$ for $M = 10^8\ M_S$, $R = 20\ \mathrm{kpc} = 20 \times 10^3\ pc$ and velocity $v = 200\ \mathrm{km/s}$. **0.2pt**

Part B. Gravitational sligshot (3.0 points)

In this part, we will consider the system of two SBHs with equal masses $M \gg m$ located in the center of the galaxy. Let's call this system a **SBH binary**. We will assume that there are no stars near the SBH binary, each SBH has a circular orbit of radius a in the gravitational field of another SBH.

B.1	Find the orbital velocity v_{bin} of each SBH. Find the total energy E of the SBH binary. Express it in terms of a, G and M.	0.25pt

There are a lot of stars at distances much larger than a from the binary. Stars travel along complex and diverse trajectories in the gravitational field of the whole galaxy. The motion of the stars can be considered chaotic, like the motion of the molecules in an ideal gas. Let us assume that stars' velocities have equal magnitudes $\sigma \ll v_{bin}$ and their average mass density is ρ. In this case dynamical friction is no longer affecting the SBH binary and energy losses are caused by other phenomenon.

B.2	Let us solve a related problem. Let a star of mass m transit by a point mass $M_2 \gg m$ being at rest. The minimal distance between the star and the point during the transit is r_m. The velocity of the star at large distance is σ. Find the exact value of impact parameter b.	0.5pt

If a star approaches the SBH binary for a distance about a, it participates in a complex 3-body interaction with the binary that almost always results in a star being shot out with the velocity about v_{bin} (the velocity of the star at the large distance after interaction). We will call such a strong interaction a collision of a star with the SBH binary. Acceleration and the shot of the star after the collision is called "**gravitational slingshot**".

B.3	Estimate the characteristic time Δt between two successive collisions of the SBH binary with stars. Take into account that $\sigma \ll v_{bin}$.	1.0pt

B.4	Estimate the SBH binary energy loss rate dE/dt. Estimate the radius variation rate da/dt. Express it in terms of a, ρ, σ, G.	0.25pt

B.5	Let us denote the initial radius of the system as a_1. Estimate the time T_{SS} for the radius to decrease by a factor of 2 due to "gravitational slingshot". Calculate T_{SS} for $\sigma = 200\ \mathrm{km/s}$, $a_1 = 1\ \mathrm{pc}$, $\rho = 10^4 M_S/\mathrm{pc}^3$.	1.0pt

Part C. Emission of gravitational waves (1.0 points)

In this part we shall study the SBH binary with equal masses which doesn't interact with the stars. Even in this case the system loses the energy due to gravitational waves emission. The energy loss rate due to gravitational waves is

$$\frac{dE}{dt} = -\frac{1024}{5}\frac{G}{c^5}(\omega^3 I)^2, \tag{2}$$

where ω is angular velocity of the binary, and $I = 2Ma^2$ is quadrupole moment of the system.

C.1	Find the SBH binary radius variation rate da/dt due to the emission of gravitational waves.	0.2pt

When the orbit radius of the SBH binary a becomes close to the gravitational radius of the black hole:

$$r_g = \frac{2GM}{c^2}, \tag{3}$$

two SBHs quickly merge.

C.2	Let us denote the initial radius of the system as $a_2 \gg r_g$. Estimate the time T_{GW} it takes for the SBH binary to shrink to the radius about r_g due to the emission of gravitational waves. Express T_{GW} as a function of a_2, M, c and G.	0.7pt

C.3	Calculate the initial radius a_H of the binary of SBHs with equal masses $M = 10^8 M_S$ if it takes it the age of the Universe to merge: $T_{GW} = t_H$.	0.1pt

Part D. Full evolution (4.4 points)

In this part we will use the results obtained above. Let us consider the **real** astrophysical situation. Two galaxies having SBH of mass $M = 10^8\ M_S$ in their centers merged into a new stellar system. Let the new galaxy be spherically symmetrical with radius $R = 20\ \mathrm{kpc} = 20 \times 10^3\ \mathrm{pc}$. Let us assume that stellar density varies with radius r to the galaxy center as

$$\rho(r) = \frac{\sigma^2}{4\pi G r^2}, \tag{4}$$

where $\sigma = 200\ \mathrm{km/s}$.

D.1	Let the body move in circular orbit of radius $a < R$ in gravitational field of the stars. Neglect the force of dynamical friction and find the velocity v of the body.	0.25pt

Immediately after the merging of galaxies two SBHs have arbitrary positions inside the new galaxy and do not affect each other. Let's consider one SBH. We assume it moves in a circular orbit of radius $a < R$ around the galaxy center and slowly loses energy due to the dynamical friction.

D.2	Estimate the orbit radius variation rate da/dt. In part A we ignored the velocities of the stars. Although stars are moving in the real galaxy, not all of them have exactly the same speed σ . Instead, the speed of the stars is only of the order of σ , and so is the relative speed of SBH with respect to the stars, hence you can use the result obtained in A.3 for estimation. You should use the density $\rho(r)$ from equation (4). Assume $\log \Lambda$ to be a constant calculated in A.4.	0.75pt

After a certain time, two SBH will approach the center of the galaxy. Let two SBH move in a circular orbit of radius a around the center galaxy in the gravitational field of the stars.

D.3	Estimate the critical radius a_1 at which gravitational interaction between two SBHs is no longer negligible and calculate it. We will say that at this moment two SBHs form a binary system (fig. 2).	0.3pt

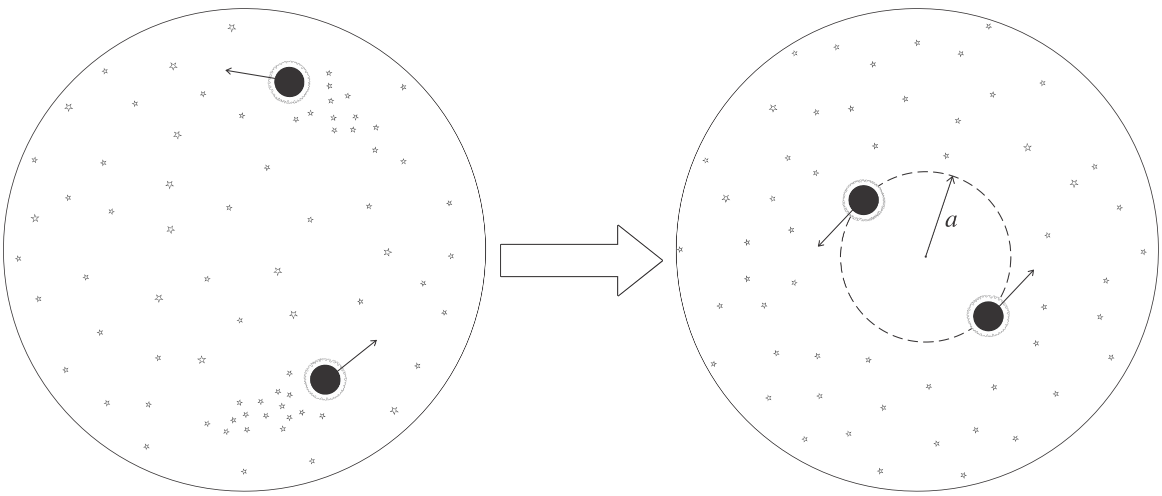


Fig. 2: The evolution of SBHs before and after the formation of the binary system

D.4	Let us assume that after the merging of galaxies two SBHs were at distances $a_0 = 2\ \mathrm{kpc} = 2 \times 10^3\ \mathrm{pc}$ from the galaxy center. Calculate the time T_1 it takes for two SBH to form a binary due to dynamical friction.	0.75pt

After forming the binary, two SBHs shoot away all the stars from the center of the galaxy and stay there alone. Since this moment, dynamical friction becomes ineffective and the binary starts to lose the energy because of the slingshot effect. You can assume that the velocities of the stars around the binary are σ and the stellar density is $\rho_1 = \rho(a_1)$ from the equation (4). Slingshot effect shrinks the radius of the system drastically and after some time the system starts to lose energy mostly due to the radiation of gravitational waves.

D.5	When the binary radius is less than some value $a < a_2$ the energy loss is caused by gravitational waves emission. Estimate the a_2 value and calculate it.	0.3pt

D.6	Estimate the time T_2 of the binary radius reduction from a_1 to a_2 (the slingshot stage). Estimate the time T_3 of binary radius reduction from a_2 to almost zero (the stage of gravitational waves emission).	1.75pt

D.7	For the parameters given above, calculate the total time T_{ev} of two SBH evolution from galaxies merging to SBH merging.	0.3pt

Historical remark. For a long time astrophysicists have been thinking the SBH binary evolution stops at the slingshot stage, since the binary has shot out all the stars with small impact parameters which might collide with it. It appeared that two SBH would never merge. This fact was called **the final parsec problem**.

Real galaxies have complicated asymmetric shapes. Few years ago it was found that in galaxies of complex shapes the stars with small impact parameters appear again and again. The SBH binary continues to lose energy, but slower than our estimation gives. The final parsec problem was successfully solved.

Space Debris

Introduction

In more than half a century of space operations quite a large number of man-made objects have been amassed near Earth. The objects that do not serve any particular purpose are called **space debris**. The most attention is usually paid to the larger debris objects, i.e. defunct satellites and spent rocket upper stages, which stay in orbit after delivering their payload. Collisions of such objects with each other may result in thousands of fragments endangering all current space missions.

There is a well-known hypothetical scenario, according to which certain collisions may cause a cascade where each subsequent collision generates more space debris that increase the likelihood of new collisions. Such a chain reaction, resulting in the loss of all near-Earth satellites and making impossible further space programs, is called the **Kessler syndrome**.

To prevent such undesirable outcome special missions are planned to remove large debris object from their present orbits either by tugging them to the Earth's atmosphere or to graveyard orbits. To this end a specially designed spacecraft – a space tug – must capture a debris object. However, before capturing an uncontrolled object it is important to understand its rotational dynamics.

We suggest you to take part in planning of such a mission and find out how the rotational dynamics of a debris object changes in time under the influence of different factors.

Rocket Stage Schematic

The debris object to be considered is a "Kerbodyne 42" rocket upper stage, whose schematic is shown in Fig. 1. The circle line in Fig. 1 marks the outline of a spherical fuel tank.

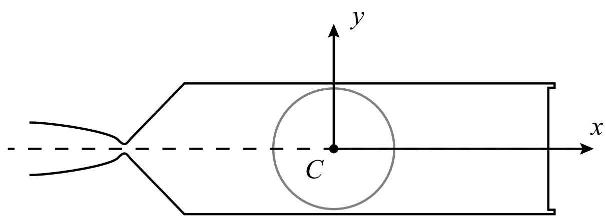


Fig. 1: "Kerbodyne 42" upper stage

We introduce a body-fixed reference frame Cxy with the origin in the center of mass C, x being the symmetry axis of the stage, and y perpendicular to x. The inertia moments with respect to x and y axes are J_x and J_y ($J_x < J_y$).

Part A. Rotation (3.8 points).

Consider an arbitrary initial rotation of the stage with angular momentum L (Fig. 2), where θ is the angle between the symmetry axis and the direction of angular momentum. Fuel tank at this point is assumed to be empty. No forces or torques act upon the stage.

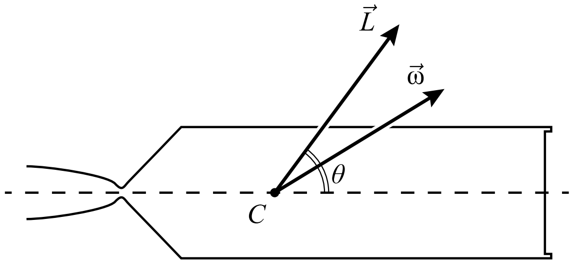


Fig. 2: Rocket stage rotation

A.1	Find the projections of angular velocity $\vec{\omega}$ on x and y, given that $\vec{L} = J_x\omega_x\vec{e}_x + J_y\omega_y\vec{e}_y$ for material symmetry axes x and y with unit vectors $\vec{e}_x$ and $\vec{e}_y$. Provide the answer in terms of $L = \lvert\vec{L}\rvert$, angle θ, and inertia moments J_x, J_y.	0.2pt

A.2	Find the rotational energy E_x associated with rotation ω_x and E_y associated with rotation ω_y. Find total rotational kinetic energy $E = E_x + E_y$ of the stage as a function of the angular momentum L and $\cos\theta$.	0.4pt

In the following questions of Section A consider the stage's free rotation with the initial angular momentum L and $\theta(0) = \theta_0$.

A.3	Let us denote by x_0 the initial orientation of the stage's symmetry axis Cx with respect to inertial reference frame. Using conservation laws find the maximum angle ψ, which the stage's symmetry axis Cx makes with x_0 during the stage's free rotation. *Note:* Since there are no external torques acting upon the stage, the angular momentum vector remains constant.	1.2pt

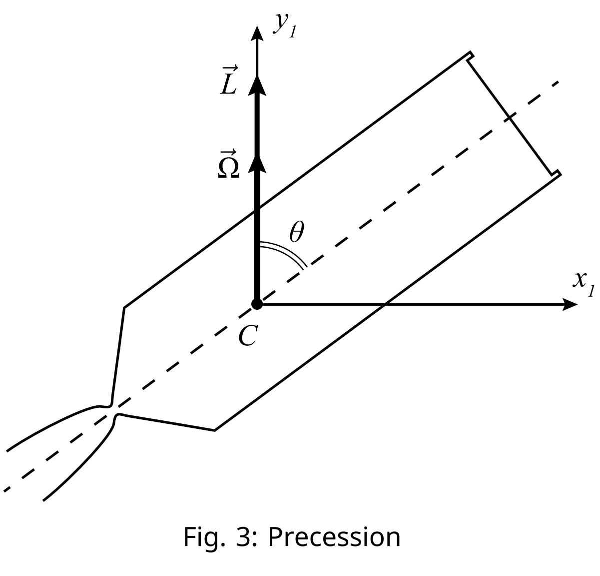


Fig. 3: Precession

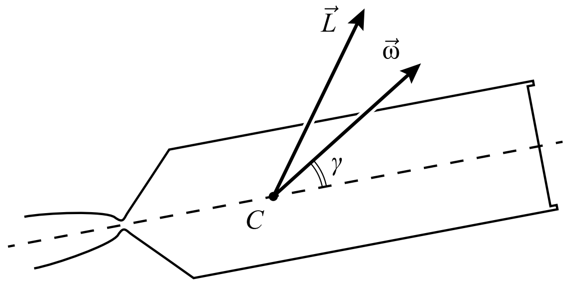
Let us now introduce the reference frame $Cx_1y_1z_1$ with y_1 along the constant angular momentum vector $\vec{L}$ (Fig. 3). This reference frame rotates about y_1 in such a way, that the stage's symmetry axis always belongs to the Cx_1y_1 plane.

A.4	Given L, $\theta(0) = \theta_0$, and inertia moments J_x, J_y, find the angular velocity $\Omega(t)$ of the reference frame Cx_1y_1 about y_1 and direction and absolute value of angular velocity of the stage $\vec{\omega}_s(t)$ relative to the reference frame Cx_1y_1 as functions of time. Provide the answer for $\vec{\omega}_s(t)$ direction in terms of angle $\gamma_s(t)$ it makes with the symmetry axis Cx. *Note*: angular velocity vectors are additive $\vec{\omega} = \vec{\omega}_x + \vec{\omega}_y = \vec{\Omega} + \vec{\omega}_s$.	2.0pt

Part B. Transient Process (1.6 points).

Most of the propellant is used during the ascent, however, after the payload has been separated from the stage, there still remains some fuel in its tank. Mass m of residual fuel is negligible in comparison to the stage's mass M. Sloshing of the liquid fuel and viscous friction forces in the fuel tank result in energy losses, and after a transient process of irregular dynamics the energy reaches its minimum.

B.1	Find the value θ_2 of angle θ after the transient process for arbitrary initial values of L and $\theta(0) = \theta_1 \in (0, \pi/2)$.	0.6pt



Angle between the stage's angular velocity and the symmetry axis

B.2	Calculate the value ω_2 of angular velocity ω after the transient process, given that initial angular velocity $\omega(0) = \omega_1 = 1\, rad/s$ makes an angle of $\gamma(0) = \gamma_1 = 30^\circ$ with the stage's symmetry axis. The moments of inertia are $J_x = 4200\ kg \cdot m^2$ and $J_y = 15\,000\ kg \cdot m^2$.	0.6pt

Part C. Magnetic Field (4.6 points).

Another important factor in rotational dynamics of a debris rocket stage, which is orbiting the Earth, is its interaction with the Earth's magnetic field. Let us first consider an auxiliary problem.

Torque due to Eddy Currents.

Let us place a thin-walled nonmagnetic spherical shell with wall thickness D and radius R in a uniform magnetic field $\vec{B}$, which slowly changes so that its derivative $\dot{\vec{B}}$ is a constant vector making angle α with the direction of $\vec{B}$ (Fig. 4). Electrical resistivity of the shell's material is ρ.

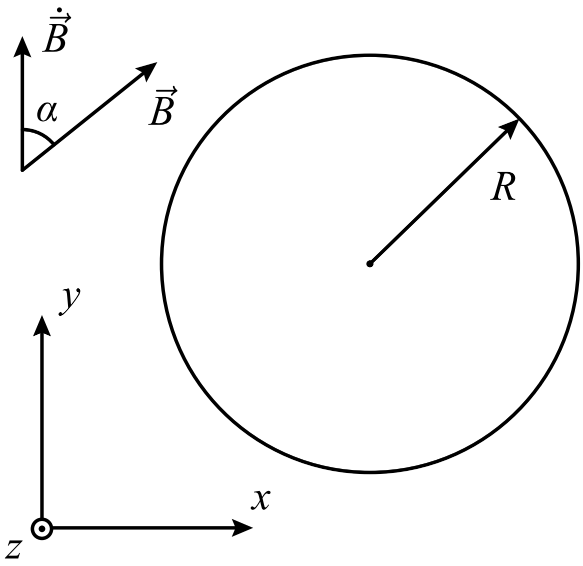


Fig. 4: Spherical shell in magnetic field

C.1	Find the induced magnetic moment $\vec{\mu}$ of the shell, neglecting its self-inductance. Provide the answer for $\vec{\mu}$ in the form of projections on xyz (see Fig. 4).	1.0pt

C.2	Find the torque $\vec{M}$ acting on the spherical shell. Provide the answer for $\vec{M}$ in the form of projections on xyz (see Fig. 4).	0.3pt

Attitude Motion Evolution in the Earth's Magnetic Field

Let us find out how the rotation changes for a rocket stage, which moves in a circular polar orbit with orbital period $T = 100\,min$ (Fig. 5). It transpires that the characteristic times of dynamics due to interaction with the geomagnetic field are much greater than the duration of the transient process. We will now study what happens to the rocket stage after the transient process has completed. To start our analysis consider the stage rotating with angular velocity ω_2 about the axis perpendicular to the orbital plane.

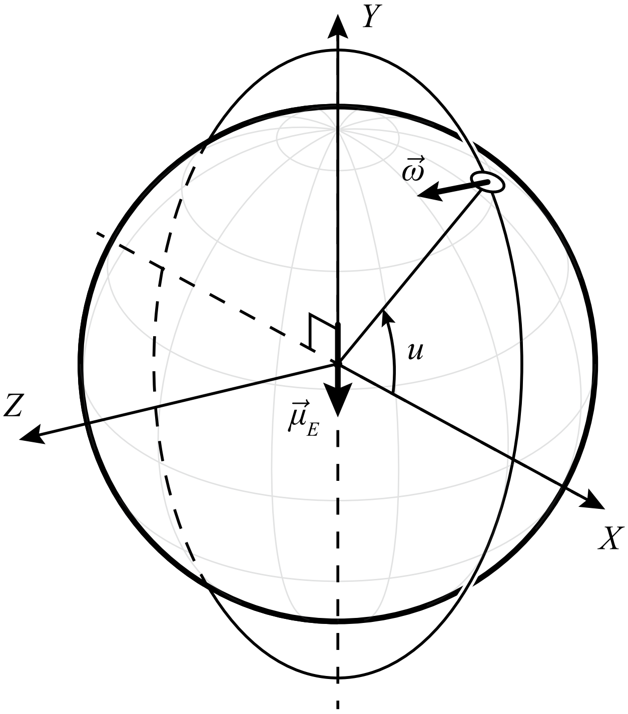


Fig. 5: The orbit

C.3 The Earth's magnetic field $\vec{B}_E$ can be modeled as the magnetic field of a point dipole in the Earth's center. Its dipole moment $\vec{\mu}_E$ is directed opposite to Y axis. The absolute value of the Earth's magnetic field B at the point where the orbit crosses the equatorial plane XZ is $B_0 = 20\,\mu T$. Find $\vec{B}_E(u)$ at a current position of the stage in the orbit defined by the angle u as shown in Fig. 5. The positive direction of u is along with the orbital motion. Provide the answer in the form of the projections of $\vec{B}_E(u)$ on XYZ. **0.4pt**

Note: It may facilitate subsequent calculations if projections of $\vec{B}_E(u)$ are given as functions of $2u$ instead of u.

Note: Magnetic field of a dipole at point $\vec{r}$ is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{\mu}}{r^3}\right). \qquad (1)$$

The "Kerbodyne 42" rocket upper stage is mostly made of wood, and the only conductive material is used for its cryogenic fuel tank. We, therefore, consider the stage’s interaction with the geomagnetic field as that of the spherical shell with wall thickness $D = 2\ \mathrm{mm}$, radius $R = 4\ \mathrm{m}$ and resistivity $\rho = 2.7\cdot 10^{-8}\ \Omega\cdot\mathrm{m}$.

C.4 Find the torque $\vec{M}(u)$ acting on the stage, as it rotates about the axis perpendicular to the orbital plane with angular velocity ω collinear to Z. Provide the answer in the form of the projections of $\vec{M}(u)$ on XYZ. **1.3pt**

C.5	Find the absolute value of angular velocity $\omega(t)$ as a function of time, given that the change in the stage's angular velocity over one orbital period is negligibly small.	1.0pt

C.6	Find the ratio of the orbital period T and the rocket stage's rotation period T_s in the steady-state regime, which sets in after a long time.	1.0pt