Vortices in Superfluid

MODD-Problems

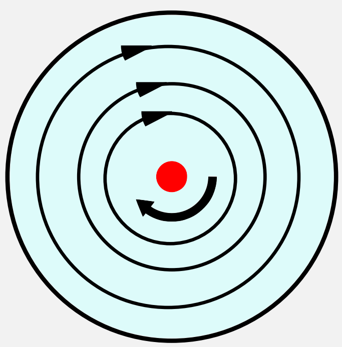
May 5, 2017

A. Steady filament (0.75)

Consider a cylindrical beaker (radius $R_0 \gg a$) of superfluid helium and a straight vertical vortex filament in its center Fig. 2.

A1 (0.25)

Plot the streamlines. Find out the velocity v at a point $\vec{r}$.



The streamlines are circular. From the circulation identity (1) it is obvious that $v = \kappa/r$.

- Streamlines are plotted correctly (one at least) 0.1
- $v = \frac{k}{r}$ 0.15

A2 (0.5)

Work out the free surface shape (height as a function of coordinate $z(\vec{r})$) around the vortex. Free fall acceleration is g. Surface tension can be neglected.

Consider a thin circular layer of the radius r. Equilibrium condition for its surface is given by the requirement

$$g\frac{dz}{dr} = \frac{v^2}{r} = \frac{\kappa^2}{r^3}. \tag{1}$$

This equation is satisfied by the surface profile

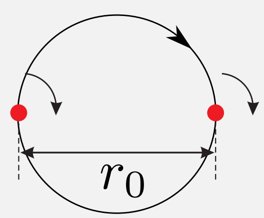
$$z(r) = [z_0] - \frac{\kappa^2}{2gr^2}. \tag{2}$$

- $\tan\alpha = \frac{k^2}{gr^3}$ or equivalent 0.25
- $z = [z_0] - \frac{k^2}{2gr^2}$ 0.25

B. Vortex motion (1.4)

B1 (0.25)

Consider two identical straight vortices initially placed at distance r_0 from each other as shown in Fig. 4. Find initial velocities of the vortices and draw their trajectories.

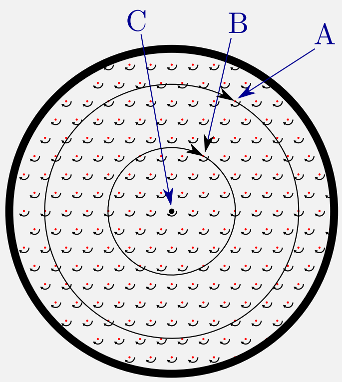


Being advected by each other's flow field, filaments will rotate around a point halfway between them. The velocity is given by $v_0 = \kappa/r_0$.

- Trajectories are plotted correctly 0.15
- Correct expression for velocity 0.1

B2 (0.15)

Draw the trajectories of vortices A, B, and C (located in the center).



- Trajectories are plotted correctly 0.15

B3 (0.4)

Find velocity $v(\vec{r})$ of a vortex positioned at $\vec{r}$.

Consider a circular path of radius $r \gg u$ around the beaker center. The circulation along this path is given by the number of vortices within it (vortex density per unit area is $(u^2\sqrt{3}/2)^{-1}$):

$$2\pi r v = 2\pi\kappa \frac{\pi r^2}{u^2\sqrt{3}/2}. \tag{3}$$

The velocity field

$$v = \frac{2\pi\kappa r}{u^2\sqrt{3}}. \tag{4}$$

- Expression for vortex density 0.2
- Correct expression for $v(r)$ 0.2

B4 (0.35)

Find the distance AB(t) between the vortices A and B at time t. Treat AB(0) as given.

This velocity pattern corresponds to the rotation of the lattice as a whole around the beaker center with angular velocity

$$\omega = \frac{2\pi\kappa}{u^2\sqrt{3}}. \tag{5}$$

$\mathrm{AB}(t) = \mathrm{AB}(0)$

- Correct answer 0.35

B5 (0.25)

Work out the "smoothed out" (omitting the lattice structure) free helium surface shape $z(\vec{r})$.

The surface shape is

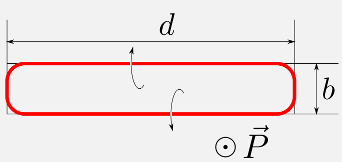
$$z(r) = [z_0] + \frac{\omega^2 r^2}{2g} = [z_0] + \frac{2\pi^2\kappa^2 r^2}{3gu^4}. \tag{6}$$

- Correct answer 0.25

C. Momentum and Energy (1.75)

C1 (0.3)

Consider a nearly rectangular vortex loop $b \times d$, $b \ll d$, Fig. 7. Indicate the direction of its momentum $\vec{P}$. Find out the momentum magnitude.



Momentum of a flat loop (see Introduction) is perpendicular to its plane and proportional to its area. For a rectangular loop the magnitude is $P = 2\pi\kappa\rho bd$.

- Correct direction of momentum 0.15
- Correct expression for momentum magnitude 0.15

C2 (0.7)

Calculate its energy U.

To produce equal magnetic and kinetic energy densities $B^2/(2\mu_0) = \rho v^2/2$, the magnetic field has to be $B = v\sqrt{\mu_0\rho} = \kappa\sqrt{\mu_0\rho}/r$. This field is generated by a current $I = 2\pi\kappa\sqrt{\rho/\mu_0}$. Energy of the wire loop can be found from the inductance $U = LI^2/2$. Inductance of a nearly rectangular wire loop:

$$L = \frac{\Phi}{I} = 2dI^{-1}\int_a^b \frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 d}{\pi}\log\frac{b}{a}. \tag{7}$$

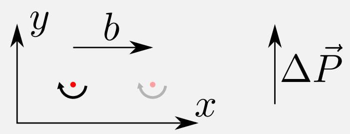
This gives for the energy

$$U = 2\pi\kappa^2\rho d\log\frac{b}{a} \tag{8}$$

- Integration limits are $\sim a$ and b 0.2
- Analogy with a magnitude field is used ($U = \frac{LI^2}{2}$, $L = \frac{\Phi}{I}$) or energy is calculated as $W = \int F\,dr$, where $F = \frac{dP}{dt}$ 0.2
- Correct expression for energy 0.3

C3 (0.75)

Suppose we shift a long straight vortex filament by a distance b in x direction, see Fig. 8. How much does the fluid momentum change? Indicate the momentum change direction. The filament length (constrained by the vessel walls) is d.



The momentum change is equal to the momentum of a long rectangular loop $P = 2\pi\kappa\rho bd$.

- The result of C1 used 0.3
- Momentum change is parallel to Y axis 0.1
- Correct direction of momentum change 0.15
- Correct expression for momentum change magnitude 0.2

Interestingly, this provides an alternative approach to find the energy of such a loop. Namely, if we slowly move one straight vortex in the velocity field of another, then we apply a force

$$F = 2\pi\kappa\rho d v = 2\pi\kappa\rho d\frac{\kappa}{r} = \frac{2\pi\kappa^2\rho d}{r}. \tag{9}$$

The work

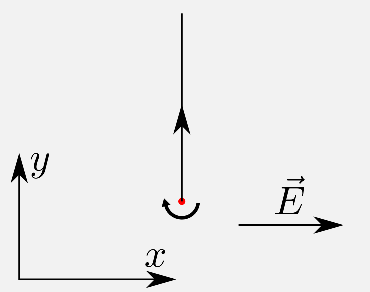
$$W = \int_a^b \frac{2\pi\kappa^2\rho d}{r} dr = 2\pi\kappa^2\rho d \log\frac{b}{a} \tag{10}$$

has to be performed to move it from distance a to b.

D. Trapped charges (2.85)

D1 (0.5)

Consider a straight vortex charged with uniform linear density $\lambda < 0$ in a uniform electric field $\vec{E}$. Draw the vortex trajectory. Find its velocity as a function of time.



Electric force $F = E\lambda d$ moves the vortex with velocity

$$v = \frac{F}{2\pi\kappa\rho d} = \frac{E\lambda}{2\pi\kappa\rho} \tag{11}$$

perpendicular to $\vec{E}$.

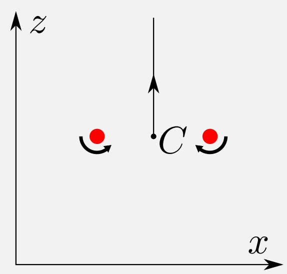
- Trajectory is straight line parallel to Y axis 0.1
- Correct direction of velocity 0.2
- Correct expression for velocity magnitude 0.2

D234

A circular vortex loop of radius R_0 initially charged with uniform linear density $\lambda < 0$ is placed in a uniform electric field $\vec{E}$ perpendicular to its plane, opposite to its momentum $\vec{P}_0$.

D2 (0.6)

Draw the trajectory of the loop center C. Find the radius of the loop as a function of time.



Electric force upon the loop $F = -2\pi E R_0|\lambda|$ is constant and fluid momentum linearly depends on time

$$P = P_0 + 2\pi E R_0|\lambda| t = 2\pi^2 \rho R^2 \kappa. \tag{12}$$

The loop is growing and its radius is increasing with time t

$$R = \sqrt{R_0^2 + \frac{E R_0|\lambda| t}{\pi \rho \kappa}}. \tag{13}$$

- Trajectory is straight line along y 0.1
- Correct velocity direction 0.15
- $P(t)$ 0.15
- $2\pi^2 \rho R^2 \kappa$ 0.15
- Correct expression for $R(t)$ 0.05

D3 (1.5)

Find its velocity $v(t)$ as a function of time.

The loop velocity v can be easily found from a relationship between the energy change rate and the momentum change rate

$$\frac{dU}{dt} = Fv = \frac{dP}{dt}v. \tag{14}$$

This gives for the velocity

$$v = \frac{dU}{dP} \approx \frac{\kappa}{2R}\log\frac{R}{a} = \frac{\kappa\log\left(\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}/a\right)}{2\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}} \approx \frac{\kappa\log(R_0/a)}{2\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}}. \tag{15}$$

This means that the vortex is moving in the direction of the force but its velocity is decreasing.

- Expression for $v \propto \frac{1}{R}\log(R)$ 1.0
- Correct expression for $v(t)$ 0.5

D4 (0.25)

The field is switched off at a time t^* when the velocity reaches the value $v^* = v(t^*)$. Find the loop velocity $v(t)$ at a later time $t > t^*$.

When $E = 0 \Rightarrow P = \text{const} \Rightarrow R = \text{const} \Rightarrow v = \text{const} \Rightarrow v(t) = v^*$.

- Correct expression for $v(t)$ 0.25

E. Influence of the boundaries (3.25)

Draw the trajectory of a straight vortex, initially placed at a distance h_0 from a flat wall. Find its velocity as a function of time.

E1 (0.5)

Well known technique of image charges (currents) in electrostatics (magnetostatics) can be directly used to solve this problem. Namely, the wall can be "substituted" with a reflected fictitious vortex on the other side of the wall. The velocity distribution of two vortices together in the upper semi-space is identical to the one produce by a single vortex above the wall. Indeed, the symmetry of the problem ensures that there is no flow through the plane of symmetry. Thus, a straight vortex line situated a distance h_0 above a flat wall with its image behave as a pair of vortices of opposite circulation a distance $2h_0$ apart. This means that the vortex moves along the wall with velocity

$$v = \frac{\kappa}{2h_0}. \tag{16}$$

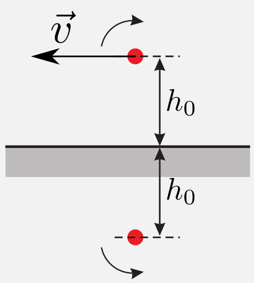


Illustration of the image method for the straight vortex filament near a flat wall

- Trajectory is plotted correctly 0.15
- Correct direction of velocity 0.1
- Correct expression for velocity magnitude 0.25

E234

Consider a straight vortex placed in a corner at a distance h_0 from both walls.

E2 (0.75)

What is the initial velocity v_0 of the vortex?

The velocity of the filament is given by superposition of the velocities $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ induced by the image vortices 1, 2 and 3, respectively (see Fig. in E3 solution). One readily obtains

$$v_1 = \frac{\kappa}{2h_0}, \quad v_2 = \frac{\kappa}{2\sqrt{2}h_0}, \quad v_3 = \frac{\kappa}{2h_0}.$$

The modulus of the filament velocity at the initial moment is

$$v_0 = |\vec{v}_1 + \vec{v}_2 + \vec{v}_3| = \sqrt{2}v_1 - v_2 = \boxed{\frac{\kappa}{2\sqrt{2}h_0}}$$

- Ideas of using superposition principal and technique of image charges 0.25
- Correct expression for initial velocity magnitude 0.5

E3 (0.5)

Draw the trajectory of the vortex.

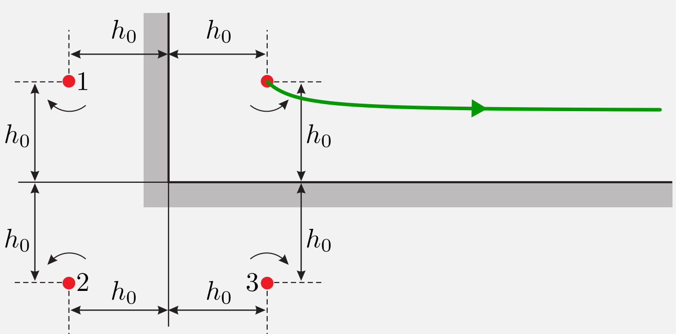


Image vortices in the corner.

- The trajectory has correct form 0.3
- Correct direction of initial velocity 0.2

E4 (1.5)

What is the velocity of the vortex v_∞ after very long time?

Energy for the system of vortices is proportional to

$$U_{\text{tot}} \propto \log \frac{\sqrt{x^2+y^2}}{a} - \log \frac{x}{a} - \log \frac{y}{a}. \tag{17}$$

The energy conservation implies that

$$C = \frac{x^2+y^2}{x^2y^2} = \frac{2}{h_0^2} \tag{18}$$

is constant along the trajectory. After very long time $y \to h_0/\sqrt{2}$ and the vortex velocity is

$$v_\infty = \frac{\kappa}{h_0\sqrt{2}}. \tag{19}$$

- $E = \text{const}$ 0.5
- Correct expression for velocity after very long time 1.5

Marker ____________________ **Theory 1** Student ____________________

TOTAL ____________________

Subquestion		Statement	Points	Marker		Consensus	
				Statement	Subquestion	statement	Subquestion
A1	a	Correct streamlines (at least one)	0.1				
	b	$v = \kappa / r$	0.15		(0.25)		
A2	a	$\tan\,\alpha \; = \; \frac{\kappa^2}{gr^3}$ or equivalent ***	0.25				
	b	$z \; = \; z_0 - \frac{\kappa^2}{2gr^2}$ (if at A1b $v \sim \frac{1}{r}$ PEP applied)	0.25		(0.5)		
B1	a	Correct trajectories	0.15				
	b	$v_0 = \kappa / r_0$	0.1		(0.25)		
B2	a	All trajectories correct	0.15				
B3	a	Expression for vortex density $(u^2\sqrt{3}/2)^{-1}$ ***	0.2				
	b	$v \; = \frac{2\pi\kappa r}{u^2\sqrt{3}}$ (if B3a is incorrect PEP applied)	0.2		(0.4)		
B4	a	AB(t) = AB(0)	0.35				
B5	a	$z(r) = z_0 + \frac{2\pi^2\kappa^2 r^2}{3gu^4}$ (if at B3b $v \sim r$ PEP applied)	0.25				
C1	a	Correct direction of momentum $\odot$	0.15				
	b	$P \; = \; 2\pi\kappa\rho bd$	0.15		(0.3)		
C2	a	Integration limits are βa and b where $\beta \approx 1$ ***	0.2				
	b	Case 1: Analogy with a magnitude field is used ($U = \frac{LI^2}{2}$ and $L = \frac{\Phi}{I}$) Case 2: Energy is calculated as $W \; = \; \int F dr$ and $F = \frac{dP}{dt}$ ***	0.2				
	c	$U \; = \; 2\pi\kappa^2\rho d \log\frac{b}{a}$ $(\log\left(\frac{b}{\beta a}\right) - \; correct)$	0.3		(0.7)		
C3	a	The result of C1 used ***	0.3				
	b	Momentum change is parallel to Y axis	0.1				
	c	Correct direction of momentum change $\Uparrow$	0.15				
	d	$\Delta P \; = \; 2\pi\kappa\rho bd$ (correct if similar to C1b)	0.2		(0.75)		
D1	a	Trajectory is straight line parallel to Y axis	0.1		(0.5)		

	b	Correct direction of velocity (upwards)	0.2				
	c	$v = \dfrac{E\lvert\lambda\rvert}{2\pi\kappa\rho}$ (if at C3d $\Delta P \sim b$ PEP applied)	0.2				
D2	a	Trajectory is straight line parallel to Z axis	0.1				
	b	Correct direction of velocity (upwards)	0.15				
	c	$P(t) = P_0 + 2\pi E R_0 \lvert\lambda\rvert t$ ***	0.15				
	d	$P = 2\pi^2 \rho R^2 \kappa$ ***	0.15				
	e	$R = \sqrt{R_0^2 + \dfrac{ER_0\lvert\lambda\rvert t}{\pi\rho k}}$	0.05		(0.6)		
D3	a	$v \sim \dfrac{1 + \log R/a}{R} \left(\approx \dfrac{\log R/a}{R}\right)$	1				
	b	$v = \begin{bmatrix} \dfrac{k}{2\sqrt{R_0^2 + \frac{ER_0\lvert\lambda\rvert t}{\pi\rho k}}}\left(1 + \log\dfrac{\sqrt{R_0^2 + \frac{ER_0\lvert\lambda\rvert t}{\pi\rho k}}}{a}\right) \\ \dfrac{k}{2\sqrt{R_0^2 + \frac{ER_0\lvert\lambda\rvert t}{\pi\rho k}}}\left(1 + \log\dfrac{R_0}{a}\right) \\ \dfrac{k}{2\sqrt{R_0^2 + \frac{ER_0\lvert\lambda\rvert t}{\pi\rho k}}}\log\dfrac{\sqrt{R_0^2 + \frac{ER_0\lvert\lambda\rvert t}{\pi\rho k}}}{a} \\ \dfrac{k}{2\sqrt{R_0^2 + \frac{ER_0\lvert\lambda\rvert t}{\pi\rho k}}}\log\dfrac{R_0}{a} \end{bmatrix}$ (if at D2e $R = \sqrt{a + bt}$ PEP applied)	0.5		(1.5)		
D4	a	$v(t) = v^*$	0.25				
E1	a	Trajectory is plotted correctly	0.15				
	b	Correct direction of velocity (⇦)	0.1				
	c	$v = \kappa/2h_0$	0.25		(0.5)		
E2	a	Ideas of using superposition principle and technique of image charges ***	0.25				
	b	$v_0 = \dfrac{\kappa}{2\sqrt{2}h_0}$	0.5		(0.75)		
E3	a	The trajectory has correct curvature	0.3				
	b	Correct direction of initial velocity (⇘)	0.2		(0.5)		
E4	a	Using of energy conservation law (or equivalent)	0.5				
	b	$v_\infty = \dfrac{\kappa}{h_0\sqrt{2}}$ (0.5 points – arithmetical error)	1.0		(1.5)		
			Total				

PEP (Propagation Error Principle): incorrect answers with right dimension obtained from earlier wrong results are to be accepted in case of right course of solution. That principle applied only in indicated cases.

Trajectory with the wrong direction indicated is considered as incorrect.

Evolution of Supermassive Black Holes Binary
Solution

A. Dynamical Friction

A1. The deflection angle is defined from: $\tan\alpha \approx \alpha = \frac{p_y}{p_x}$, assuming that $\alpha \ll 1$. One can find $p_y = \int F_y\, dt$, and according to Newton's gravity law

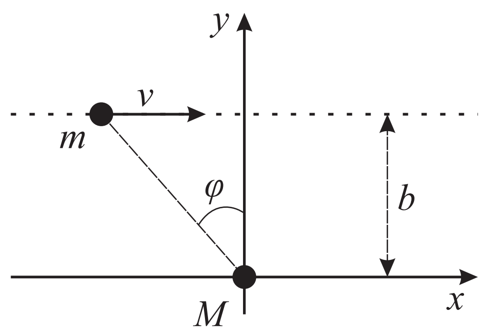
$$F_y = \frac{GMm}{b^2}\cos^3\varphi$$

The geometry: $x = b\tan\varphi$, so we change the variable $dt = \frac{dx}{v} = \frac{b}{v}\frac{d\varphi}{\cos^2\varphi}$ and we have

$$p_y = \frac{GMm}{bv}\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi = \frac{2GMm}{bv}.$$

Here we assume that the body moves along the stright line, due to $\alpha \ll 1$, see Fig 1. So $\alpha = \frac{p_y}{p}$ and

$$\boxed{\alpha = \frac{2GM}{bv^2} = \frac{2b_1}{b}}, \qquad \boxed{k = 2}$$



A2. During the transit of a massive body, star's energy remains constant: $p_x^2 + p_y^2 = \text{const}$. Hence

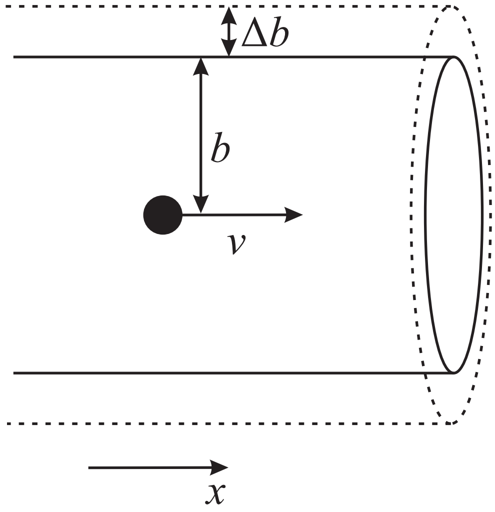
$$(p + \Delta p_x)^2 + p_y^2 = p^2.$$

We know that $p_y \ll p$, so the SBH momentum change along the x-axis $\Delta p_x = -\frac{p_y^2}{2p} = -\frac{\alpha^2}{2}p$, so

$$\boxed{\Delta p_x = -\frac{2G^2M^2m}{b^2v^3}}.$$

A3. To calculate net force we might integrate over stars with different impact parameters. The number of stars' transits during the time Δt equals $\Delta N = 2\pi bvn\, db\, \Delta t$, so force, decelerating the object along the x-axis,

(1) $$F_{DF} = \frac{1}{\Delta t}\int \Delta p_x\, dN = -4\pi G^2M^2\frac{nm}{v^2}\int_{b_{min}}^{b_{max}}\frac{db}{b} = -4\pi G^2M^2\frac{\rho}{v^2}\log\Lambda$$



The above formulas are true only for $b > b_1$, so the lower integration limit is $b_{min} = b_1$, and the upper limit is determined by the galaxy size $b_{max} = R$. So we have

$$F_{DF} = -4\pi G^2 M^2 \frac{\rho}{v^2} \log \Lambda \tag{2}$$

where $\Lambda = R/b_1$.

A4. We calculate: $b_1 = \frac{GM}{v^2} = 10.7$ pc, $\log \Lambda = 7.5$.

B. Gravitational slingshot

B1. From the second Newton's law

$$\frac{Mv^2}{a} = \frac{GM^2}{4a^2},$$

and we have for the orbital velocity $\boxed{v_{bin} = \sqrt{\frac{GM}{4a}}}$. The system energy is

$$E = E_{\text{kin}} + U = 2 \cdot \frac{Mv^2}{2} - \frac{GM^2}{2a}.$$

The answer is

$$\boxed{E = -\frac{GM^2}{4a}} \tag{3}$$

B2. From angular momentum conservation law

$$b\sigma = r_m v_0,$$

express v_0. Write down the energy conservation law

$$\frac{\sigma^2}{2} = \frac{v_0^2}{2} - \frac{GM_2}{r_m}$$

and derive

$$b = r_m\sqrt{1+\frac{2GM_2}{\sigma^2 r_m}}.$$

B3. To estimate the time between collisions let us use an analogy with the gas. As known from the molecular kinetic theory, given that molecules have radii r, thermal velocities v, and the molecular concentration n, the time Δt between collisions of one molecule with the others can be estimated from the relation $\pi r^2 v n \Delta t = 1$. In our problem b_{max} stands in place of the molecule radius, therefore for estimation it can be written

$$(\Delta t)^{-1} = \pi\sigma b_{max}^2 n.$$

Estimate the maximal impact parameter b_{max}, corresponding to the star collision with the binary system. The star should reach the distance of a to the binary system to collide. The star at large distances from the SBH binary interacts with it as with a point object of mass $M_2 = 2M$. From the results of B.2, assuming $r_m = a$, we obtain $b_{max} = a\sqrt{1+\frac{4GM}{\sigma^2 a}}$. Taking into account that $\sigma^2 \ll \frac{GM}{a}$, simplify:

$$b_{max} = \frac{2}{\sigma}\sqrt{GMa},$$

so we have

$$\boxed{\Delta t = \frac{m\sigma}{4\pi GM\rho a}}$$

B4. During the one act of gravitational slingshot, star energy increases at average by

$$\Delta E_{star} = \frac{mv_{bin}^2}{2} - \frac{m\sigma^2}{2}.$$

So the binary energy decreases by the same magnitude $\Delta E_{bin} = -\Delta E_{star}$. Taking into account that $\sigma \ll v_{bin}$, we derive

$$\Delta E_{bin} = -\frac{m}{2}v_{bin}^2 = \frac{GmM}{8a}.$$

Average binary system energy loss rate equals

$$\boxed{\frac{dE}{dt} = \frac{\Delta E}{\Delta t} = -\frac{\pi G^2 M^2 \rho}{2\sigma}} \tag{4}$$

Taking the time derivative of (3), we have

$$\frac{dE}{dt} = \frac{d}{dt}\left(-\frac{GM^2}{4a}\right) = \frac{GM^2}{4a^2}\frac{da}{dt}, \tag{5}$$

From (4) and (5) the orbit radius variation rate can be estimated as

$$\boxed{\frac{da}{dt} = -\frac{2\pi G\rho a^2}{\sigma}} \tag{6}$$

B5. Equation (6) can be easily integrated

$$\frac{da}{a^2} = -\frac{2\pi G\rho}{\sigma}dt. \tag{7}$$

To reduce the radius twice it takes time

$$\boxed{T_{SS} = \frac{\sigma}{2\pi G\rho a_1} = 7.3 \times 10^{-4}\,\mathrm{Gy}}$$

C. Emission of gravitational waves

C1. Using that $\omega = \dfrac{v_{bin}}{a} = \sqrt{\dfrac{GM}{4a^3}}$ and formulas from the problem text one can obtain:

$$\frac{dE}{dt} = -\frac{1024 \times 4}{5} \times \frac{GM^2 v_{bin}^6}{c^5 a^2} = \frac{64}{5} \cdot \frac{G^4 M^5}{c^5 a^5}. \tag{8}$$

Combining (5) and (8) we get the desirable result:

$$\boxed{\frac{da}{dt} = -\frac{256}{5} \cdot \frac{G^3 M^3}{c^5 a^3}} \tag{9}$$

C2. Integrating the equation (9) one can obtain:

$$a^3\, da = -\frac{256}{5} \cdot \frac{G^3 M^3}{c^5} dt \qquad \Longrightarrow \qquad \frac{a_2^4 - r_g^4}{4} = \frac{256}{5} \cdot \frac{G^3 M^3}{c^5} \cdot T_{GW}; \tag{10}$$

And taking into account $a_2 \gg r_g$ we derive the final result for T_{GW}:

$$\boxed{T_{GW} = \frac{5}{1024} \cdot \frac{a_2^4 c^5}{G^3 M^3}} \tag{11}$$

C3. From the previous equation and $T_{GW} = t_H$:

$$\boxed{a_H = \sqrt[4]{\frac{1024}{5} \cdot \frac{G^3 M^3\, t_H}{c^5}} = 0.098\,\mathrm{pc}} \tag{12}$$

D. Full evolution

D1. The galaxy is spherically symmetric, so mass enclosed within a sphere of radius r equals

$$m(r) = \int_0^r 4\pi x^2 \rho(x)\, dx = \frac{\sigma^2 r}{G}. \tag{13}$$

Thus the free fall acceleration of the body equals in the gravitational field of stars is

$$g(r) = \frac{Gm(r)}{r^2} = \frac{\sigma^2}{r}. \tag{14}$$

Therefore the body velocity is determined by relation

$$\frac{v^2}{r} = g = \frac{\sigma^2}{r},$$

which means

$$\boxed{v = \sigma} \tag{15}$$

So the velocity is constant.

D2. The energy of SBH in this gravitational field is

$$E = \frac{M\sigma^2}{2} + U$$

So the kinetic energy is constant and

$$\frac{dE}{dt} = \frac{dU}{dt} = \frac{dU}{da}\frac{da}{dt}$$

From the definition of potential energy we have

$$\frac{dU}{da} = g(a)M = \frac{M\sigma^2}{a}$$

Using the result of A3 we have

$$\frac{dE}{dt} = -F_{Df}v = -4\pi G^2 M^2 \frac{\rho(a)}{\sigma} \log\Lambda = -\frac{GM^2\sigma\log\Lambda}{a^2}.$$

Combining this equations we get the answer

$$\frac{da}{dt} = -\frac{GM\log\Lambda}{a\sigma} \tag{16}$$

D3. To estimate one can assume that SBHs form a binary when the mass of stars inside the sphere of radius a equals to M:

$$m(a) = \frac{\sigma^2 a}{G} = M,$$

so

$$\boxed{a_1 = \frac{GM}{\sigma^2} = 10.8\text{pc}}$$

Alternative variant: the force from another SBH is equal to force from all stars:

$$\frac{Gm(a)}{a^2} = \frac{GM}{4a^2}$$

so the answer is

$$\boxed{a_1 = \frac{GM}{4\sigma^2} = 2.7\text{pc}}$$

D4. Integrating the equation (16) we have

$$\frac{a_0^2 - a_1^2}{2} = \frac{GM\log\Lambda}{\sigma}T_1$$

and using that $a_1 \ll a_0$ we have

$$T_1 = \frac{a_0^2\sigma}{2GM\log\Lambda} = 0.121\,\text{Gy}.$$

D5. Total energy losses are caused by gravitational slingshot and gravitational waves emission, so combining equations (4) and (8):

$$\frac{dE}{dt} = -\frac{\pi G^2 M^2 \rho_1}{2\sigma} - \frac{64}{5}\cdot\frac{G^4M^5}{c^5a^5} \tag{17}$$

where

$\rho_1 = \rho(a_1) = \rho(10.8\text{pc}) = 6.3 \times 10^3 M_s/pc^3$, alternative: $\rho_1 = \rho(2.7\text{pc}) = 1.0 \times 10^5 M_s/\text{pc}^3$

Energy losses due to GW dominates when $\frac{\pi G^2 M^2 \rho_1}{2\sigma} < \frac{64 G^4 M^5}{5 c^5 a^5}$ i.e. $a < a_2$ where

$$a_2^5 = \frac{128}{5\pi} \cdot \frac{G^2 M^3 \sigma}{c^5 \rho_1} = \frac{512}{5} \cdot \frac{G^3 M^3 a_1^2}{c^5 \sigma}$$

Numerical answer is $\boxed{a_2 = 0.018\,\text{pc}}$ (alternative: $a_2 = 0.010\,\text{pc}$).

D6. For rough approximation it can be considered that at the slingshot stage ($a > a_2$) energy losses are caused only by slingshot, so T_2 is calculated analogiously to B5: $\frac{da}{a^2} = -\frac{2\pi G\rho}{\sigma} dt$ and

$$T_2 \approx \frac{\sigma}{2\pi G \rho_1 a_2} = 0.063\,\text{Gy} \qquad (T_2 \approx 0.0068\,\text{Gy})$$

And at the GW emission stage ($a < a_2$) energy losses are caused only by GW emission, so T_3 is calculated directly from C2:

$$T_3 \approx \frac{5}{1024} \cdot \frac{a_2^4 c^5}{G^3 M^3} = \frac{1}{8\pi} \cdot \frac{\sigma}{G \rho_1 a_2} = 0.016\,\text{Gy} \qquad (T_3 \approx 0.0017\,\text{Gy})$$

D7. Total time of SBH binary evolution from the moment of galaxies merging to SBH merging equals

$$T_{ev} = T_1 + T_2 + T_{GW} = 0.12 + 0.06 + 0.02\,\text{Gy} = 0.20\,\text{Gy} \qquad (T_{ev} = 0.13\,\text{Gy})$$

Theory, Task Q2

Marker________________________ Student________________________

TOTAL______________________

N	**Statement**	**Points**	**Marker**		**Consensus**	
A1	Idea that $\alpha = p_y / p_x$	0.1				
	$p_y = \int F dt$	0.1				
	$F_y = \frac{GMm}{b^2}\cos^3\varphi$	0.2				
	$dt = \frac{dx}{v}$ or $dt = \frac{b\,d\varphi}{v\cos^2\varphi}$, or $p_y = \frac{GMm}{bv}\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi$, or alternative correct equation	0.1				
	Answer $\alpha = \frac{2GM}{bv^2} = \frac{2b_1}{b}$ or k=2	0.25		**(0.75)**		
A2	$\Delta p_x = p(1-\cos\alpha)$ or $\Delta p_x = \mp\frac{\Delta p_y^2}{2p}$	0.1				
	Answer $\Delta p_x = \mp\frac{2G^2M^2m}{b^2v^3} = \frac{2b_1^2}{b^2}p$	0.15		**(0.25)**		
A3	$\Delta N = 2\pi b\Delta b v n\,\Delta t$, up to factor 10	0.2				
	Answer $F_{DF} = \mp 4\pi G^2 M^2\frac{\rho}{v^2}\log\Lambda$, up to factor 10	0.2		**(0.4)**		
A4	$\log\Lambda = 7.4\ldots7.6$	**0.2**		**(0.2)**		
A		**1.6**				
B1	$v_{bin} = \sqrt{\frac{GM}{4a}}$	0.1				
	Answer $E = -\frac{GM^2}{4a}$, with correct sign *Incorrect numerical factor doesn't influence following scores.	0.15		**(0.25)**		
B2	$b\sigma = r_m v_0$	0.1				
	$\frac{\sigma^2}{2} = \frac{v_0^2}{2} - \frac{GM_2}{r_m}$	0.1				
	Answer $b = r_m\sqrt{1+\frac{2GM_2}{\sigma^2 r_m}}$	0.3		**(0.5)**		
B3	$(\Delta t)^{-1} = \pi\,\sigma r^2 n$	0.2				
	$r = b_{max}$	0.3				
	$b_{max} = a\sqrt{1+\frac{4GM}{\sigma^2 a}} \approx \frac{2}{\sigma}\sqrt{GMa}$	0.3				
	Answer $\Delta t = \frac{m\sigma}{4\pi GM\rho a}$, up to numeric coefficient	0.2		**(1.0)**		
B4	$\frac{dE}{dt} = -\frac{\pi G^2 M^2\rho}{2\sigma}$, up to numeric coefficient	0.15				
	$\frac{da}{dt} = -\frac{2\pi G\rho a^2}{\sigma}$, up to numeric coefficient	0.1		**(0.25)**		
B5	$T_{SS} = \frac{\sigma}{2\pi G\rho a_1}$, up to numeric coefficient	0.7				
	$T_{SS} = 7.3\times10^{-4}\,Gy$, up to factor 10	0.3		**(1.0)**		
B		**3.0**				

C1	$\frac{da}{dt}=-\frac{256}{5}\cdot\frac{G^3M^3}{c^5a^3}$, with correct sign	**0.2**		**(0.2)**		
C2	Integral is calculated $\frac{a_2^4-r_g^4}{4}=\frac{256}{5}\cdot\frac{G^3M^3}{c^5}\cdot T_{GW}$	0.3				
	Answer $T_{GW}=\frac{5}{1024}\cdot\frac{a_2^4c^5}{G^3M^3}$, up to factor 10	0.4				
		0.7		**(0.7)**		
C3	$a_H=0.098\text{pc}$, up to factor 10	**0.1**		**(0.1)**		
C		**1.0**				
D1	$m(r)=\frac{\sigma^2 r}{G}$	0.1				
	$v=\sigma$	0.15		**(0.25)**		
D2	$\frac{dE}{dt}=\frac{dU}{dt}$ *alt. solution:* using angular momentum equation $mv\dot{a}=Fa$	0.3				
	$\frac{dU}{da}=g(a)M$	0.2				
	$\frac{dE}{dt}=-F_{DF}v$	0.15				
	$\frac{da}{dt}=-\frac{GM\log\Lambda}{a\,\sigma}$	0.1		**(0.75)**		
D3	$\mathrm{m}(a)=M$	0.1				
	$a_1=\frac{GM}{\sigma^2}$, up to factor 10	0.1				
	$a_1=1...100\text{pc}$	0.1		**(0.3)**		
D4	$\frac{a_0^2-a_1^2}{2}=\frac{GM\log\Lambda}{\sigma}T_1$	0.4				
	$T_1=\frac{a_0^2\,\sigma}{2GM\log\Lambda}$, up to factor 10	0.25				
	$T_1=0.121Gy$, up to factor 10	0.1		**(0.75)**		
D5	$\dot{E}_{SS}=\dot{E}_{GW}$	0.1				
	$a_2^5=\frac{512}{5}\cdot\frac{G^3M^3a_1^2}{c^5\sigma}$, up to factor 10	0.1				
	$a_2=0.001...0.1\text{pc}$	0.1		**(0.3)**		
D6	Idea of neglecting $\dot{E}_{GW}$ in slingshot stage and neglecting $\dot{E}_{SS}$ in gravitational waves stage	0.25				
	$T_2\approx\frac{\sigma}{2\pi G\rho_1 a_2}$	0.2				
	$T_2\approx 10^{-3}...10^{-1}Gy$	0.65				
	$T_3\approx 10^{-4}...10^{-1}Gy$	0.65		**(1.75)**		
D7	$T_{ev}=T_1+T_2+T_{GW}$	0.1				
	Scored if D6 score > 0 $\mathrm{T}_{ev}=0.02...2.00\text{Gy}$	0.2		**(0.3)**		
D		**4.4**				
	TOTAL	**10**				

Space Debris

APhO 2017

Introduction

In more than half a century of space operations quite a large number of man-made objects have been amassed near Earth. The objects that do not serve any particular purpose are called *space debris*. The most attention is usually paid to the larger debris objects, i.e. defunct satellites and spent rocket upper stages, which stay in orbit after delivering their payload. Collisions of such objects with each other may result in thousands of fragments endangering all current space missions.

There is a well-known hypothetical scenario, according to which certain collisions may cause a cascade where each subsequent collision generates more space debris that increase the likelihood of new collisions. Such a chain reaction, resulting in the loss of all near-Earth satellites and making impossible further space programs, is called the *Kessler syndrome*.

To prevent such undesirable outcome special missions are planned to remove large debris object from their present orbits either by tugging them to the Earth's atmosphere or to graveyard orbits. To this end a specially designed spacecraft – a space tug – must capture a debris object. However, before capturing an uncontrolled object it is important to understand its rotational dynamics.

We suggest you to take part in planning of such a mission and find out how the rotational dynamics of a debris object changes in time under the influence of different factors.

Rocket Stage Schematic

The debris object to be considered is a "Kerbodyne 42" rocket upper stage, whose schematic is shown in Fig. 1. The circle line in Fig. 1 marks the outline of a spherical fuel tank.

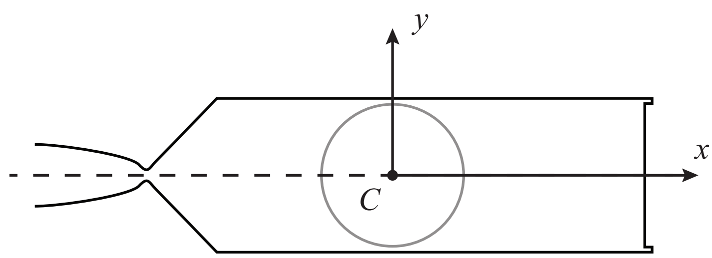


Figure 1: "Kerbodyne 42" upper stage

We introduce a body-fixed reference frame Cxy with the origin in the center of mass C, x being the symmetry axis of the stage, and y perpendicular to x. The inertia moments with respect to x and y axes are J_x and J_y ($J_x < J_y$).

A. Rotation

Consider an arbitrary initial rotation of the stage with angular momentum L (Fig. 2), where θ is the angle between the symmetry axis and the direction of angular momentum. Fuel tank at this point is assumed to be empty. No forces or torques act upon the stage.

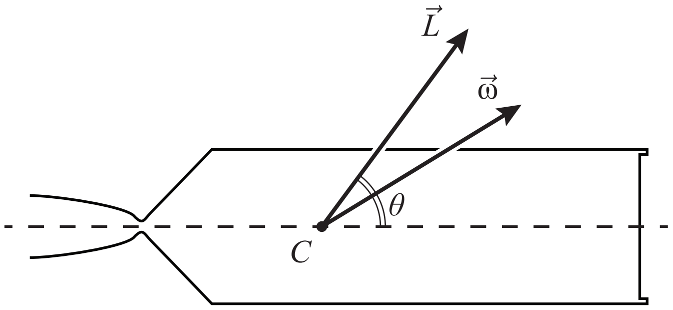


Figure 2: Rocket stage rotation

1.(0.2 pts) Find the projections ω_x and ω_y of angular velocity $\vec{\omega}$ on x and y, given that $\vec{L} = J_x \omega_x \vec{e_x} + J_y \omega_y \vec{e_y}$ for material symmetry axes x and y with unit vectors $\vec{e_x}$ and $\vec{e_y}$. Provide the answer in terms of $L = |\vec{L}|$, angle θ, and inertia moments J_x, J_y.

$$\omega_x = \frac{L\cos\theta}{J_x}, \tag{A1}$$

0.1 point

$$\omega_y = \frac{L\sin\theta}{J_y}. \tag{A2}$$

0.1 point

2.(0.4 pts) Find the rotational energy E_x associated with rotation ω_x and E_y associated with rotation ω_y. Find total rotational kinetic energy $E = E_x + E_y$ of the stage as a function of the angular momentum L and $\cos\theta$.

$$E_x = \frac{J_x \omega_x^2}{2}, \tag{A3}$$

0.1 points

$$E_y = \frac{J_y \omega_y^2}{2}, \tag{A4}$$

0.1 points

$$E(\theta) = E_x + E_y = \frac{J_x\omega_x^2}{2} + \frac{J_y\omega_y^2}{2} = \frac{L_x^2}{2J_x} + \frac{L_y^2}{2J_y} = \frac{L^2}{2J_y} + \frac{L^2}{2}\left(\frac{1}{J_x} - \frac{1}{J_y}\right)\cos^2\theta. \qquad \text{(A5)}$$

0.2 points

In the following questions of Section A consider the stage's free rotation with the initial angular momentum L and $\theta(0) = \theta_0$.

3. (1.2 pts) Let us denote by x_0 the initial orientation of the stage's symmetry axis Cx with respect to the inertial reference frame. Using conservation laws find the maximum angle ψ, which the stages symmetry axis Cx makes with x_0 during the stage's free rotation.

Note: Since there are no external torques acting upon the stage, the angular momentum vector remains constant.

Both kinetic energy and angular momentum are conserved, and $\cos^2\theta$ can be obtained from equation A5.

Consequently, the set of values that θ can take is discrete (one value in each quadrant for every value of $\cos^2\theta$), and in the process of continuous motion θ cannot change its initial value. Therefore the stage's axis of symmetry moves around $\vec{L}$ making a conic surface with aperture $2\theta_0$. Consequently

$$\psi = 2\theta_0. \qquad \text{(A6)}$$

1.2 points for the correct answer for ψ.

If the correct answer is not provided 1.0 point is given for the proof that $\theta(t) = \theta_0$ and does not change in time.

If this is not done:

- 0.2 points for the formula expressing the angular momentum conservation,
- 0.2 points for the formula expressing the energy conservation,
- 0.2 points for the formula expressing θ through any given constant parameters of the problem

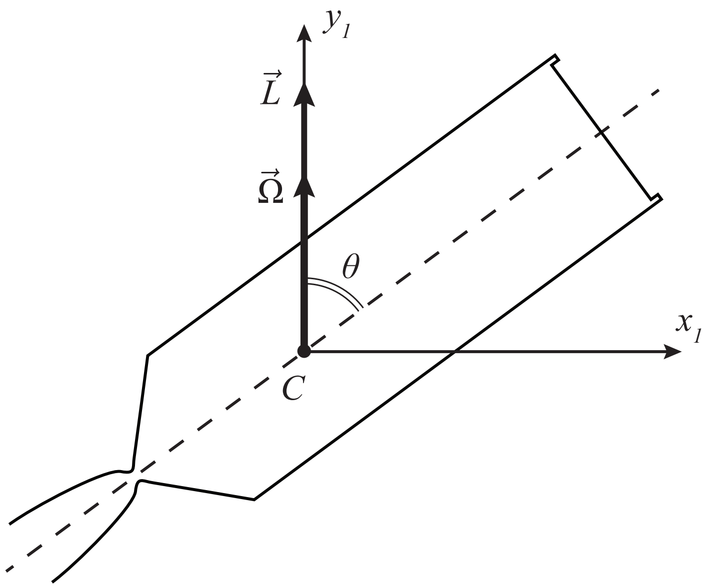


Figure 3:

Let us now introduce the reference frame $Cx_1y_1z_1$ with y_1 along the constant angular momentum vector $\vec{L}$ (Fig. 3). This reference frame rotates about y_1 in such a way, that the stage's symmetry axis always belongs to the Cx_1y_1 plane.

4. (2.0 pts) Given L, $\theta(0) = \theta_0$ and inertia moments J_x, J_y, find the angular velocity $\Omega(t)$ of the reference frame Cx_1y_1 about y_1 and direction (i.e. angle $\gamma_s(t)$ that $\vec{\omega}_s(t)$ makes with the symmetry axis Cx) and absolute value of angular velocity of the stage $\vec{\omega}_s(t)$ relative to the reference frame Cx_1y_1 as functions of time.

Note: angular velocity vectors are additive $\vec{\omega} = \vec{\omega}_x + \vec{\omega}_y = \vec{\Omega} + \vec{\omega}_s$.

The symmetry axis is at rest with respect to the rotating reference frame, because $\theta(t) = \theta_0$ and the symmetry axis always belongs to the Cx_1y_1 plane. Hence, $\vec{\omega}_s$ must be collinear to the symmetry axis at all times. Thus

$$\gamma_s(t) = 0.$$

0.5 points

Projecting the sum $\vec{\Omega} + \vec{\omega}_s$ onto Cx and Cy yields for any t:

$$\omega_s + \Omega\cos\theta = \omega_x = \frac{L\cos\theta}{J_x}, \tag{A7}$$

$$\Omega\sin\theta = \omega_y = \frac{L\sin\theta}{J_y}. \tag{A8}$$

0.25 points for each of the equations A7 and A8

Whence

$$\Omega = \frac{L}{J_y}. \tag{A9}$$

Thus Ω does not depend on time.

1.0 points

Taking into account that $\theta(t) = \theta_0$:

$$\omega_s = \left(\frac{1}{J_x} - \frac{1}{J_y}\right) L \cos\theta_0. \tag{A10}$$

And ω_s also does not depend on time.

0.5 points

NB:
1.0 points is given for the correct answer for Ω.
For ω_s 0.5 points is given for the direction of $\vec{\omega_s}$ (along Cx)
0.5 points is given for A10

Alternatively:
0.25 points is given for any of the A7, A8 equations

B. Transient Process

Most of the propellant is used during the ascent, however, after the payload has been separated from the stage, there still remains some fuel in its tank. Mass m of residual fuel is negligible in comparison to the stage's mass M. Sloshing of the liquid fuel and viscous friction forces in the fuel tank result in energy losses, and after a transient process of irregular dynamics the energy reaches its minimum.

1.(0.6 pts) Find the value θ_2 of angle θ after the transient process, for arbitrary initial values of L and $\theta(0) = \theta_1 \in (0, \pi/2)$.

Interaction of the residual fuel with the fuel tank walls can be considered an internal force. Hence, as before, no external forces or torques act upon the system, and the angular momentum is conserved.
For the given initial value of θ and knowing that $J_x < J_y$, it is easily shown from A5 that $E(\cos\theta)$ reaches its minimum for $\theta = \pi/2$.
Thus

$$\theta_2 = \frac{\pi}{2}. \tag{B1}$$

0.6 points

2.(0.6 pts) Calculate the value ω_2 of angular velocity ω after the transient process, given that initial angular velocity $\omega(0) = \omega_1 = 1\ rad/s$ makes an angle of $\gamma(0) = \gamma_1 = 30°$ with the stage's symmetry axis. The moments of inertia are $J_x = 4200\ kg \cdot m^2$ and $J_y = 15\ 000\ kg \cdot m^2$.

B1 implies that after the transient process the stage rotates about the axis perpendicular to its symmetry axis.

0.2 points

Final angular velocity value ω_2 can be obtained from the angular momentum conservation law:

$$\omega_2 = \frac{L}{J_y} = \frac{\sqrt{J_x^2 \cos^2 \gamma_1 + J_y^2 \sin^2 \gamma_1}}{J_y} \omega_1 \tag{B2}$$

0.2 points

$$\omega_2 \approx 0.56\ rad/s. \tag{B3}$$

0.2 points

C. Magnetic Field

Another important factor in rotational dynamics of a debris rocket stage, which is orbiting the Earth, is its interaction with the Earth's magnetic field. Let us first consider an auxiliary problem.

Torque due to Eddy Currents

Let us place a thin-walled nonmagnetic spherical shell with wall thickness D and radius R in a uniform magnetic field $\vec{B}$, which slowly changes so that its derivative $\dot{\vec{B}}$ is a constant vector making angle α with the direction of $\vec{B}$ (Fig. 4). Electrical resistivity of the shell's material is ρ.

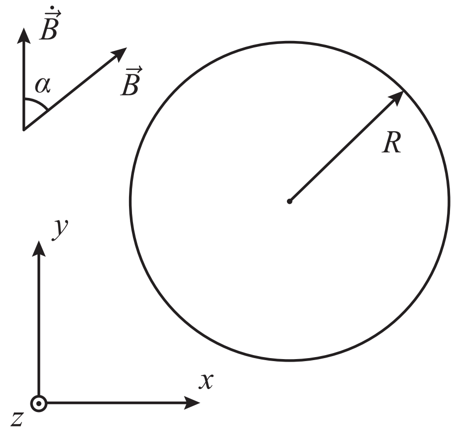


Figure 4: Spherical shell in magnetic field

1. (1.0 pts) Find the induced magnetic moment $\vec{\mu}$ of the shell, neglecting its self-inductance. Provide the answer for $\vec{\mu}$ in the form of projections on xyz (see Fig. 4).

Let us cut the sphere into ring slices so that $\dot{\vec{B}}$ is perpendicular to their planes and introduce angle φ as shown in Fig. 5.

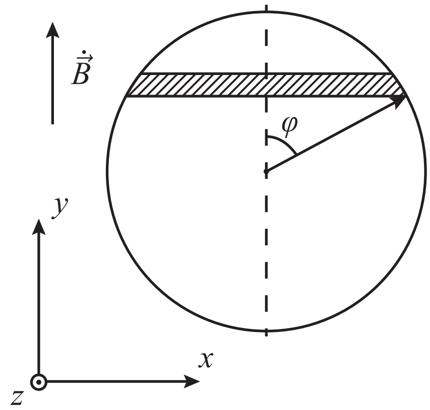


Figure 5:

According to Faraday's law the absolute value of eddy current EMF, induced in such a slice by the varying magnetic field is

$$\mathcal{E} = \dot{\Phi} = S\dot{B} = \pi R^2 \sin^2 \varphi \dot{B} \tag{C1}$$

0.2 points

The ring slice resistance is

$$dr = \frac{2\pi\rho R \sin\varphi}{DRd\varphi}. \tag{C2}$$

0.2 points

Current in the ring slice

$$dI = \mathcal{E}/dr = \frac{1}{2\rho} DR^2 \dot{B} \sin\varphi d\varphi \tag{C3}$$

0.1 points

And, finally, magnetic moment:

$$d\mu = SdI = \frac{\pi}{2\rho} DR^4 \dot{B} \sin^3\varphi d\varphi = \frac{1}{4\rho} \dot{B} dJ, \tag{C4}$$

where dJ is the moment of inertia for a slice ring of unit density with respect to the central axis, which is parallel to y.

0.2 points

Thus

$$\mu = \frac{1}{4\rho} J\dot{B} = \frac{2\pi}{3\rho} DR^4 \dot{B}, \tag{C5}$$

where J is the moment of inertia of the sphere with respect to the axis, passing through its center. Taking into account the direction:

$$\mu_x = 0, \tag{C6}$$

$$\mu_y = -\frac{2\pi}{3\rho} DR^4 \dot{B}, \tag{C7}$$

$$\mu_z = 0. \tag{C8}$$

0.1 points for each $\vec{\mu}$ component

2. (0.3 pts) Find the torque $\vec{M}$ acting on the spherical shell. Provide the answer for $\vec{M}$ in the form of projections on xyz (see Fig. 4).

The torque is given by $\vec{M} = [\vec{\mu}, \vec{B}]$. It is directed along the z axis and equals

$$M_z = \mu B \sin\alpha = \frac{2\pi}{3\rho} D R^4 B \dot{B} \sin\alpha. \tag{C9}$$

0.1 points for each $\vec{M}$ component

NB: Alternatively, if the task of the previous assignment (find $\vec{\mu}$) is not completed, but the answer for $\vec{M}$ is, nevertheless, provided, the points for intermediate steps from the previous assignment (except 0.3 points for $\vec{\mu}$ components) are redistributed for the actions to find $\vec{M}$.

Attitude Motion Evolution in the Earth's Magnetic Field

Let us find out how the rotation changes for a rocket stage, which moves in a circular polar orbit with orbital period $T = 100\ min$ (Fig. 6). It transpires that the characteristic times of dynamics due to interaction with the geomagnetic field are much greater than the duration of the transient process. We will now study what happens to the rocket stage after the transient process has completed. To start our analysis consider the stage rotating with angular velocity ω_2 about the axis perpendicular to the orbital plane.

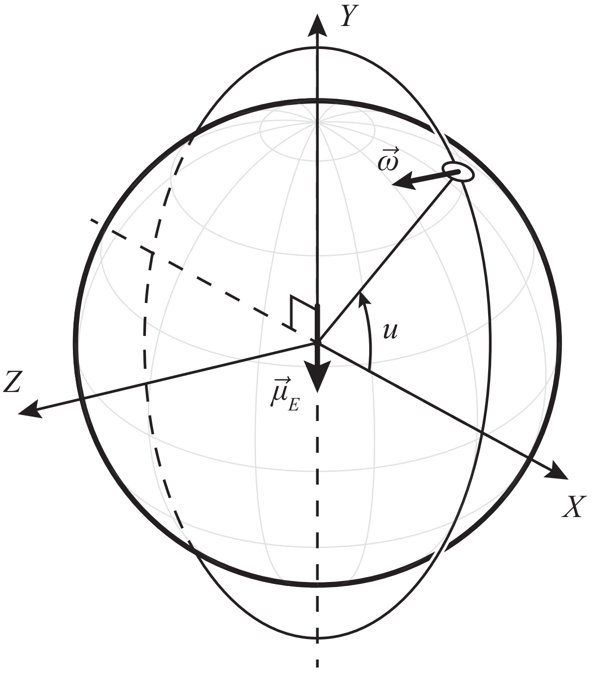


Figure 6: The orbit

1.(0.4 pts) The Earth's magnetic field $\vec{B}_E$ can be modeled as the magnetic field of a point dipole in the Earth's center. Its dipole moment $\vec{\mu}_E$ is directed opposite to Y axis. The absolute value of the Earth's magnetic field B at the point where the orbit crosses the equatorial plane

XZ is $B_0 = 20\ \mu T$. Find $\vec{B}_E(u)$ at a current position of the stage in the orbit defined by the angle u as shown in Fig. 6. The positive direction of u is along with the orbital motion. Provide the answer in the form of the projections of $\vec{B}_E(u)$ on XYZ axes.

Note: Magnetic field of a dipole at point $\vec{r}$ is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3\,(\vec{\mu}\cdot\vec{r})\,\vec{r}}{r^5} - \frac{\vec{\mu}}{r^3}\right).$$

Note: It may facilitate subsequent calculations if projections of $\vec{B}_E(u)$ are given as functions of $2u$ instead of u.

Let R_O be the orbit radius. The dipole field formula at point $\vec{r} = (R_O \cos u, R_O \sin u, 0)$ and $\vec{\mu} = (0, -\vec{\mu}_E, 0)$ yield

$$B_X = -\frac{3}{2}\frac{\mu_0\mu_E}{4\pi R_O^3}\sin 2u, \tag{C10}$$

$$B_Y = \left(1 - 3\sin^2 u\right)\frac{\mu_0\mu_E}{4\pi R_O^3}, \tag{C11}$$

$$B_Z = 0. \tag{C12}$$

0.05 for each $\vec{B}$ component, if no final answer (see below) is obtained.

At point, where the orbit passes through the equatorial plane ($u = 0$) the magnetic field is

$$B_X = 0, \tag{C13}$$

$$B_Y = \frac{\mu_0\mu_E}{4\pi R_O^3}, \tag{C14}$$

$$B_Z = 0. \tag{C15}$$

Thus $B_0 = \frac{\mu_0\mu_E}{4\pi R_O^3}$.

0.1 points for B_0

Finally, the Earth's magnetic field is:

$$B_X(u) = -\frac{3}{2}B_0 \sin 2u, \tag{C16}$$

$$B_Y(u) = \frac{1}{2}\left(3\cos(2u) - 1\right)B_0, \tag{C17}$$

$$B_Z(u) = 0. \tag{C18}$$

0.1 points for each component of $\vec{B}$

The "Kerbodyne 42" rocket upper stage is mostly made of wood, and the only conductive material is used for its cryogenic fuel tank. We, therefore, consider the stage's interaction with the geomagnetic field as that of the spherical shell with wall thickness $D = 2\ mm$, radius $R = 4\ m$ and resistivity $\rho = 2.7 \cdot 10^{-8}\ \Omega \cdot m$.

2.(**1.3 pts**) Find the torque $\vec{M}(u)$ acting on the stage, as it rotates with angular velocity ω collinear to Z. Provide the answer for $\vec{M}(u)$ in the form of projections on XYZ.

Using C9 requires us to find the magnetic field derivative in the body frame.
Consider a body frame xyz, whose axis z is collinear to Z and plane xy is rotated by angle β with respect to XY. Magnetic field in this reference frame is

$$B_x = B_X \cos\beta + B_Y \sin\beta, \tag{C19}$$
$$B_y = -B_X \sin\beta + B_Y \cos\beta, \tag{C20}$$
$$B_z = B_Z = 0. \tag{C21}$$

0.1 point for any idea that provides understanding that there are two processes in which B changes with respect to body-frames – orbital motion and rotational dynamics. The same 0.1 point is given for any approach overcoming this issue.

The derivative of magnetic field is therefore

$$\begin{aligned}
\dot{B}_x =& \dot{B}_X \cos\beta + \dot{B}_Y \sin\beta + (-B_X \sin\beta + B_Y \cos\beta)\,\dot{\beta} = \\
&= (B'_X(u)\cos\beta + B'_Y(u)\sin\beta)\,\dot{u} + (-Bh_X \sin\beta + B_Y \cos\beta)\,\dot{\beta}, \\
\dot{B}_y =& -\dot{B}_X \sin\beta + \dot{B}_Y \cos\beta + (-B_X \cos\beta - B_Y \sin\beta)\,\dot{\beta} = \\
&= (-B'_X(u)\sin\beta + B'_Y(u)\cos\beta)\,\dot{u} + (-B_X \cos\beta - B_Y \sin\beta)\,\dot{\beta}, \\
\dot{B}_z =& 0.
\end{aligned}$$

0.1 points for each component of $\dot{\vec{B}}$ related to the orbital motion
0.1 points for each component of $\dot{\vec{B}}$ related to the rotational dynamics
Full points are also given if $\dot{\vec{B}}$ is found in the vector form

Substituting $\dot{u} = 2\pi/T$ and $\dot{\beta} = \omega$ and using the expressions C9, C16, and C17, we obtain that the torque is directed along z and equals

$$M_z = \frac{2\pi}{3\rho} D B_0^2 R^4 \left(\frac{3\pi}{T}(3 - \cos 2u) - \frac{\omega}{2}(5 - 3\cos 2u) \right) = M_Z. \tag{C22}$$

0.1 points for M_x and M_y,
0.5 points for M_z (for complicated calculations)

3. (**1.0**)Find the absolute value of angular velocity $\omega(t)$ as a function of time, given that the change in the stage's angular velocity over one orbital period is negligibly small.

We will average M_Z over u and use the obtained expression instead of M_Z. This helps getting rid of the members, containing $\cos 2u$:

$$\langle M_Z \rangle = \frac{2\pi}{3\rho} D B_0^2 R^4 \left(\frac{9\pi}{T} - \frac{5\omega}{2} \right). \tag{C23}$$

0.25 points for the explicit idea to average M_Z over u
0.25 for the correct expression for $\langle M_Z \rangle$

As the torque is directed along with the rotation axis, it does not change the axis' direction, which means that the obtained formula always holds for the rocket stage rotational dynamics. As we consider the transient process to have completed. It follows from B1 that the rocket stage rotates about the axis, which is perpendicular to its symmetry axis. Thus the angular momentum of the stage is

$$L_Z = J_y \omega. \tag{C24}$$

0.25 for the correct equation with the correct inertia moment

As $\dot{L}_Z = M_Z$ is the governing equation for the angular velocity:

$$\dot{\omega} = \frac{2\pi}{3 J_y \rho} D B_0^2 R^4 \left(\frac{9\pi}{T} - \frac{5\omega}{2} \right). \tag{C25}$$

Its solution is:

$$\omega(t) = \frac{18\pi}{5T} + \left(\omega_2 - \frac{18\pi}{5T} \right) e^{-\delta t}, \tag{C26}$$

where $\delta = \frac{5\pi}{3 J_y \rho} D B_0^2 R^4$.

0.25 points for the correct solution of the differential equation for ω
Alternatively 0.15 for the exponential dependence of ω from t.

4. (**1.0**) Find the ratio of the orbital period T and the rocket stage's rotation period T_s in the steady-state regime, which sets in after a long time.

From C26 it follows that the angular velocity asymptotically tends to $18\pi/2T$. Thus the ratio of the two periods

$$\frac{T}{T_s(\infty)} = \frac{T\omega(\infty)}{2\pi} = 9/5 = 1.8. \tag{C27}$$

1.0 for the correct result.

Q3: Space Debris

Marker____________________ Student______________________

TOTAL______________________

Subq.	№	Statement	Pts.	Marker Stat.	Marker Subq.	Consensus Stat.	Consensus Subq.
A1 (0.2)	1	$\omega_x = \frac{L\cos\theta}{J_x}$	0.1				
	2	$\omega_y = \frac{L\sin\theta}{J_y}$	0.1				
A2 (0.4)	3	$E_x = \frac{J_x\omega_x^2}{2}$	0.1				
	4	$E_y = \frac{J_y\omega_y^2}{2}$	0.1				
	5	$E(\theta) = \frac{L^2}{2J_y} + \frac{L^2}{2}\left(\frac{1}{J_x} - \frac{1}{J_y}\right)\cos^2\theta$	0.2				
	6	*if no (5)* Correct energy not in terms of L, J, $\cos\theta$	0.1*				
A3 (1.2)	7	$\psi = 2\theta_0$	1.2				
	8	*if no (7)* $\theta = \mathrm{const}$	1.0*				
	9	*if no (7,8)* $L = \mathrm{const}$	0.2*				
	10	*if no (7,8)* $E = \mathrm{const}$	0.2*				
	11	*if no (7,8)* Correct formula for θ in terms of given parameters	0.2*				
A4 (2.0)	12	$\Omega(\mathrm{t}) = \frac{L}{J_y}$	1.0				
	13	$\gamma_s(t) = 0$	0.5				
	14	$\omega_s(t) = \left(\frac{1}{J_x} - \frac{1}{J_y}\right) L\cos\theta_0$	0.5				
	15	*if no (12,14)* $\omega_s + \Omega\cos\theta = \frac{L\cos\theta}{J_x}$	0.25*				
	16	*if no (12,14)* $\Omega\sin\theta = \frac{L\sin\theta}{J_y}$	0.25*				
B1 (0.6)	17	$\theta_2 = \frac{\pi}{2}$	0.6				
	18	*if no (17)* $\cos\theta_2 = 0$	0.3*				

Subq.	№	Statement	Pts.	Marker		Consensus	
				Stat.	Subq.	Stat.	Subq.
B2 (0.6)	19	$\omega_2 \approx \frac{5}{9}\,\mathrm{rad/s} \approx 0.556\,\mathrm{rad/s}$	0.6				
	20	*if no (19)* $\omega_2 = \frac{\omega_1}{J_y}\sqrt{J_x^2\cos^2\gamma_1 + J_y^2\sin^2\gamma_1}$	0.4*				
	21	*if no (19,20)* rotation at $t = \infty$ about *y* axis	0.2*				
C1 (1.0)	22	$\mu_x = 0$	0.1				
	23	$\mu_y = -\frac{2\pi}{3\rho}DR^4\dot{B}$	0.8				
	24	$\mu_z = 0$	0.1				
	25	*if no (23)* $d\mu = \frac{\pi}{2\rho}DR^4\dot{B}\sin^3\varphi\,d\varphi$	0.7*				
	26	*if no (23,25)* Expression for elementary Ampere's torque which explicitly corresponds to (34)	0.7*				
	27	*if no (23,25,26)* Expression for elementary Ampere's force which explicitly corresponds to (34)	0.6*				
	28	*if no (23,25-27)* $dI = \frac{1}{2\rho}DR^2\dot{B}\sin\varphi\,d\varphi$	0.5*				
	29	*if no (23,25-28)* $r(d\varphi) = \frac{2\pi\rho R\sin\varphi}{DRd\varphi}$	0.2*				
	30	*if no (23,25-28)* Faraday's law ($\mathcal{E} = -\dot{\Phi}$)	0.2*				
	31	if no (22-30) and 34 is correct	0.7*				
C2 (0.3)	32	$M_x = 0$	0.1				
	33	$M_y = 0$	0.1				
	34	$M_z = \frac{2\pi}{3\rho}DR^4B\dot{B}\sin\alpha$	0.1				

Subq.	№	Statement	Pts.	Marker Stat.	Marker Subq.	Consensus Stat.	Consensus Subq.
C3 (0.4)	35	$B_{EX}(u) = -\frac{3}{2} B_0 \sin 2u$	0.15				
	36	$B_{EY}(u) = \frac{1}{2}(3\cos 2u - 1)B_0$	0.15				
	37	$B_{EZ}(u) = 0$	0.1				
	38	*if no (35) or (36)* $B_0 = \frac{\mu_0 \mu_E}{4\pi} \frac{1}{R_O^3}$	0.1*				
	39	*if no (35, 38)* $B_{EX}(u) = -\frac{3\mu_0 \mu_E}{8\pi} \frac{1}{R_O^3} \sin 2u$	0.05*				
	40	*if no (36, 38)* $B_{EY}(u) = -\frac{\mu_0 \mu_E}{8\pi} \frac{1}{R_O^3}(3\cos 2u - 1)$	0.05*				
C4 (1.3)	41	$M_X = 0$	0.1				
	42	$M_Y = 0$	0.1				
	43	$M_Z = \frac{2\pi}{3\rho} D B_0^2 R^4 \left(\frac{3\pi}{T}(3 - \cos 2u) - \frac{\omega}{2}(5 - 3\cos 2u)\right)$	1.1				
	44	*if no (43)* vector form for $\dot{\vec{B}}$ explicitly equivalent to (45-47)	0.6*				
	45	*if no (43,44)* $\dot{B}_x = (48) + (51)$	0.2*				
	46	*if no (43,44)* $\dot{B}_y = (49) + (52)$	0.2*				
	47	*if no (43,44)* $\dot{B}_z = 0$	0.2*				
	48	*if no (43,44,45)* orbital part of $\dot{B}_x$: $(B_X'(u)\cos\beta + B_Y'(u)\sin\beta)\frac{2\pi}{T}$	0.1*				
	49	*if no (43,44,46)* orbital part of $\dot{B}_y$: $(-B_X'(u)\sin\beta + B_Y'(u)\cos\beta)\frac{2\pi}{T}$	0.1*				
	50	*if no (43,44,47)* orbital part of $\dot{B}_z = 0$	0.1*				
	51	*if no (43,44,45)* rotational part of $\dot{B}_x$: $(-B_X(u)\sin\beta + B_Y(u)\cos\beta)\omega$	0.1*				
	52	*if no (43,44,46)* rotational part of $\dot{B}_y$: $(-B_X(u)\cos\beta - B_Y(u)\sin\beta)\omega$	0.1*				
	53	*if no (43,44,47)* rotational part of $\dot{B}_z = 0$	0.1*				
	54	*if no (43)* $\dot{\vec{B}} = \dot{\vec{B}}_{orbital} + \dot{\vec{B}}_{rotational}$	0.1*				
	55						

C5 (1.0)	56	$\omega(\mathrm{t}) = \frac{18\pi}{5T} + \left(\omega_2 - \frac{18\pi}{5T}\right) e^{-\delta t}, \delta = \frac{5\pi}{3J_y\rho} D B_0^2 R^4$	1.0				
	57	*if no (56)* $\langle M_Z \rangle = \frac{2\pi}{3\rho} D B_0^2 R^4 \left(\frac{9\pi}{T} - \frac{5\omega}{2}\right)$	0.5*				
	58	*if no (56, 57)* $\langle M_Z \rangle \sim (A - B\omega), A \neq 0, B > 0$	0.25*				
	59	*if no (56-58)* explicit attempt to average M_Z is present	0.15*				
	60	*if no (56)* $L_Z = J_y \omega$	0.25*				
	61	*if no (56)* $\omega(\mathrm{t}) = \omega_2 e^{-\delta t}, \delta = \frac{5\pi}{3J_y\rho} D B_0^2 R^4$	0.15*				
	62	*if no (56,61)* $\omega(\mathrm{t}) = \omega_2 e^{-\kappa t}, \kappa > 0$	0.1*				
C6 (1.0)	63	1.8	1.0				

Notes:

* mark the lines that are applied only if the points for the answer sheet questions are not given

Error propagation rules:

Rule 1. Errors are traced back to the origin and are penalized only in those statements, where they occur
Rule 2. Rule 1 does not apply whenever there is a clear **physical** explanation, why the obtained erroneous results cannot be true (e.g., angular velocity tends to infinity in C5, or $\mu \sim \rho$ in C1)
Rule 3. If rule 1 does not apply, all points are halved for the statements influenced by the error and following the question, where the physical explanation can be observed.
Special rule for C6. Points are given only for the exact result (no remorse).