(**Full Mark = 20**)

Part	Model Answer	Marks
A1		**2**

The potential energy for $N = 2$ is:

$$E_\mathrm{p}(\alpha) = Mg \cdot y_{\mathrm{c{\cdot}m.}(0,0)} \times 4 + Mg \cdot \Delta y \times 2 \quad \textbf{(0.5 points)} \qquad \text{- Eq. (1)}$$

where

$$y_{\mathrm{c{\cdot}m.}(0,0)} = -\frac{\sqrt{3}l}{3} \sin\left(\frac{\pi}{6} + \alpha\right) \quad \textbf{(0.5 points)} \qquad \text{- Eq. (2)}$$

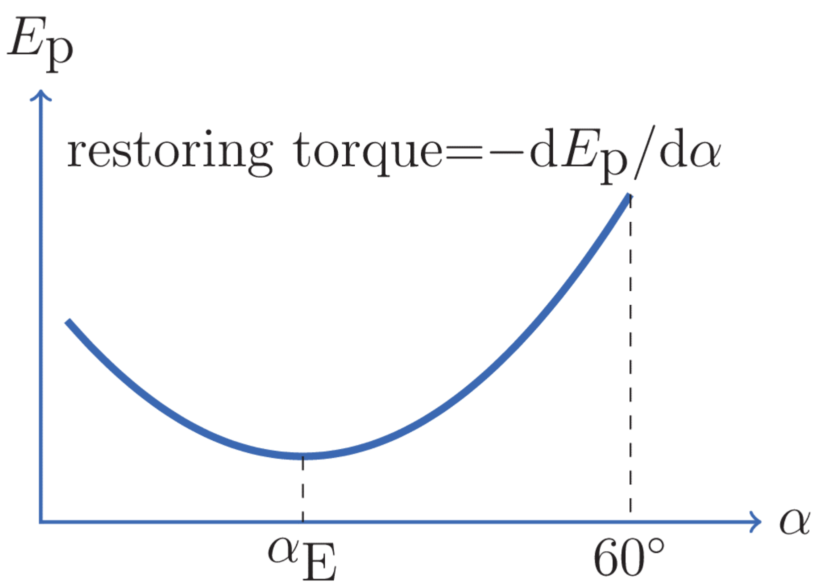
is the y coordinate of center of mass of triangle (0,0), and

$$\begin{aligned} \Delta y &= y_{\mathrm{A}(0,1)} - y_{\mathrm{A}(0,0)} \\ &= -l\left[\sin\left(\frac{\pi}{3} + \alpha\right) + \sin\left(\frac{\pi}{3} - \alpha\right)\right] \\ &= -\sqrt{3}l \cos\alpha \quad \textbf{(0.5 points)} \qquad \text{- Eq. (3)} \end{aligned}$$

is the translational difference of two neighbouring triangles in y-direction. Solving Eqs. (1), (2) and (3), we obtain

$$E_\mathrm{p}(\alpha) = -\frac{2}{3} Mgl\left(4\sqrt{3}\cos\alpha + 3\sin\alpha\right) \quad \textbf{(0.5 points)} \qquad \text{- Eq. (4)}$$

Part	Model Answer	Marks
A2		**1**



At equilibrium, the potential energy reaches a minimum, which gives:

$$\left.\frac{dE_\mathrm{p}(\alpha)}{d\alpha}\right|_{\alpha=\alpha_\mathrm{E}} = 0 \quad \textbf{(0.5 points)} \qquad \text{- Eq. (5)}$$

$$\sqrt{3}\sin\alpha_\mathrm{E} + 3\cos\alpha_\mathrm{E} = 0 \qquad \text{- Eq. (6)}$$

	or $$\alpha_{\mathrm{E}} = \tan^{-1}\frac{\sqrt{3}}{4}$$ **(0.5 point)** - Eq. (7)	
A3	If the total energy of the oscillation has the following form $$E(\Delta\alpha, \Delta\dot{\alpha}) = E_{\mathrm{p}} + E_{\mathrm{k}} = \frac{1}{2}K(\Delta\alpha)^2 + \frac{1}{2}I(\Delta\dot{\alpha})^2 ,$$ **(0.5 points)** - Eq. (8) where E_{p} and E_{k} are the potential and kinetic energies of the system respectively, then the motion is a simple harmonic oscillation with angular frequency $\omega = \sqrt{K/I}$. Here $\Delta\alpha = \alpha - \alpha_{\mathrm{E}}$. Under a small perturbation, the potential energy change is: $$\Delta E_{\mathrm{p}} \approx \frac{1}{2}\frac{d^2E_{\mathrm{p}}}{\mathrm{d}\alpha^2}\bigg\rvert_{\alpha=\alpha_{\mathrm{E}}}(\Delta\alpha)^2$$ $$= \left(\frac{1}{2}\right)\left(\frac{2}{3}Mgl\right)(4\sqrt{3}\cos\alpha_{\mathrm{E}} + 3\sin\alpha_{\mathrm{E}})(\Delta\alpha)^2$$ $$= \frac{\sqrt{57}}{3}Mgl(\Delta\alpha)^2$$ **(1 point)** - Eq. (9) The total kinetic energy of the system includes the translational kinetic energy of every plate and the rotational kinetic energy of every plate relative to its center of mass $$E_{\mathrm{k}} = \sum E_{\mathrm{k}}^{\mathrm{trans}} + \sum E_{\mathrm{k}}^{\mathrm{rot}}$$ - Eq. (10) The rotational kinetic energy is $$\sum E_{\mathrm{k}}^{\mathrm{rot}} = 4 \times \frac{1}{2}\frac{Ml^2}{12}(\Delta\dot{\alpha})^2 = \frac{1}{6}Ml^2(\Delta\dot{\alpha})^2$$ **(0.5 points)** - Eq. (11) $E_{\mathrm{k}}^{\mathrm{trans}}$ can be obtained by considering the motion of the center of mass of each triangle and setting $N = 2$. $$x_{\mathrm{c.m.}(m,n)} = m(2l\cos\alpha) + n(2l\cos\alpha)\cos\frac{\pi}{3} + \frac{l}{\sqrt{3}}\cos\left(\alpha + \frac{\pi}{6}\right),$$ $$y_{\mathrm{c.m.}(m,n)} = -n(2l\cos\alpha)\sin\frac{\pi}{3} - \frac{l}{\sqrt{3}}\sin\left(\alpha + \frac{\pi}{6}\right).$$ **(0.5 point)** Differentiating and substituting $$\sin\alpha = \frac{\sqrt{3}}{\sqrt{19}}, \cos\alpha = \frac{4}{\sqrt{19}}, \sin\left(\alpha + \frac{\pi}{6}\right) = \frac{7}{2\sqrt{19}}, \cos\left(\alpha + \frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2\sqrt{19}},$$	**5**

$$\dot{x}_{\text{c.m.}(m,n)}=-\left(2m+n+\frac{7}{6}\right)\frac{3}{\sqrt{57}}l\Delta\dot{\alpha},\qquad \dot{y}_{\text{c.m.}(m,n)}=\frac{3(2n-1)}{2\sqrt{19}}l\Delta\dot{\alpha}.$$

$$v^2_{\text{c.m.}(m,n)}=\dot{x}^2_{\text{c.m.}(m,n)}+\dot{y}^2_{\text{c.m.}(m,n)}=\frac{(12m+6n+7)^2+27}{228}l^2(\Delta\dot{\alpha})^2,$$ **(1 point)**

$$E^{\text{trans}}_{\text{c.m.,k}}=\frac{M}{2}\left[v^2_{\text{c.m.}(0,0)}+v^2_{\text{c.m.}(0,1)}+v^2_{\text{c.m.}(1,0)}+v^2_{\text{c.m.}(1,1)}\right]=\frac{164}{57}Ml^2(\Delta\dot{\alpha})^2.$$

$$E^{\text{trans}}_{\text{k}}=E^{\text{trans}}_{\text{c.m.,k}}+E^{\text{rot}}_{\text{k}}=\frac{347}{114}Ml^2(\Delta\dot{\alpha})^2.$$ **(1 point)**

Alternatively, another way to get E_k^{trans} is based on the center of mass of the whole system:

$$E_{\text{k}}=\sum E^{\text{trans}}_{\text{c.m.,k}}+\sum E^{\text{rot}}_{\text{r.c.,k}}$$ **(0.5 points)** - Eq. (12)

where

$$E^{\text{trans}}_{\text{r.c.,k}}=\frac{M}{2}\left[v^2_{\text{r.c.}(0,0)}+v^2_{\text{r.c.}(1,0)}+v^2_{\text{r.c.}(0,1)}+v^2_{\text{r.c.}(1,1)}\right]$$ - Eq. (13)

is the translational kinetic energy <u>relative</u> to the center of mass of the system and

$$E^{\text{trans}}_{\text{c.m.,k}}=\frac{4M}{2}v^2_{\text{c.m.}}$$ - Eq. (14)

is the translational kinetic energy of the center of mass of the system.

The center of mass of each of the 2×2 = 4 triangles always form diamond shape with lateral length $2l\cos\alpha$. The center of mass of the whole system is at the center of the diamond shape. Hence

$$v_{\text{r.c.}(0,0)}=v_{\text{r.c.}(1,1)}=\left.\frac{d(\sqrt{3}l\cos\alpha)}{d\alpha}\right|_{\alpha=\alpha_{\text{E}}}\Delta\dot{\alpha}$$

$$v_{\text{r.c.}(1,0)}=v_{\text{r.c.}(0,1)}=\left.\frac{d(l\cos\alpha)}{d\alpha}\right|_{\alpha=\alpha_{\text{E}}}\Delta\dot{\alpha}$$ - Eq. (15)

Substituting Eqs. (14) and (15) into Eq. (13), we obtain

$$E^{\text{trans}}_{\text{r.c.,k}}=4\sin\alpha^2_{\text{E}}Ml^2(\Delta\dot{\alpha})^2$$ - Eq. (16)

For $E^{\text{trans}}_{\text{c.m.,k}}$,

$$v_{\text{c.m.}} = \sqrt{\left(\frac{\mathrm{d}x_{\text{c.m.}}}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}y_{\text{c.m.}}}{\mathrm{d}\alpha}\right)^2}\Bigg|_{\alpha=\alpha_{\mathrm{E}}} \Delta\dot{\alpha} \qquad \text{- Eq. (17)}$$

is the velocity of the center-of-mass of the four triangular plates, with

$$x_{\text{c.m.}} = x_{\text{c.m.}(0,0)} + \frac{1}{2}\left(x_{\mathrm{B}(0,0)} + x_{\mathrm{A}(1,0)}\right)$$

$$= \frac{\sqrt{3}l}{3}\cos\left(\frac{\pi}{6} + \alpha\right) + \frac{3}{2} l \cos\alpha \qquad \text{- Eq. (18)}$$

$$y_{\text{c.m.}} = y_{\text{c.m.}(0,0)} + \frac{1}{2}\Delta y$$

$$= -\frac{\sqrt{3}l}{3}\sin\left(\frac{\pi}{6} + \alpha\right) - \frac{\sqrt{3}}{2} l \cos\alpha \qquad \text{- Eq. (19)}$$

Substituting Eqs. (17), (18) and (19) and into Eq. (14), we obtain

$$E_{\text{c.m.,k}}^{\text{trans}} = \left(\frac{2}{3} + 10\sin^2\alpha_E\right) Ml^2 (\Delta\dot{\alpha})^2 \quad \textbf{(0.5 points)} \qquad \text{- Eq. (20)}$$

Combining Eqs. (12), (16) and (20), we obtain

$$E_{\mathrm{k}} = E_{\mathrm{k}}^{\text{rot}} + E_{\text{r.c.,k}}^{\text{trans}} + E_{\text{c.m.,k}}^{\text{trans}}$$

$$= \left(\frac{5}{6} + 14\sin^2\alpha_E\right) Ml^2 (\Delta\dot{\alpha})^2$$

$$= \frac{347}{114} Ml^2 (\Delta\dot{\alpha})^2 \quad \textbf{(1.5 points)} \qquad \text{- Eq. (21)}$$

According to Eqs. (8), (9) and (21),

$$f = \frac{1}{2\pi}\sqrt{\frac{\frac{\sqrt{57}}{3}Mgl}{\frac{347}{114}Ml^2}} = \frac{1}{2\pi}\sqrt{\frac{38\sqrt{57}}{347}\frac{g}{l}} \quad \textbf{(0.5 points)} \qquad \text{- Eq. (22)}$$

[Note 1: 0.5 point should be deducted if there are numerical mistakes, but all steps are correct.

Note 2: A rough estimate of $f \sim \sqrt{\frac{g}{l}}$ can get 0.5 points out of 5 points.]

B1	For arbitrary N, the total potential energy $$E_{\mathrm{p}} = \sum_{m,n=0}^{N-1} E_{\mathrm{p}}(m,n) \quad \text{- Eq. (23)}$$ where $$E_{\mathrm{p}}(m,n) = \frac{1}{3}Mg\left[y_{\mathrm{A}(m,n)} + y_{\mathrm{B}(m,n)} + y_{\mathrm{C}(m,n)}\right] \quad \text{- Eq. (24)}$$ **(0.5 points for Eqs. (23) and (24))** and $$y_{\mathrm{A}(m,n)} = -nl\sin\left(\frac{\pi}{3}-\alpha\right) - nl\sin\left(\frac{\pi}{3}+\alpha\right) = -\sqrt{3}nl\cos\alpha$$ $$y_{\mathrm{B}(m,n)} = y_{\mathrm{A}(m,n)} - l\sin\alpha = -\sqrt{3}nl\cos\alpha - l\sin\alpha$$ $$y_{\mathrm{C}(m,n)} = y_{\mathrm{A}(m,n)} - l\sin\left(\frac{\pi}{3}+\alpha\right) = -\sqrt{3}nl\cos\alpha - l\sin\left(\frac{\pi}{3}+\alpha\right) \quad \text{- Eq. (25)}$$ **(0.5 points for all three correct coordinates)** Thus, $$E_{\mathrm{p}}(m,n) = -\frac{1}{3}Mgl\left[3\sqrt{3}n\cos\alpha + \sin\alpha + \sin\left(\frac{\pi}{3}+\alpha\right)\right] \quad \text{- Eq. (26)}$$ and $$E_{\mathrm{p}} = \sum_{m,n=0}^{N-1} E_{\mathrm{p}}(m,n)$$ $$= -\frac{1}{3}Mgl\sum_{m,n=0}^{N-1}\left[3\sqrt{3}n\cos\alpha + \sin\alpha + \sin\left(\frac{\pi}{3}+\alpha\right)\right] \quad \text{(0.5 points)} \quad \text{- Eq. (27)}$$ Using the mathematical relations $$\sum_{m=0}^{N-1} 1 = \sum_{n=0}^{N-1} 1 = N$$ and $$\sum_{m=0}^{N-1} m = \sum_{n=0}^{N-1} n = \frac{N(N-1)}{2} \quad \text{- Eq. (28),}$$ Eq. (27) becomes	**3**

$$E_{\mathrm{p}} = -\frac{1}{3}N^2Mgl\left[\frac{3\sqrt{3}(N-1)\cos\alpha}{2} + \sin\alpha + \sin\left(\frac{\pi}{3}+\alpha\right)\right]$$

or $\quad = -\frac{1}{3}N^2Mgl\left[\frac{\sqrt{3}(3N-2)\cos\alpha}{2} + \frac{3}{2}\sin\alpha\right]$ **(1 points)** - Eq. (29)

At equilibrium, $\frac{\mathrm{d}E_{\mathrm{p}}}{\mathrm{d}\alpha} = 0$, therefore

$$-\frac{3\sqrt{3}(N-1)\sin\alpha'_{\mathrm{E}}}{2} + \cos\alpha'_{\mathrm{E}} + \cos\left(\frac{\pi}{3}+\alpha'_{\mathrm{E}}\right) = 0 \qquad \text{- Eq. (30)}$$

$$\alpha'_{\mathrm{E}} = \tan^{-1}\left(\frac{\sqrt{3}}{3N-2}\right) \qquad \text{- Eq. (31)}$$

(0.5 points)

[Remark: Increasing α lowers each triangle relative to its vertex A, but globally raises the system, i.e. the bottom tube is raised higher. When $N \to \infty$, the global displacement dominates, consequently $\alpha \to 0$.]

B2 (3)

Under a small perturbation, the potential energy change, according to Eq. (29) is

$$\Delta E_{\mathrm{p}} \approx \frac{1}{2}\frac{d^2E_{\mathrm{p}}}{\mathrm{d}\alpha^2}\bigg|_{\alpha=\alpha'_{\mathrm{E}}}(\Delta\alpha)^2 \sim N^3 \quad \text{or} \quad \gamma_1 = 3 \qquad \text{- Eq. (32)}$$

(0.5 points)

[Remark: There are N^2 triangles and the y coordinate of the total center of mass is proportional to N, hence $E_{\mathrm{p}} \sim N^3$ and $\gamma_1 = 3$. Using this argument to derive the correct γ_1 can also get 0.5 points.]

The kinetic energy of a triangle includes the translational energy of its center of mass and the rotational energy about its center of mass. Hence the total kinetic energy of the N^2 triangles is

$$E_{\mathrm{k}} = \sum_{m,n} E_{\mathrm{c.m.}(m,n)} + \sum_{m,n} E_{\mathrm{r.c.}(m,n)} \qquad \text{- Eq. (33)}$$

where

$$E_{\mathrm{r.c.}(m,n)} = \frac{1}{2}\frac{Ml^2}{12}(\Delta\dot{\alpha})^2 = \frac{1}{24}Ml^2(\Delta\dot{\alpha})^2 \sim 1 \qquad \text{- Eq. (34)}$$

and

$$E_{\mathrm{c.m.}(m,n)} = \frac{M}{2}v^2_{\mathrm{c.m.}(m,n)}$$

$$= \frac{M(\Delta\dot{\alpha})^2}{2}\left[\left(\frac{\mathrm{d}x_{\mathrm{c.m.}(m,n)}}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}y_{\mathrm{c.m.}(m,n)}}{\mathrm{d}\alpha}\right)^2\right]_{\alpha=\alpha'_{\mathrm{E}}} \qquad \text{- Eq. (35)}$$

(0.5 points)

Since

$$x_{\text{c.m.}(m,n)} = x_{\text{A}(m,n)} + \frac{\sqrt{3}l}{3}\cos\left(\frac{\pi}{6}+\alpha\right)$$

$$= (2m+n)l\cos\alpha + \frac{l}{2}\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha$$

and

$$y_{\text{c.m.}(m,n)} = y_{\text{A}(m,n)} + \frac{\sqrt{3}l}{3}\sin\left(\frac{\pi}{6}+\alpha\right)$$

$$= \sqrt{3}nl\cos\alpha + \frac{\sqrt{3}l}{6}\cos\alpha + \frac{l}{2}\sin\alpha \qquad \text{- Eq. (36)}$$

(0.5 points for correct *x* and *y*)

$$\frac{\mathrm{d}x_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-(2m+n)\sin\alpha - \frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{6}\cos\alpha\right]l$$

$$\frac{\mathrm{d}y_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-\sqrt{3}n\sin\alpha - \frac{\sqrt{3}}{6}\sin\alpha + \frac{1}{2}\cos\alpha\right]l$$

we have

$$E_{\text{c.m.}(m,n)} = \frac{1}{2}Ml^2(\Delta\dot{\alpha})^2\begin{bmatrix}(4m^2+4n^2+4mn+2m+2n)\sin^2\alpha_{\text{E}}' \\ +\frac{2\sqrt{3}}{3}(m-n)\sin\alpha_{\text{E}}'\cos\alpha_E' + \frac{1}{3}\end{bmatrix} \qquad \text{- Eq. (37)}$$

Since $\alpha_{\text{E}}' \sim \frac{1}{N}$ in Eq. (31), we have

$$E_{\text{c.m.}(m,n)} = A\cdot N^2\cdot\frac{1}{N^2} + B\cdot N\cdot\frac{1}{N} + C \sim 1 \quad \textbf{(0.5 points)} \qquad \text{- Eq. (38)}$$

According to Eqs. (33), (34) and (38), we have

$$E_{\text{k}} = \sum_{m,n} E_{\text{c.m.}(m,n)} + \sum_{m,n} E_{\text{r.c.}(m,n)} \sim N\times N\times 1 \sim N^2$$

or $\gamma_2 = 2$ **(0.5 points)** - Eq. (39)

[Remarks: $E_k \sim N^2$ because there are N^2 triangles, each contribute $E_{r.c.}(m,n) \sim 1$ (relative-to-center-of-mass kinetic energy) and $E_{c.m.}(m,n) \sim 1$ (center-of-mass kinetic energy).]

Note that $E_{r.c.}(m,n) \sim 1$ is true for arbitrary α while $E_{c.m.}(m,n) \sim 1$ is only true for the special case of $\alpha'_E \to 0$ or $N \to \infty$.

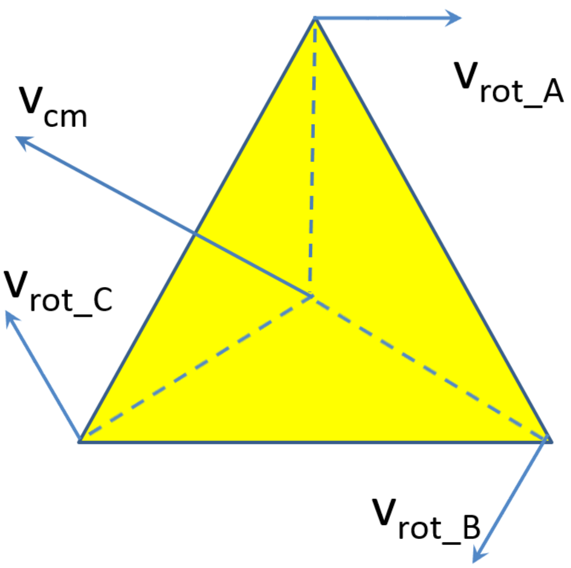
Therefore

$$f'_E \sim \sqrt{\frac{E_p}{E_k}} \sim \sqrt{N}$$

or $\gamma_3 = 0.5$ **(0.5 points)** - Eq. (40)

C1

The minimum force should act on the farthest triangle ($N-1$, $N-1$), whose motion can be decomposed into the motion of the center of mass and the rotation around the center of mass: $\vec{v} = \vec{v}_{c.m.} + \vec{v}_{rot}$. As shown in the figure, $\vec{v}_{rot}$ of vertex C makes the smallest angle relative to the direction of $\vec{v}_{c.m.}$ near $\alpha_m \equiv \pi/3$. Hence its displacement is the largest and its corresponding force is minimum, i.e. the minimum force should act on vertex C($N-1$, $N-1$). **(1 point)**



[Remarks: A rigorous calculation is given in Appendix 3.]

Marks: 1

C2	At $\alpha = \alpha_\mathrm{m} \equiv \pi/3$, a small change in α will change the potential energy by: $$\Delta E_\mathrm{p}(\alpha_\mathrm{m}) = \left.\frac{dE_\mathrm{p}}{d\alpha}\right\rvert_{\alpha=\alpha_\mathrm{m}} \Delta\alpha$$ $$= \frac{1}{3}N^2 Mgl\left[\left(\frac{3\sqrt{3}N}{2} - \sqrt{3}\right)\sin\alpha_\mathrm{m} - \frac{3}{2}\cos\alpha_\mathrm{m}\right]\Delta\alpha$$ $$= \frac{3}{4}(N-1)N^2 Mgl\Delta\alpha$$ **(1 point)** - Eq. (41) The displacement of $C(m,n)$ point is $$\Delta x_{C(m,n)} = -\left[(2m+n)\sin\alpha_\mathrm{m} - \sin\left(\frac{\pi}{3}+\alpha_\mathrm{m}\right)\right] l\Delta\alpha$$ $$= \frac{(2m+n+1)\sqrt{3}}{2} l\Delta\alpha$$ **(0.5 points)** $$\Delta y_{C(m,n)} = -\left[\sqrt{3}n\sin\alpha_\mathrm{m} - \cos\left(\frac{\pi}{3}+\alpha_\mathrm{m}\right)\right] l\Delta\alpha$$ $$= \frac{(3n+1)}{2} l\Delta\alpha$$ **(0.5 points)** For C(N-1,N-1), $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2} = (3N-2)(l\Delta\alpha)$. **(1 point)** Hence $$F_\mathrm{min} = \frac{\Delta E_\mathrm{p}(\alpha_\mathrm{m})}{\Delta r_\mathrm{max}} = \frac{3(N-1)N^2}{4(3N-2)} Mg$$ **(1 point)** - Eq. (42) and $$\theta_{F_\mathrm{min}} = \tan^{-1}\left[\frac{\Delta y_{C(N-1,N-1)}}{\Delta x_{C(N-1,N-1)}}\right] + \pi$$ $$= -\tan^{-1}\frac{\sqrt{3}}{3} + \pi = \frac{5\pi}{6}$$ **(1 point)** - Eq. (43) **[Remarks: This θ_{F_min} is not perpendicular to the $C(N$-1,N-1)–A(0,0) direction because of the constraints of the tunes, e.g. A(1,0), A(2,0), A(3,0), $\cdots$, are also the holding points.]**	**5**

N/A

Appendix 1:

(a) Calculation of the exact E_{p}, E_{k} and f_{E}' in Parts (C), (D) and € for arbitrary N

Under a small perturbation, the potential energy change is

$$\Delta E_{\mathrm{p}} \approx \frac{1}{2}\frac{d^2 E_{\mathrm{p}}}{d\alpha^2}\bigg|_{\alpha=\alpha_{\mathrm{E}}'}(\Delta\alpha)^2$$

$$= \frac{1}{3}N^2 Mgl\left(\frac{3\sqrt{3}N - 2\sqrt{3}}{2}\cos\alpha_{\mathrm{E}}' + \frac{3}{2}\sin\alpha_{\mathrm{E}}'\right)\frac{(\Delta\alpha)^2}{2}$$

$$= \frac{\sqrt{3(3N-2)^2+9}}{12}N^2 Mgl(\Delta\alpha)^2 \qquad \text{- Eq. (44)}$$

The kinetic energy of a triangle includes the translational energy of its center of mass and the rotational energy around its center of mass. Hence the total kinetic energy of the N^2 triangles is

$$E_{\mathrm{k}} = \textstyle\sum_{m,n} E_{\mathrm{c.m.}(m,n)} + \sum_{m,n} E_{\mathrm{r.c.}(m,n)} \qquad \text{- Eq. (45)}$$

where

$$E_{\mathrm{r.c.}(m,n)} = \frac{1}{2}\frac{Ml^2}{12}(\Delta\dot{\alpha})^2 = \frac{1}{24}Ml^2(\Delta\dot{\alpha})^2 \qquad \text{- Eq. (46)}$$

and

$$E_{\mathrm{c.m.}(m,n)} = \frac{M}{2}v_{\mathrm{c.m.}(m,n)}^2$$

$$= \frac{M(\Delta\dot{\alpha})^2}{2}\left[\left(\frac{\mathrm{d}x_{\mathrm{c.m.}(m,n)}}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}y_{\mathrm{c.m.}(m,n)}}{\mathrm{d}\alpha}\right)^2\right]_{\alpha=\alpha_{\mathrm{E}}'} \qquad \text{- Eq. (47)}$$

Since

$$x_{\mathrm{c.m.}(m,n)} = x_{\mathrm{A}(m,n)} + \frac{\sqrt{3}l}{3}\cos\left(\frac{\pi}{6}+\alpha\right)$$

$$= (2m+n)l\cos\alpha + \frac{l}{2}\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha$$

and

$$y_{\mathrm{c.m.}(m,n)} = y_{\mathrm{A}(m,n)} - \frac{\sqrt{3}l}{3}\sin\left(\frac{\pi}{6}+\alpha\right)$$

$$= -\sqrt{3}nl\cos\alpha - \frac{\sqrt{3}l}{6}\cos\alpha - \frac{l}{2}\sin\alpha$$ - Eq. (48)

Hence,

$$\frac{\mathrm{d}x_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-(2m+n)\sin\alpha - \frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{6}\cos\alpha\right]l$$

$$\frac{\mathrm{d}y_{\text{c.m.}(m,n)}}{\mathrm{d}\alpha} = \left[-\sqrt{3}n\sin\alpha + \frac{\sqrt{3}}{6}\sin\alpha - \frac{1}{2}\cos\alpha\right]l$$

We have

$$E_{\text{c.m.}(m,n)} = \frac{1}{2}Ml^2(\Delta\dot{\alpha})^2\begin{bmatrix}(4m^2+4n^2+4mn+2m+2n)\sin^2\alpha'_{\mathrm{E}} \\ +\frac{2\sqrt{3}}{3}(m-n)\sin\alpha'_{\mathrm{E}}\cos\alpha'_{\mathrm{E}}+\frac{1}{3}\end{bmatrix}$$ - Eq. (49)

and

$$E_k = \sum_{m,n} E_{\text{c.m.}(m,n)} + \sum_{m,n} E_{\text{r.c.}(m,n)}$$

$$= \left[\frac{1}{6}(11N-1)(N-1)\sin^2\alpha'_{\mathrm{E}} + \frac{5}{24}\right]N^2Ml^2(\Delta\dot{\alpha})^2$$

$$= \left[\frac{(11N-1)(N-1)}{2(3N-2)^2+6} + \frac{5}{24}\right]N^2Ml^2(\Delta\dot{\alpha})^2$$ - Eq. (50)

With Eqs. (44) and (50), we have

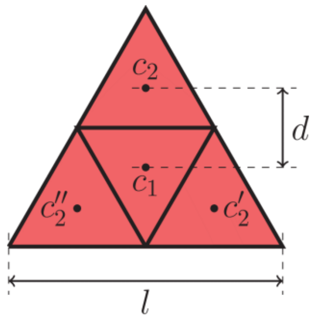
$$f'_{\mathrm{E}} = \frac{1}{2\pi}\sqrt{\frac{\frac{\sqrt{3(3N-2)^2+9}}{12}N^2Mgl}{\left[\frac{(11N-1)(N-1)}{2(3N-2)^2+6}+\frac{5}{24}\right]N^2Ml^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\sqrt{3(3N-2)^2+9}}{\left[\frac{12(11N-1)(N-1)}{(3N-2)^2+3}+5\right]}\frac{g}{l}}$$ - Eq. (51)

(b) <u>Center of mass movement of the whole system</u>

According to Eq. (48), we have

$$x_{\text{c.m.(sys.)}}(\alpha) = \frac{\sum_{m,n} x_{\text{c.m.}(m,n)}}{N^2}$$

	$$= \frac{\sum_{m,n}\left[(2m+n)l\cos\alpha + \frac{l}{2}\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha\right]}{N^2}$$ $$= \left(\frac{3N-2}{2}\right) l\cos\alpha - \frac{\sqrt{3}l}{6}\sin\alpha$$ and $$y_{\text{c.m.}(m,n)}(\alpha) = \frac{\sum_{m,n} y_{\text{c.m.}(m,n)}}{N^2}$$ $$= -\frac{\sum_{m,n}\left[\sqrt{3}nl\cos\alpha + \frac{\sqrt{3}l}{6}\cos\alpha + \frac{l}{2}\sin\alpha\right]}{N^2}$$ $$= -\left(\frac{3N-2}{6}\right)\sqrt{3}l\cos\alpha - \frac{l\sin\alpha}{2} \quad \text{- Eq. (52)}$$ Eq. (52) is the trajectory of the center of mass for the whole system, which is not a straight line.	
	Appendix 2: <u>Calculation of the moment of inertia of a triangular plate</u>  An equilateral triangle with lateral length l can be divided into four small equilateral triangles with lateral length $l/2$. For the central small triangle centered at c_1, its moment of inertia is $$I_1 = \beta\frac{M}{4}\left(\frac{l}{2}\right)^2 \quad \text{- Eq. (53)}$$ For the non-central small triangle centered at c_2, c_2' and c_2'', $$I_2 = I_1 + \frac{M}{4}d^2 \quad \text{- Eq. (54)}$$ where $d = \sqrt{3}l/6$ is the distance between the centers of triangles 1 and 2. The second term is from the parallel-axis theorem. The moment of inertia of the whole triangle is the sum of the moment of inertia of the four sub-triangles:	**N/A**

	$$\beta M l^2 = 4 \times \beta \frac{M}{4}\left(\frac{l}{2}\right)^2 + 3 \times \frac{M}{4} d^2$$ - Eq.(55) Thus $$\beta = \frac{1}{12}$$ - Eq. (56)	

Appendix 3: The minimum force corresponds to the maximum displacement of the exerting point of this force.

N/A

Consider the position of vertices A, B, C of a triangle (*m*,*n*) :

$$x_{A(m,n)} = (2m+n)\cos\alpha_m\, l$$

$$y_{A(m,n)} = -\sqrt{3}n\cos\alpha_m\, l$$

$$x_{B(m,n)} = (2m+n+1)\cos\alpha_m\, l$$

$$y_{B(m,n)} = -(\sqrt{3}n\cos\alpha_m + \sin\alpha_m) l$$

$$x_{C(m,n)} = \left[(2m+n)\cos\alpha_m + \cos\left(\frac{\pi}{3}+\alpha_m\right)\right] l$$

$$y_{C(m,n)} = -\left[\sqrt{3}n\cos\alpha_m + \sin\left(\frac{\pi}{3}+\alpha_m\right)\right] l \qquad \text{- Eq. (57)}$$

Taking derivatives on α on the above coordinates we get

$$\Delta x_{A(m,n)} = -(2m+n)\sin\alpha_m\, l\Delta\alpha = -\frac{(2m+n)\sqrt{3}}{2} l\Delta\alpha$$

$$\Delta y_{A(m,n)} = \sqrt{3}n\sin\alpha_m\,(l\Delta\alpha) = \frac{3n}{2} l\Delta\alpha$$

$$\Delta x_{B(m,n)} = -(2m+n+1)\sin\alpha_m\, l\Delta\alpha = -\frac{(2m+n+1)\sqrt{3}}{2} l\Delta\alpha$$

$$\Delta y_{B(m,n)} = -(-\sqrt{3}n\sin\alpha_m + \cos\alpha_m) l\Delta\alpha = \frac{3n-1}{2} l\Delta\alpha$$

$$\Delta x_{C(m,n)} = \left[-(2m+n)\sin\alpha_m - \sin\left(\frac{\pi}{3}+\alpha_m\right)\right] l\Delta = -\frac{(2m+n+1)\sqrt{3}}{2} l\Delta\alpha$$

$$\Delta y_{C(m,n)} = -\left[-\sqrt{3}n\sin\alpha_m + \cos\left(\frac{\pi}{3}+\alpha_m\right)\right] l\Delta\alpha = \frac{(3n+1)}{2} l\Delta\alpha \qquad \text{- Eq. (58)}$$

For $\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, we have

$$\Delta r_{A(m,n)} = \sqrt{3m^2+3n^2+3mn}(l\Delta\alpha)$$

$$\Delta r_{B(m,n)} = \sqrt{3m^2+3n^2+3mn+3m+1}(l\Delta\alpha)$$

$$\Delta r_{C(m,n)} = \sqrt{3m^2+3n^2+3mn+3m+3n+1}(l\Delta\alpha) \qquad \text{- Eq. (59)}$$

Thus we find

$$\Delta r_{\mathrm{C}(m,n)} > \Delta r_{\mathrm{B}(m,n)} > \Delta r_{\mathrm{A}(m,n)} \qquad \text{- Eq. (60)}$$

Therefore, we should choose point C of the triangle $(N-1, N-1)$ to obtain

$$\Delta r_{\max} = (3N-2)l\Delta\alpha \qquad \text{- Eq. (61)}$$

so that the force is minimal.

(Full Marks: 20)

Part	Model Answer	Marks
A	The physical volume is $$V_p = a^3(t)V.$$ **(0.5 points).** The comoving number density is a constant, thus the physical number density is $$\frac{n(t)}{n(t_0)} = \left(\frac{a(t_0)}{a(t)}\right)^3.$$ **(0.5 points)** The kinetic energy for non-relativistic particles are negligible, thus the energy density is $$\rho_m(t) = m\,n(t),$$ **(0.5 points)** where m is the mass of a particle. Thus $$\rho_m(t) = \rho_m(t_0)\left(\frac{a(t_0)}{a(t)}\right)^3$$ **(0.5 points)** **[Remarks: It is acceptable if the student just writes $\rho_m \propto 1/a^3$ and full points will be given.]**	**2**
B	The Einstein's energy relation for a massless particle is $$E = pc.$$ **(0.5 points)** From de Brogile's relation: $$p \propto 1/\lambda_P \propto 1/a(t).$$ **(0.5 points)** **[Remarks: No point if only $\lambda_p \propto a(t)$ is written because already given.]** Thus $$E \propto 1/a(t).$$ **(0.5 points)** Physical number density is $n \propto 1/a^3$. Energy density is $n\,E$. Thus $$\rho_r(t) = \rho_r(t_0)\left(\frac{a(t_0)}{a(t)}\right)^4$$ **(0.5 points)** **[Remarks: It is acceptable if the student just write $\rho_r \propto 1/a^4$.]**	**2**

C	The photons in thermal equilibrium satisfy Boltzmann distribution $$n(E(a)) \propto e^{-\frac{E(a)}{k_B T(a)}},$$ **(1 point)** where $E \propto 1/a(t)$. Condition of being non-interacting implies that there is no energy transfer. Thus the energy distribution must be stable. To be explicit, for two different comoving wavelengths, $$\frac{n(E_1(a))}{n(E_2(a))} = e^{[E_2(a)-E_1(a)]/[k_B T(a)]} = \text{const.}$$ **[Remarks: All the above steps can be replaced by the intuition of $E \propto T$, if the students realize it, the above 1 point can be given.]** Thus $$T(a) \propto 1/a, \text{ i.e. } \gamma = -1.$$ **(1 point)**	**2**
D	The 1st law of thermodynamics is $$\mathrm{d}E_X = -p_X \mathrm{d}V_p.$$ **(1 point)** Here no entropy term appears, because S =const. No chemical potential appears, because of particle number conservation. Here $V_p = a^3 V$. $$\mathrm{d}V_p = 3a^2 V \mathrm{d}a.$$ **(1 point)** $$E_X = \rho_X V_p.$$ **(0.5 points)** $$\mathrm{d}E_X = \mathrm{d}(\rho_X V_p) = a^3 V \mathrm{d}\rho_X + 3\rho_X a^2 V \mathrm{d}a.$$ **(0.5 points)** Thus $$\mathrm{d}\rho_X + 3\left(\frac{\mathrm{d}a}{a}\right)(\rho_X + p_X) = 0.$$ **(0.5 points)** $$\dot{\rho}_X + 3\left(\frac{\dot{a}}{a}\right)(\rho_X + p_X) = 0.$$ **(0.5 points)** **[Remarks: 0.5 point for relating variation and time derivative no matter in which step it is being used.]**	**4**

E	With lens area A, we only receive part of the starlight. The area ratio is $$A/(4\pi a^2(t_0)r^2).$$ **(1 point)** The wavelength of each photon emitted from the star gets stretched. Thus energy per photon is lowered, contributing a ratio $$a(t_e)/a(t_0).$$ **(1 point)** The separation among the photons also increases due to cosmic expansion, contributing a ratio $$a(t_e)/a(t_0).$$ **(1 point)** As a result, the power that the telescope receives is $$P_r = \frac{A\,a^2(t_e)}{4\pi a^4(t_0)r^2} \times P_e.$$ **(1 point)**	**4**
F	The kinetic energy and gravitational energy of the shell adds up to a constant: $$E = \frac{1}{2}m(\dot{r}_p)^2 - \frac{GMm}{r_p},$$ **(2 points)** where $$M = \frac{4\pi}{3}{r_p}^3\frac{\rho}{c^2},$$ **(1 point)** (Note: energy conservation without evolving pressure requires the assumption of non-relativistic matter.) $$r_p = a(t)r,$$ **(1 point)** **[Remarks: The point is given because the student understand that the shell is not pulled gravitationally from the outside, because the force due to the mass outside cancels.]** Thus $$\frac{2E}{mr^2} = \dot{a}^2 - \frac{8\pi G}{3c^2}\rho a^2.$$ **(1 point)** ***<u>Alternative Solution</u>*:** For the gravitational force due to the mass inside: $$m\,\ddot{r}_p = -\frac{G\,M\,m}{r_p^2} = -\frac{4\pi}{3c^2}Gm\rho r_p,$$ **(2 points)** where m is mass of shell.	**5**

	$r_{\mathrm{p}} = a(t)r$, **(1 point)** **[Remarks: The point is given because the student understand that the shell is not pulled gravitationally from the outside, because the force due to the mass outside cancels.]** and $\rho = \rho(t_0)a^3(t_0)/a^3(t)$. Thus $$\ddot{a} = -\frac{4\pi}{3c^2}G\rho(t_0)a^3(t_0)a^{-2}.$$ **(1 point)** Integrate the above equation. One gets $$c = \frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3c^2}\rho(t_0)a^{-1} = \frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3c^2}\rho a^2 ,$$ **(1 point)** where c is an integration constant.	
G	(b) decelerating. This is because gravity is attractive for the matter that we are considering here. As a result, $\mathrm{d}a(t)/\mathrm{d}t$ is a decreasing function of t.	**1**
	Appendix: Notes about the physics behind this set of problems: To reduce students' reading load, we have not mentioned in the problems, that those problems set up the framework of researches in modern cosmology: A theory of gravity (especially Einstein's general relativity) contains two aspects: Gravity tells matter how to move (kinematics of matter motion in a gravitational field); and matter determines the gravitational field (dynamics of the gravitational field). Parts (A)-(E) are about kinematics and part (F) is about dynamics in this sense. The two key equations in cosmology are derived in part (D) (this is known as the continuity equation, containing parts (A) and (B) as special cases) and (F), upon which the whole theory of modern cosmology is built. The equation derived in part (F) is known as the Friedmann equation, which is conventionally written as $\left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = \frac{8\pi G}{3}\rho$. This equation governs the dynamics of cosmic expansion and actually not only applies for non-relativistic matter but also for general matter components (which needs general relativity to derive). The constant k is related to the curvature of 3-dimensional space, which is observed to be vanishingly small. Part (C) indicates that the universe was hotter at earlier ages. The hot universe in local thermal equilibrium determines the whole thermal history of our universe, which answers questions such as where the light elements come from, and when the universe becomes transparent for light. Part (E) defines the luminosity distance, which relates the telescope observations to the cosmic reality.	**N/A**

Part	Model Answer (Full mark = 20)	Marks
A1	The angular momentum should be $\vec{L} = r\vec{e}_r \times \vec{p} = r\vec{e}_r \times \vec{e}_\varphi \int_0^{2\pi} \frac{mv}{2\pi r} r\, d\varphi$ (1 point for the definition of angular momentum) Here $\vec{e}_r$ is the unit vector pointing from the center of the ring to the mass point on the ring and $\vec{e}_\varphi$ is the unit vector parallel to the direction of the linear velocity at the mass point. We know that $v = \omega r$, so finally we can get $\vec{L} = m\omega r^2 \vec{e}_z$, with $\vec{e}_z = \vec{e}_r \times \vec{e}_\varphi$. (1 point for the correct answer: 0.5 points for the magnitude and 0.5 points for the direction)	**2**
A2	For a current loop, the magnetic moment is defined as $\vec{M} = I\vec{A}$ The current can be expressed as $I = -ef = -e\frac{\omega}{2\pi}$ (1 point for the current expression) Finally $\vec{M} = -e\frac{\omega}{2\pi}\pi r^2 \vec{e}_z = -\frac{e\vec{L}}{2m}$ (1 point for the answer)	**2**
A3	For a current loop, under a uniform magnetic field the total torque should be $\vec{\tau} = \vec{M} \times \vec{B}$ (0.5 point for the torque definition) The work done by the magnetic field on the torque should be $W = \int_{\frac{\pi}{2}}^{\theta} \vec{\tau} \cdot \mathrm{d}\vec{\theta}' = \int_{\frac{\pi}{2}}^{\theta} -\lvert\vec{\tau}\rvert \mathrm{d}\theta' = \int_{\theta}^{\frac{\pi}{2}} \lvert\vec{M}\rvert\lvert\vec{B}\rvert \sin\theta' \mathrm{d}\theta' = \vec{M} \cdot \vec{B}$ (1.5 points for the work on the torque)	**2**

	$U = -W$ $= -\vec{M}\cdot\vec{B}$ (0.5 point for the answer) $= \frac{1}{2}e\omega r^2 B_z \cos\theta$	
A4	We assume that the magnetic field is along z direction such that $\vec{B} = B\vec{e}_z$, then in general $U = -\vec{M}\cdot\vec{B} = -M_z B$ The magnetic torque of an electron should be $M_z = \frac{-e}{2m_e} S_z$ (0.5 points for the electron torque) Thus $U = -\vec{M}\cdot\vec{B}$ $= -\frac{-e}{2m_e} S_z B$ $= \frac{\mu_B}{\hbar} S_z B$ (0.5 points for the answer) $= \frac{1}{2}\mu_B B$ Here $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton. $\mu_B = 5.788\times10^{-5}\,\mathrm{eV\cdot T^{-1}}$	1
A5	Thus for spin parallel state $S_z = \frac{1}{2}\hbar$, we have $U = 5.788\times10^{-5}\,\mathrm{eV}$ (0.5 points) For spin anti-parallel state $S_z = -\frac{1}{2}\hbar$, we have $U = -5.788\times10^{-5}\,\mathrm{eV}$ (0.5 points)	1

B1	In the superconductivity state, electrons forming a Cooper pair have opposite spins, thus the external magnetic field cannot have any effect on the cooper pair. Thus the energy of the Cooper pair does not change. $E_S = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2D$ (1 point for the answer)	**1**
B2	In the normal state, the two electrons will align their magnetic moments parallel to the external magnetic field. Therefore we have $E_N = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{2\mu_B S_{1x} B_x}{\hbar} + \frac{2\mu_B S_{2x} B_x}{\hbar}$ Here the potential energy of electrons should be twice as the classical estimation according to quantum mechanics. Because $S_{1x} = S_{2x} = -\frac{1}{2}\hbar$ can make the magnetic moment aligned along *x* direction, eventually we have $E_N = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\mu_B B_x = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{e\hbar}{m_e} B_x$ (1 point)	**1**
B3	$E_N < E_S \Rightarrow 2B_x m_B > 2D \Rightarrow B_x > \frac{D}{m_B}$ Thus $B_P = \frac{\Delta}{\mu_B} = \frac{2m_e\Delta}{e\hbar}$ (1 points) Note: The above simple consideration for the upper critical field B_P over estimates its value. The strict derivation considering the Pauli magnetization and superconductivity condensation energy will give $B_P = \frac{\Delta}{\sqrt{2}\mu_B} = \sqrt{2}\frac{m_e\Delta}{e\hbar}$.	**1**
C1	Method 1:	**3**

Substituting $\psi(x)=\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}\mathrm{e}^{-\lambda x^2}$ into the $F(y)$, we have

$$F(\psi)=\sqrt{\frac{2\lambda}{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda x^2}\left[-\alpha\mathrm{e}^{-\lambda x^2}-\frac{\hbar}{4m_\mathrm{e}}\left(-2\lambda e^{-\lambda x^2}+4\lambda^2x^2\mathrm{e}^{-\lambda x^2}\right)+\frac{e^2B_z^2x^2}{m_\mathrm{e}}\mathrm{e}^{-\lambda x^2}\right]\mathrm{d}x$$

$$=\sqrt{\frac{2\lambda}{\pi}}\int_{-\infty}^{+\infty}\left[\left(-\alpha+\frac{\hbar^2\lambda}{2m_\mathrm{e}}\right)\mathrm{e}^{-2\lambda x^2}+\left(\frac{e^2B_z^2}{m_e}-\frac{\hbar^2\lambda^2}{m_e}\right)x^2\mathrm{e}^{-2\lambda x^2}\right]\mathrm{d}x$$

$$=-\alpha+\frac{\hbar^2\lambda}{2m_\mathrm{e}}+\left(\frac{e^2B_z^2}{m_\mathrm{e}}-\frac{\hbar^2\lambda^2}{m_\mathrm{e}}\right)\cdot\frac{1}{4\lambda}$$

$$=-\alpha+\frac{\hbar^2\lambda}{4m_\mathrm{e}}+\frac{e^2B_z^2}{4\lambda m_\mathrm{e}}$$

(1.5 points for the correct expression of $F(y)$ as a function of /)

We can treat $F(y)$ as a function of / . Thus we have

$$F(\psi)=-\alpha+\frac{\hbar^2\lambda}{4m_\mathrm{e}}+\frac{e^2B_z^2}{4\lambda m_\mathrm{e}},\ \text{and}\ \frac{\mathrm{d}F}{\mathrm{d}\lambda}=\frac{\hbar^2}{4m_\mathrm{e}}-\frac{e^2B_z^2}{4m_\mathrm{e}\lambda^2}.$$

$F(y)$ takes the minimum value when $\frac{\mathrm{d}F}{\mathrm{d}\lambda}=0$ and $\frac{\mathrm{d}^2F}{\mathrm{d}\lambda^2}>0$, thus

$\frac{\hbar^2}{4m_\mathrm{e}}-\frac{e^2B_z^2}{4m_\mathrm{e}\lambda^2}=0$ (0.5 point for the way to minimize $F(y)$)

Finally, we can get

$\lambda=\frac{eB_z}{\hbar}$ (1 point for the correct answer)

We can check that $\frac{d^2F}{d/^2}>0$ when $\lambda=\frac{eB_z}{\hbar}$, which guarantees that $F(y)$ takes the minimum value when $\lambda=\frac{eB_z}{\hbar}$.

Method 2:

	$$F(\psi)=\int_{-\infty}^{+\infty}\psi\left(-\alpha\psi-\frac{\hbar^2}{4m_e}\frac{d^2\psi}{dx^2}+\frac{e^2B_z^2x^2}{m_e}\psi\right)dx$$ $$=\int_{-\infty}^{+\infty}\psi\left(-\alpha-\frac{\hbar^2}{4m_e}\frac{d^2}{dx^2}+\frac{e^2B_z^2x^2}{m_e}\right)\psi dx$$ (1 point) $$=\int_{-\infty}^{+\infty}\psi\tilde{H}\psi dx$$ In this way, for normalized wave function ψ the $F(\psi)$ is simply the energy expectation $\langle\tilde{H}\rangle$, the eigenvalue of the Hamiltonian $$\tilde{H}=-\frac{\hbar^2}{4m_e}\frac{d^2}{dx^2}+\frac{e^2B_z^2}{m_e}x^2-\alpha$$ The first two terms correspond to the quantum simple harmonic oscillator Hamiltonian. Thus the ground state energy should be $$F_{min}=\frac{1}{2}\hbar\omega-\alpha$$ Here $\omega=\frac{eB_z}{m_e}$ and ground state wave function becomes $$\psi=\left(\frac{2m_e\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m_e\omega}{\hbar}x^2}$$ $$=\left(\frac{2eB_z}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{eB_z}{\hbar}x^2}$$ (1 point) Therefore, we have $$\lambda=\frac{eB_z}{\hbar}.$$ (1 point)	
C2	From Part (C1) we know $F_{min}(\psi)=\frac{\hbar eB_z}{2m_e}-\alpha$. At the critical value for B_z, it makes the energy difference zero. It means that the critical value B_z satisfies $$\frac{\hbar eB_z}{2m_e}-\alpha=0.$$ (1 point for this equation) Consequently,	**2**

	$B_z = \frac{2m_e\alpha}{e\hbar}$. (1 point for the correct answer)	
D1	$E_\mathrm{I} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_\mathrm{B}S_{1z}B_z}{\hbar} + \frac{2\mu_\mathrm{B}S_{2z}B_z}{\hbar}$ Here $S_{1z} = \frac{1}{2}\hbar$, $S_{2z} = -\frac{1}{2}\hbar$ $E_\mathrm{I} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_\mathrm{B}S_{1z}B_z}{\hbar} + \frac{2\mu_\mathrm{B}S_{2z}B_z}{\hbar}$ $= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{2\mu_\mathrm{B}B_z}{2} - \frac{2\mu_\mathrm{B}B_z}{2}$ (1 point) $= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - 2\mu_\mathrm{B}B_z$ $= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\Delta - \frac{e\hbar}{m_e}B_z$	1
D2	In the normal state, the electrons will align the magnetic moment parallel to the total magnetic field, thus $E_{\|\|} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{2\mu_\mathrm{B}\vec{S}_1\cdot\vec{B}_1}{\hbar} + \frac{2\mu_\mathrm{B}\vec{S}_2\cdot\vec{B}_2}{\hbar}$ For electron 1, $\vec{B}_1 = (B_x, 0, -B_z)$ For electron 2, $\vec{B}_2 = (B_x, 0, B_z)$ Therefore, $\vec{S}_1 = -\frac{1}{2}\hbar\left(\frac{B_x}{\sqrt{B_x^2+B_z^2}}, 0, \frac{-B_z}{\sqrt{B_x^2+B_z^2}}\right)$ and $\vec{S}_2 = -\frac{1}{2}\hbar\left(\frac{B_x}{\sqrt{B_x^2+B_z^2}}, 0, \frac{B_z}{\sqrt{B_x^2+B_z^2}}\right)$ can make the their magnetic moments parallel to the total magnetic field respectively. (1 point for the correct expression of spins: 0.5 points for each respectively) Finally $E_{\|\|} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - 2\mu_\mathrm{B}\sqrt{B_x^2+B_z^2} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{e\hbar}{m_e}\sqrt{B_x^2+B_z^2}$ (1 point for the answer)	2
D3	$E_{\|\|} < E_\mathrm{Ising} \Rightarrow 2\mu_\mathrm{B}\sqrt{B_x^2+B_z^2} > 2\Delta + 2\mu_\mathrm{B}B_z \Rightarrow B_x > \frac{\sqrt{\Delta^2 + 2\Delta\mu_\mathrm{B}B_z}}{\mu_\mathrm{B}}$ (1 points)	1

	Another correct expression is: $B_I > \dfrac{2m_e\sqrt{\Delta^2 + \dfrac{e\hbar}{m_e}\Delta B_z}}{e\hbar}$.	