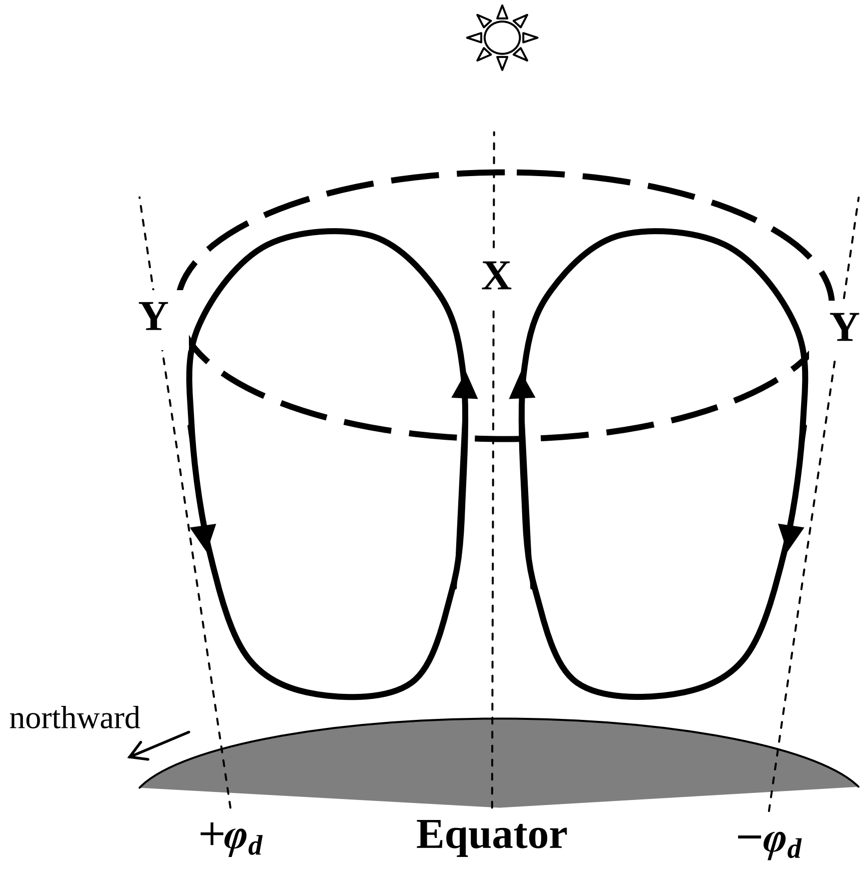
Question 1

The schematic below shows the Hadley circulation in the Earth's tropical atmosphere around the spring equinox. Air rises from the equator and moves poleward in both hemispheres before descending in the subtropics at latitudes $\pm\varphi_d$ (where positive and negative latitudes refer to the northern and southern hemisphere respectively). The angular momentum about the Earth's spin axis is conserved for the upper branches of the circulation (enclosed by the dashed oval). Note that the schematic is not drawn to scale.



(a) **(2 points)** Assume that there is no wind velocity in the east-west direction around the point X. What is the expression for the east-west wind velocity u_Y at the points Y? Convention: positive velocities point from west to east.
(The angular velocity of the Earth about its spin axis is Ω, the radius of the Earth is a, and the thickness of the atmosphere is much smaller than a.)

(b) **(1 point)** Which of the following explains ultimately why angular momentum is not conserved along the lower branches of the Hadley circulation?
Tick the correct answer(s). There can be more than one correct answer.

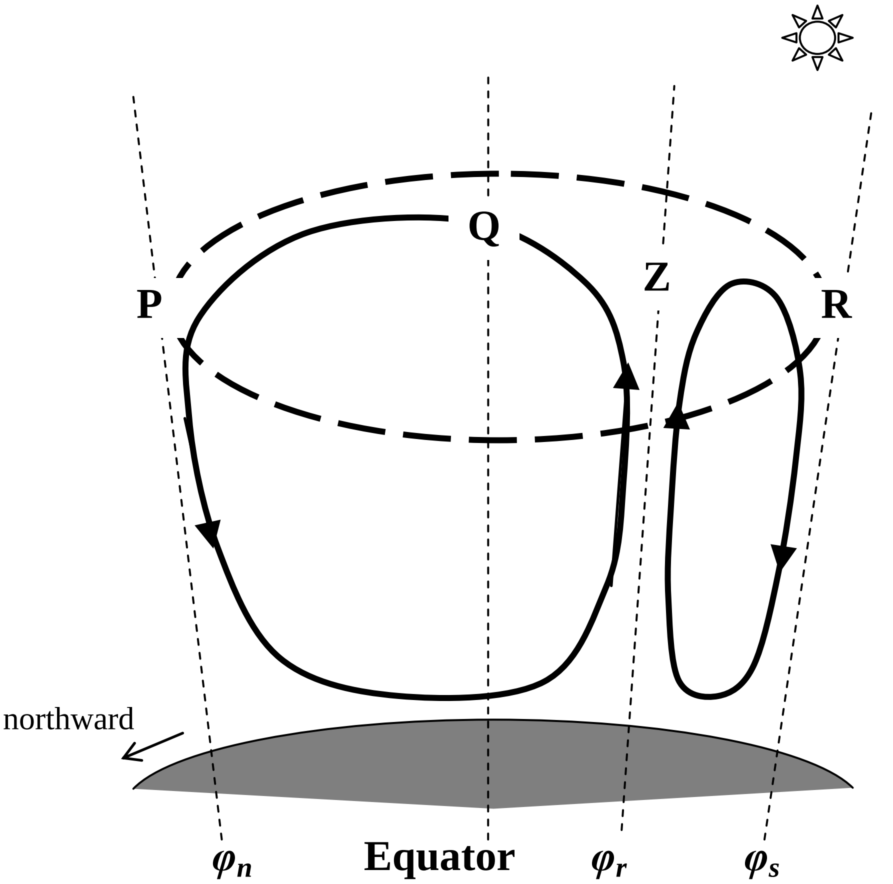
(I) There is friction from the Earth's surface.

(II) There is turbulence in the lower atmosphere, where different layers of air are mixed

(III) The air is denser lower down and so inertia slows down the motion around the spin axis of the Earth.

(IV) The air is moist at the lower levels causing retardation to the wind velocity.

Around the northern winter solstice, the rising branch of the Hadley circulation is located at the latitude φ_r and the descending branches are located at φ_n and φ_s as shown in the schematic below. Refer to this diagram for parts (c), (d) and (e).



(c) **(2 points)** Assume that there is no east-west wind velocity around the point Z. Given that $\varphi_r = -8°$, $\varphi_n = 28°$ and $\varphi_s = -20°$, what are the east-west wind velocities u_P, u_Q and u_R respectively at the points P, Q and R?
(The radius of the Earth is a = 6370 km.)
Hence, which hemisphere below has a stronger atmospheric jet stream?

(I) Winter Hemisphere

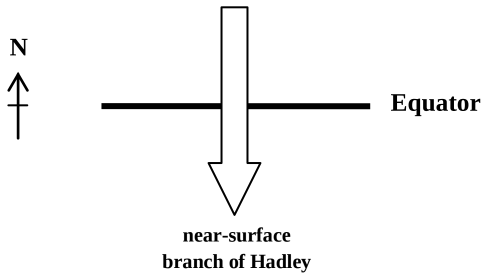
(II) Summer Hemisphere

(III) Both hemispheres have equally strong jet streams.

(d) **(1 point)** The near-surface branch of the Hadley circulation blows southward across the equator. Mark by arrows on the figure below the direction of the east-west component of the Coriolis force acting on the tropical air mass

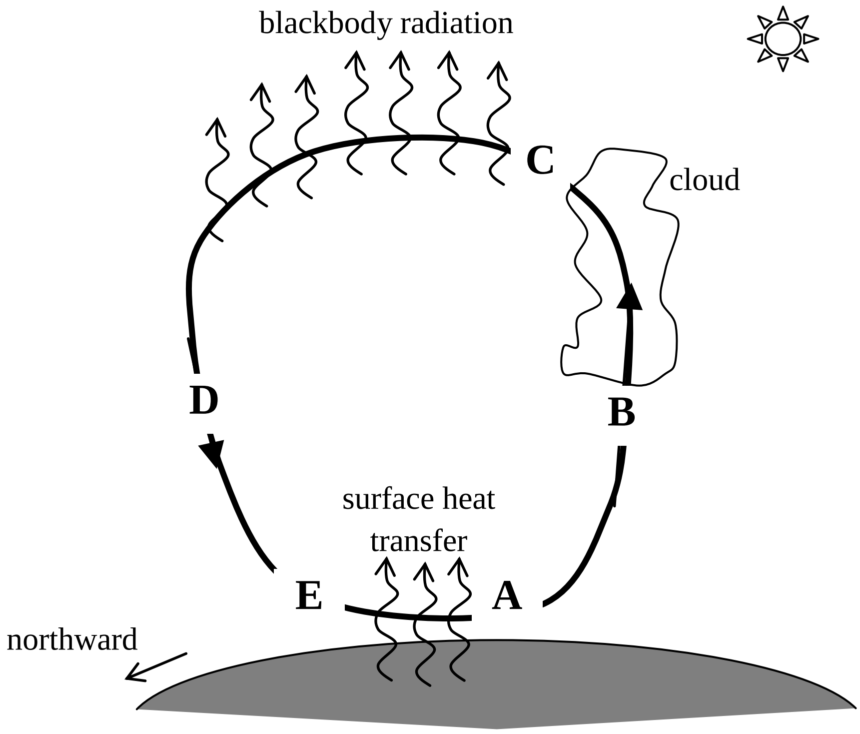
(A) north of the equator;

(B) south of the equator.



(e) **(1 point)** From your answer to part (d) and the fact that surface friction nearly balances the Coriolis forces in the east-west direction, sketch the near-surface wind pattern in the tropics near the equator during northern winter solstice.

Suppose the Hadley circulation can be simplified as a heat engine shown in the schematic below. Focusing on the Hadley circulation reaching into the winter hemisphere as shown below, the physical transformation of the air mass from A to B and from D to E are adiabatic, while that from B to C, C to D and from E to A are isothermal. Air gains heat by contact with the Earth's surface and by condensation of water from the atmosphere, while air loses heat by radiation into space.



(f) **(2 points)** Given that atmospheric pressure at a vertical level owes its origin to the weight of the air above that level, order the pressures p_A, p_B, p_C, p_D, p_E, respectively at the points A, B, C, D, E by a series of inequalities.
(Given that p_A = 1000 hPa and p_D = 225 hPa. Note that 1 hPa is 100 Pa.)

(g) **(2 points)** Let the temperature next to the surface and at the top of the atmosphere be T_H and T_C respectively. Given that the pressure difference between points A and E is 20 hPa, calculate T_C for T_H = 300 K.
Note that the ratio of molar gas constant (R) to molar heat capacity at constant pressure (c_p) for air, κ, is 2/7.

(h) **(2 points)** Calculate the pressure p_B.

(i) For an air mass moving once around the winter Hadley circulation, using the molar gas constant, R, and the quantities defined above, obtain expressions for

(A) **(2 points)** the net work done per unit mole W_{net} ignoring surface friction;

(B) **(1 point)** the heat loss per unit mole Q_{loss} at the top of the atmosphere.

(j) **(1 point)** What is the value of the ideal thermodynamic efficiency ε_i for the winter Hadley circulation?

(k) **(2 points)** Prove that the actual thermodynamic efficiency ε for the winter Hadley circulation is always smaller than ε_i, showing all mathematical steps.

(l) **(1 point)** Which of the following statements best explains why ε is less than the ideal value? Tick the correct answer(s). There can be more than one correct answer.

(I) We have ignored work done against surface friction.

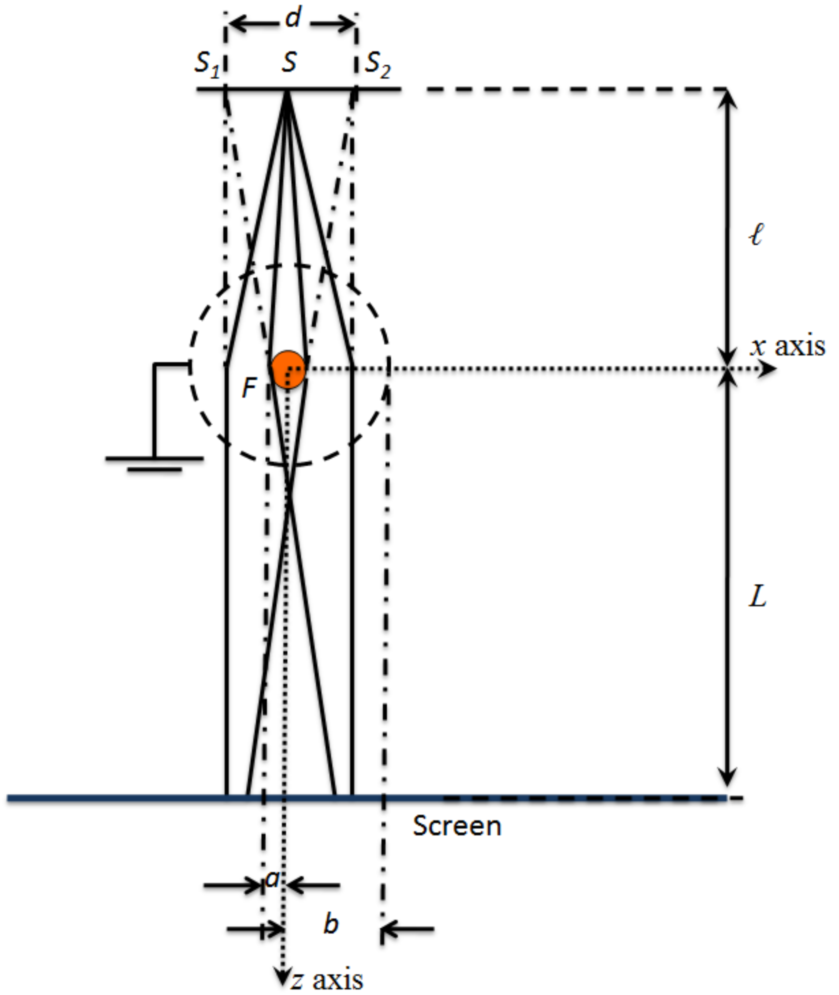
(II) Condensation occurs at a temperature lower than the temperature of the heat source.

(III) There is irreversible evaporation of water at the surface.

(IV) The ideal efficiency is applicable only when there is no phase change of water.

Question 2

The two-slit electron interference experiment was first performed by Möllenstedt *et al*, Merli-Missiroli and Pozzi in 1974 and Tonomura *et al* in 1989. In the two-slit electron interference experiment, a monochromatic electron point source emits particles at S that first passes through an electron "biprism" before impinging on an observational plane; S_1 and S_2 are virtual sources at distance d. In the diagram, the filament is pointing into the page. Note that it is a very thin filament (not drawn to scale in the diagram).



The electron "biprism" consists of a grounded cylindrical wire mesh with a fine filament F at the center. The distance between the source and the "biprism" is ℓ, and the distance between the distance between the "biprism" and the screen is L.

(a) (**2 points**) Taking the center of the circular cross section of the filament as the origin *O*, find the electric potential at any point (x,z) very near the filament in terms of V_a, a and b where V_a is the electric potential of the surface of the filament, a is the radius of the filament and b is the distance between the center of the filament and the cylindrical wire mesh. (Ignore mirror charges.)

(b) (**4 points**) An incoming electron plane wave with wave vector k_z is deflected by the "biprism" due to the x-component of the force exerted on the electron. Determine k_x the x-component of the wave vector due to the "biprism" in terms of the electron charge, e, v_z, V_a, k_z, a and b, where e and v_z are the charge and the z-component of the velocity of the electrons ($k_x \ll k_z$). Note that $\vec{k} = \frac{2\pi\vec{p}}{h}$ where h is the Planck constant.

(c) Before the point *S*, electrons are emitted from a field emission tip and accelerated through a potential V_0. Determine the wavelength of the electron in terms of the (rest) mass m, charge e and V_0,

(i) (**2 points**) assuming relativistic effects can be ignored, and
(ii) (**3 points**) taking relativistic effects into consideration.

(d) In Tonomura *et al* experiment,

v_z = $c/2$,
V_a = 10 V,
V_0 = 50 kV,
a = 0.5 μm,
b = 5 mm,
ℓ = 25 cm,
L = 1.5 m,
h = 6.6×10^{-34} Js,
electron charge, $e = 1.6 \times 10^{-19}$ C,
mass of electron, $m_0 = 9.1 \times 10^{-31}$ kg,
and the speed of light *in vacuo*, $c = 3 \times 10^{8}\ \mathrm{ms^{-1}}$

(i) (**2 points**) calculate the value of k_x,
(ii) (**2 points**) determine the fringe separation of the interference pattern on the screen,
(iii) (**1 point**) If the electron wave is a spherical wave instead of a plane wave, is the fringe spacing larger, the same or smaller than the fringe spacing calculated in (ii)?
(iv) (**2 points**) In part (c), determine the percentage error in the wavelength of the electron using non-relativistic approximation.
(v) (**2 points**) the distance d between the apparent double slits.

Question 3

Gravitational lensing is a phenomenon where light from a distant source may be deflected by the curvature of space-time caused by a massive lensing object close to or in the line of sight between an observer and a distant object. This was first directly observed during the solar eclipse of 1919 where the observed positions of stars behind the sun differed from their astrometric positions following Einstein's earlier predictions.

In the case where the observer, lensing object of mass M and source are on a straight line, light from the source is deflected by an angle α (in radians) given by

$$\alpha = \frac{4GM}{r_E c^2}$$

where G is the gravitational constant $(6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}})$, c is the speed of light $(3.00 \times 10^{8}\ \mathrm{m\,s^{-1}})$, and r_E is the Einstein radius which is the least distance between the lensing object and the apparent light path.

(a) **(4 points)** Draw a diagram to describe the physical layout of an ideal (observer, lens and point source in a straight line) lensing system. Draw the light path and mark the quantities α and r_E. Also mark the angular Einstein radius θ_E (the angular deflection of the source image as seen from earth), and the other quantities that an observer on earth can measure.

(b) **(2 points)** Sketch the image of the source (such as a star), as seen by an observer on earth, in the case where the source, lensing object and observer are on a straight line.

(c) **(3 points)** Sketch the image of the source (such as a star), as seen by an observer on earth, in the non-ideal case where the source, lensing object and observer are not in a straight line. Sketch the source-lens system to explain why this is so.

Gravitational lensing has been proposed as a method to detect massive compact halo objects (MACHOs) in our galaxy, which may be a candidate for dark matter. These objects are often dark stellar remnants such as neutron stars and black holes. As stars and MACHOs orbit in the galaxy, there is a chance that a lensing event may occur when a black hole or neutron star passes in front of a background star.

(d) **(3 points)** The Schwarzschild radius of a black hole defines the point of no return. A correct expression for the Schwarzschild radius can be obtained by taking it to be the radius where the escape speed is equal to the speed of light. This means that something inside the Schwarzschild radius cannot escape the black hole.
Using Newtonian mechanics, derive the formula for the escape speed at a distance r away from a point object of mass M. Hence, derive the Schwarzschild radius for a point object of mass M in terms of the gravitational constant G and the speed of light c. Show your steps and reasoning clearly. (This happens to give the correct expression for the Schwarzschild radius that comes from general relativity.)

(e) **(1 point)** In the case where the source, lens and observer are in a straight line, given a measurement of α and r_E, how would you calculate the Schwarzschild radius of the lensing object?

(f) **(2 points)** Consider the case where we have a lensing object of the order of a few solar masses ($M \sim$ a few $\times\ 10^{30}$ kg) in the nearby regions of the galaxy (distance $D_L \sim$ a few $\times\ 10^{18}$ m away) and a source object somewhat further out ($D_S \sim$ a few $\times\ D_L$). Which of the following apply in this case?
Choose the following conditions that apply to the case as described in the question:

- α is large and $\tan\alpha$, $\sin\alpha$, $\cos\alpha$ must be calculated exactly
- α is small and the small angle approximations to $\tan\alpha$, $\sin\alpha$, $\cos\alpha$ are permissible
- α is irrelevant and need not be calculated
- θ_E is large and $\tan\theta_E$, $\sin\theta_E$, $\cos\theta_E$ must be calculated exactly
- θ_E is small and the small angle approximations to $\tan\theta_E$, $\sin\theta_E$, $\cos\theta_E$ are permissible
- θ_E is irrelevant and need not be calculated

(g) **(3 points)** Using the conditions in part (f), rewrite your expression in part (e) in terms of measurable quantities (which are θ_E, D_S and D_L) for a lensing object of the order of a few solar masses ($M \sim$ a few $\times\ 10^{30}$ kg) and in the nearby regions of the galaxy (distance $D_L \sim$ a few $\times\ 10^{18}$ m away) with a source object somewhat further out ($D_S \sim$ a few $\times\ D_L$). Show your working.

(h) **(2 points)** Suppose we have an event where a lensing object of 6.0×10^{30} kg (3.0 solar masses), 2.6×10^{18} m away from earth passes in front of a star 9.2×10^{18} m away from earth. This happens such that the ideal configuration occurs during the event. What is the angular Einstein radius θ_E (as seen from earth) during this event when the source, lens and observer line up?