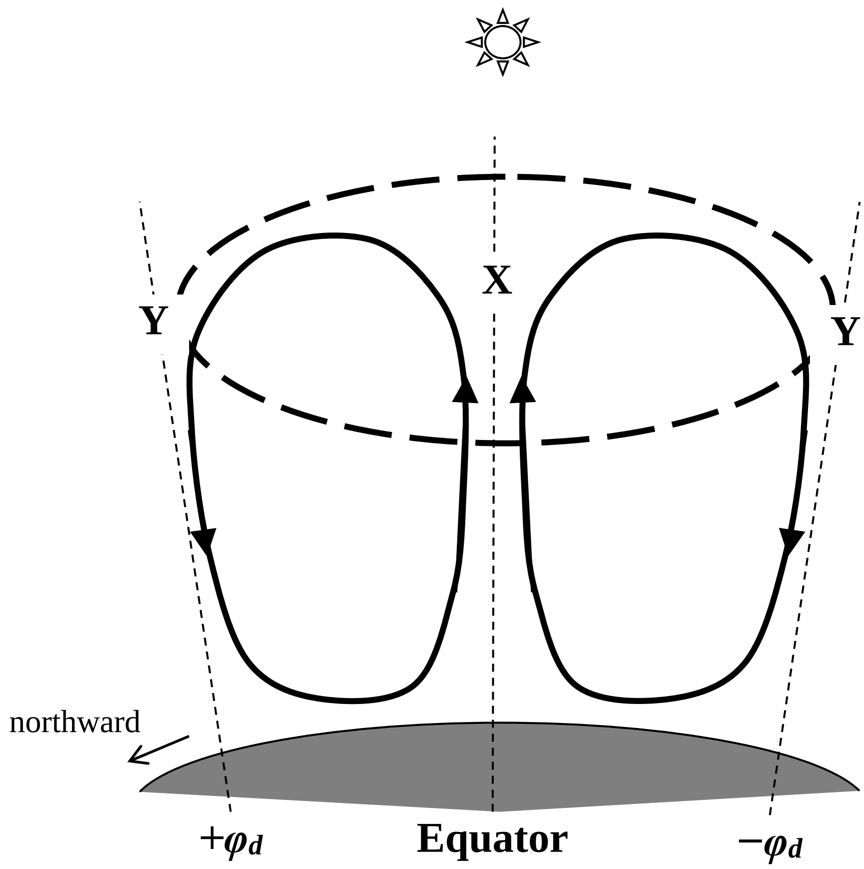
Question 1

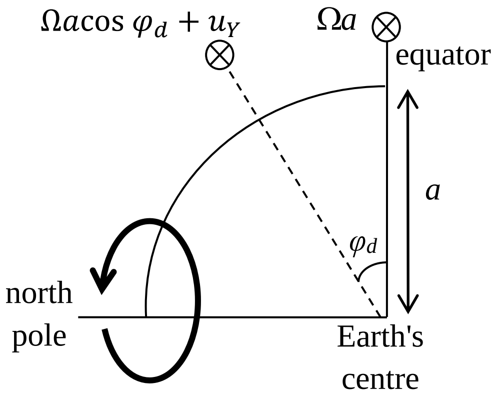
The schematic below shows the Hadley circulation in the Earth's tropical atmosphere around the spring equinox. Air rises from the equator and moves poleward in both hemispheres before descending in the subtropics at latitudes $\pm\varphi_d$ (where positive and negative latitudes refer to the northern and southern hemisphere respectively). The angular momentum about the Earth's spin axis is conserved for the upper branches of the circulation (enclosed by the dashed oval). Note that the schematic is not drawn to scale.



(a) **(2 points)** Assume that there is no wind velocity in the east-west direction around the point X. What is the expression for the east-west wind velocity u_Y at the points Y? Convention: positive velocities point from west to east.
(The angular velocity of the Earth about its spin axis is Ω, the radius of the Earth is a, and the thickness of the atmosphere is much smaller than a.)

Solution:

As the problem is symmetric about the equator, we need only consider the northern hemisphere as shown below.



Conservation of angular momentum about the Earth's spin axis implies that:

$$\Omega a^2 = (\Omega a\cos\varphi_d + u_Y)a\cos\varphi_d \qquad \textbf{(1.5 point)}$$

$$u_Y = \Omega a\left(\frac{1}{\cos\varphi_d} - \cos\varphi_d\right) \qquad \textbf{(0.5 point)}$$

(b) **(1 point)** Which of the following explains ultimately why angular momentum is not conserved along the lower branches of the Hadley circulation?
Tick the correct answer(s). There can be more than one correct answer.

(I) There is friction from the Earth's surface.

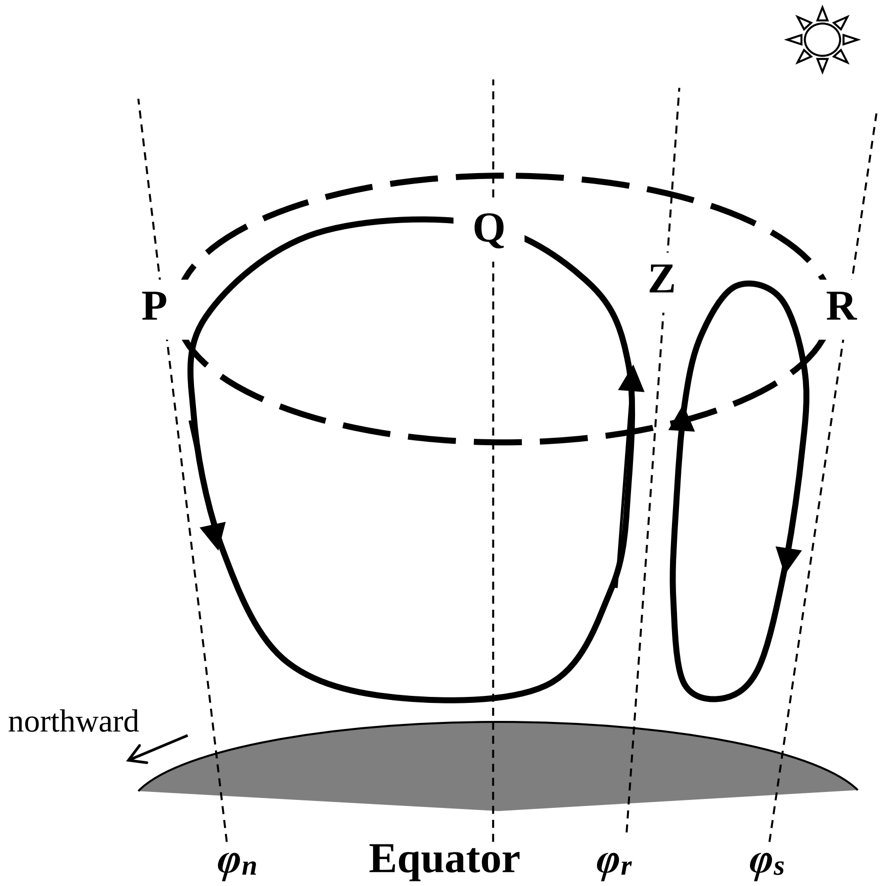
(II) There is turbulence in the lower atmosphere, where different layers of air are mixed

(III) The air is denser lower down and so inertia slows down the motion around the spin axis of the Earth.

(IV) The air is moist at the lower levels causing retardation to the wind velocity.

Solution: (I) & (II) **(0.5 point each)**
To discourage guessing, <u>minus 0.5 point</u> for each wrong answer.
The minimum points to be awarded in this part is 0.

Around the northern winter solstice, the rising branch of the Hadley circulation is located at the latitude φ_r and the descending branches are located at φ_n and φ_s as shown in the schematic below. Refer to this diagram for parts (c), (d) and (e).



(c) **(2 points)** Assume that there is no east-west wind velocity around the point Z. Given that $\varphi_r = -8°$, $\varphi_n = 28°$ and $\varphi_s = -20°$, what are the east-west wind velocities u_P, u_Q and u_R respectively at the points P, Q and R?
(The radius of the Earth is a = 6370 km.)
Hence, which hemisphere below has a stronger atmospheric jet stream?
(I) Winter Hemisphere
(II) Summer Hemisphere
(III) Both hemispheres have equally strong jet streams.

Solution:

The angular velocity of the Earth about its spin axis is:

$$\Omega = \frac{2\pi}{24 \times 60 \times 60s} = 7.27 \times 10^{-5}\ s^{-1}$$

so we have:

$$\Omega a = (7.27 \times 10^{-5} s^{-1})(6.37 \times 10^{6} m) = 463\ ms^{-1}$$

Conservation of angular momentum about the Earth's spin axis implies that the wind velocity u at latitude φ is:

$$\Omega a^2 \cos^2 \varphi_r = (\Omega a \cos\varphi + u) a \cos\varphi$$

$$u = \Omega a \left(\frac{\cos^2 \varphi_r}{\cos\varphi} - \cos\varphi\right)$$ **(0.5 point)**

The required east-west wind velocities are:
(0.5 point for each correct answer, but capped at 1 point maximum)

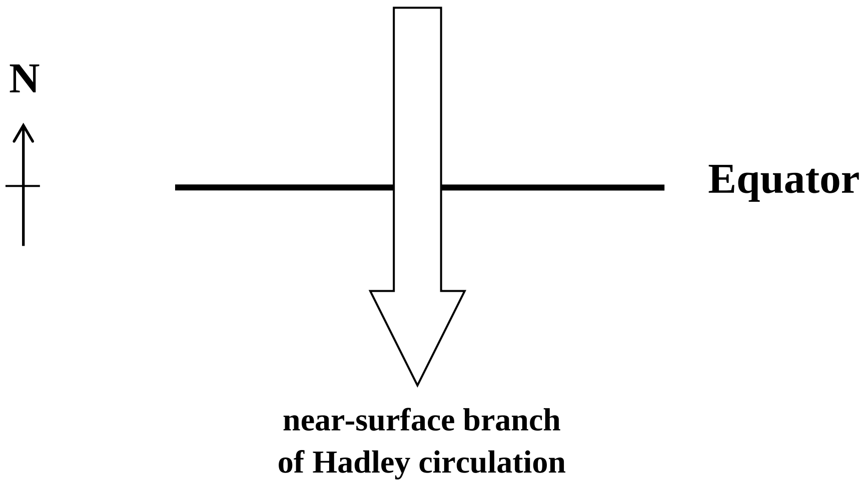
$$u_p = \Omega a \left(\frac{\cos^2 \varphi_r}{\cos \varphi_n} - \cos\varphi_n\right) = 463 ms^{-1} \times \left(\frac{\cos^2 8°}{\cos 28°} - \cos 28°\right) = 105\ ms^{-1}$$

$$u_Q = \Omega a \left(\frac{\cos^2 \varphi_r}{\cos 0°} - \cos 0°\right) = 463 ms^{-1} \times \left(\frac{\cos^2 8°}{1} - 1\right) = -8.97\ ms^{-1}$$

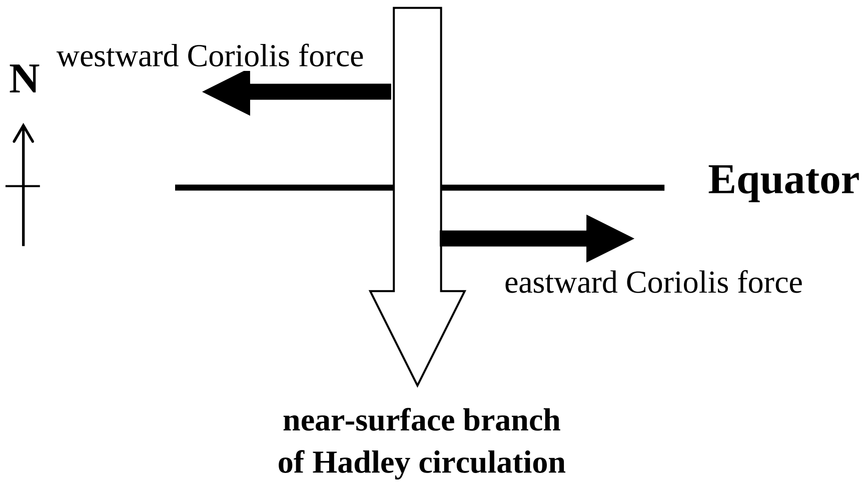
$$u_R = \Omega a \left(\frac{\cos^2 \varphi_r}{\cos \varphi_s} - \cos\varphi_s\right) = 463 ms^{-1} \times \left(\frac{\cos^2 8°}{\cos 20°} - \cos 20°\right) = 48.1\ ms^{-1}$$

Thus, the winter hemisphere (I) has a stronger atmospheric jet stream. **(0.5 point)**

(d) **(1 point)** The near-surface branch of the Hadley circulation blows southward across the equator. Mark by arrows on the figure below the direction of the east-west component of the Coriolis force acting on the tropical air mass
(A) north of the equator;
(B) south of the equator



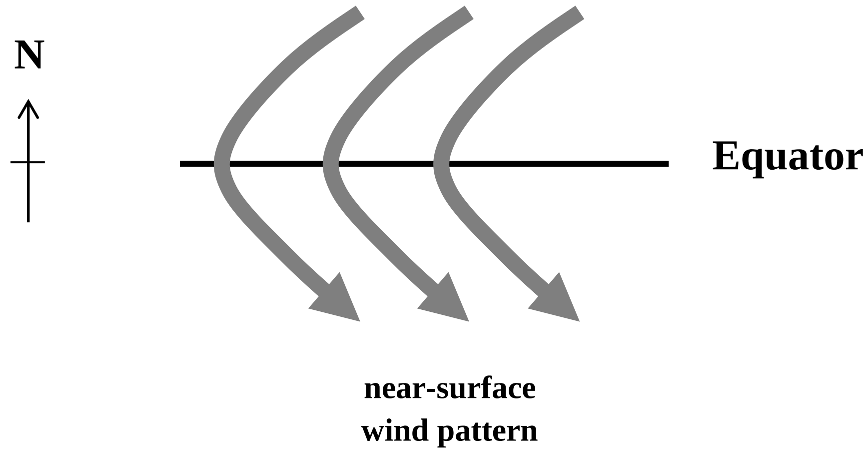
Solution: (0.5 point for each correct arrow)



(e) **(1 point)** From your answer to part (d) and the fact that surface friction nearly balances the Coriolis forces in the east-west direction, sketch the near-surface wind pattern in the tropics near the equator during northern winter solstice.

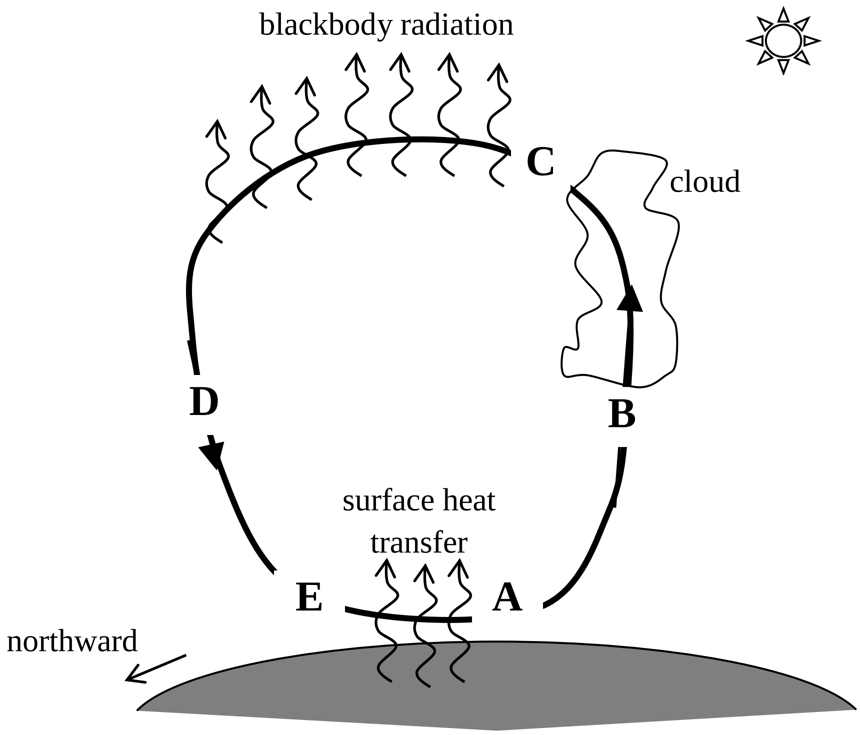
Solution:

As surface friction nearly balances the Coriolis forces in the east-west direction, the east-west component of surface friction must act eastward and westward north and south of the equator respectively. Since friction always opposes motion, the east-west wind velocity near the surface must be westward and eastward north and south of the equator respectively. So the resultant near-surface wind pattern looks like below.



(0.5 point for consistency with part (d), even if part (d) was wrong)
(0.5 point for correct answer)

Suppose the Hadley circulation can be simplified as a heat engine shown in the schematic below. Focusing on the Hadley circulation reaching into the winter hemisphere as shown below, the physical transformation of the air mass from A to B and from D to E are adiabatic, while that from B to C, C to D and from E to A are isothermal. Air gains heat by contact with the Earth's surface and by condensation of water from the atmosphere, while air loses heat by radiation into space.



(f) **(2 points)** Given that atmospheric pressure at a vertical level owes its origin to the weight of the air above that level, order the pressures p_A, p_B, p_C, p_D, p_E, respectively at the points A, B, C, D, E by a series of inequalities.
(Given that p_A = 1000 hPa and p_D = 225 hPa. Note that 1 hPa is 100 Pa.)

Solution:

Since there is less and less air above as one climbs upward in the atmosphere, atmospheric pressure must decrease upwards.
So,

$$p_A > p_B > p_C \quad and \quad p_E > p_D > p_C$$ **(0.5 point)**

The process EA represents an isothermal expansion as heat is gained from the surface. So,

$$p_E > p_A$$ **(0.5 point)**

Since the total heat gain must equal the total heat loss, more heat must be lost in the isothermal compression CD than in the isothermal expansion BC. So net heat loss occurs from B to D and hence

$$p_D > p_B$$ **(0.5 point)**

So with the values of the pressure at A and D, we deduce that:

$$p_A > p_D$$ **(0.5 point)**

Collecting all inequalities together,

$$p_E > p_A > p_D > p_B > p_C$$

(g) **(2 points)** Let the temperature next to the surface and at the top of the atmosphere be T_H and T_C respectively. Given that the pressure difference between points A and E is 20 hPa, calculate T_C for T_H = 300 K.
Note that the ratio of molar gas constant (R) to molar heat capacity at constant pressure (c_p) for air, κ, is 2/7.

Solution:

Since $p_E > p_A$ and p_A = 1000 hPa, we have p_E = 1020 hPa.
From the adiabatic compression from D to E, we have:

$$p_E^{-\kappa} T_H = p_D^{-\kappa} T_C$$ **(1 point)**

$$T_C = \left(\frac{p_D}{p_E}\right)^{\kappa} \times T_H = \left(\frac{225}{1020}\right)^{2/7} \times 300K = 195K$$ **(1 point)**

(h) **(2 points)** Calculate the pressure p_B.

Solution:

From the adiabatic expansion AB and adiabatic compression DE,

$$\left.\begin{aligned} p_A^{-\kappa}T_H &= p_B^{-\kappa}T_C \\ p_E^{-\kappa}T_H &= p_D^{-\kappa}T_C \end{aligned}\right\} \quad \frac{p_A}{p_E} = \frac{p_B}{p_D} \quad \cdots\cdots\cdots\cdots(*) \quad (\mathbf{1\,point})$$

$$\therefore p_B = \frac{p_A}{p_E} p_D = \frac{1000}{1020} 225\,hPa = 220\,hPa \quad (\mathbf{1\,point})$$

(i) For an air mass moving once around the winter Hadley circulation, using the molar gas constant, R, and the quantities defined above, obtain expressions for

(A) **(2 points)** the net work done per unit mole W_{net} ignoring surface friction;

(B) **(1 point)** the heat loss per unit mole Q_{loss} at the top of the atmosphere.

Solution:

(A) Work done per mole in an isothermal process is generally given by

$$W = \int p\, dV = \int p\, d\left(\frac{RT}{p}\right) = -RT \int p^{-1} dp = -RT \ln p + const. \quad \textbf{(1 point)}$$

Work done per mole in processes EA and BCD are respectively,

$$W_{EA} = -RT_H \ln p_A + RT_H \ln p_E = RT_H \ln\left(\frac{p_E}{p_A}\right)$$

$$W_{BCD} = RT_C \ln\left(\frac{p_B}{p_D}\right)$$

Work done in an adiabatic process is used entirely to raise the internal energy of the air mass. Since the decrease in internal energy in process AB exactly cancels the increase in internal energy in process DE because the respective decrease and increase in temperature cancel, no net work is done in the adiabatic processes.

So the net work done per mole on the air mass is:

$$W_{net} = W_{EA} + W_{BCD}$$
$$= RT_H \ln\left(\frac{p_E}{p_A}\right) + RT_C \ln\left(\frac{p_B}{p_D}\right)$$
$$= R(T_H - T_C) \ln\left(\frac{p_E}{p_A}\right) + RT_C \ln\left(\frac{p_B}{p_D}\frac{p_E}{p_A}\right)$$
$$= R(T_H - T_C) \ln\left(\frac{p_E}{p_A}\right) \quad or \quad R(T_H - T_C) \ln\left(\frac{p_D}{p_B}\right)$$

using equation (∗) in part (h)

(1 point)

(B) The heat loss per mole at the top of the atmosphere is the same as the work done per mole on the air mass because there is no change in internal energy for an isothermal process.

$$Q_{loss} = W_{CD} \qquad \textbf{(0.5 point)}$$
$$= RT_C \ln\left(\frac{p_D}{p_C}\right) \qquad \textbf{(0.5 point)}$$

(j) **(1 point)** What is the value of the ideal thermodynamic efficiency ε_i for the winter Hadley circulation?

Solution:

$$\varepsilon_i = 1 - \frac{T_C}{T_H} \qquad \textbf{(0.5 point)}$$

$$= 1 - \frac{195}{300} = 0.35 \qquad \textbf{(0.5 point)}$$

(k) **(2 points)** Prove that the actual thermodynamic efficiency ε for the winter Hadley circulation is always smaller than ε_i, showing all mathematical steps.

Solution:

$$\varepsilon = \frac{W_{net}}{Q_{loss} + W_{net}}$$

$$\frac{1}{\varepsilon} - 1 = \frac{Q_{loss}}{W_{net}} = \frac{RT_C \ln\left(\frac{p_D}{p_C}\right)}{R(T_H - T_C)\ln\left(\frac{p_E}{p_A}\right)}$$

$$= \frac{T_C \ln\left(\frac{p_D}{p_B} \times \frac{p_B}{p_C}\right)}{(T_H - T_C)\ln\left(\frac{p_E}{p_A}\right)}$$

$$> \frac{T_C \ln\left(\frac{p_D}{p_B}\right)}{(T_H - T_C)\ln\left(\frac{p_E}{p_A}\right)} \quad \text{as } \frac{p_B}{p_C} > 1 \qquad \textbf{(1 point)}$$

$$= \frac{T_C}{T_H - T_C} \qquad \text{using equation } (*) \text{ in part (h)}$$

$$\frac{1}{\varepsilon} > 1 + \frac{T_C}{T_H - T_C} = \frac{T_H}{T_H - T_C}$$

$$\varepsilon < \frac{T_H - T_C}{T_H} = \varepsilon_i \qquad \textbf{(1 point)}$$

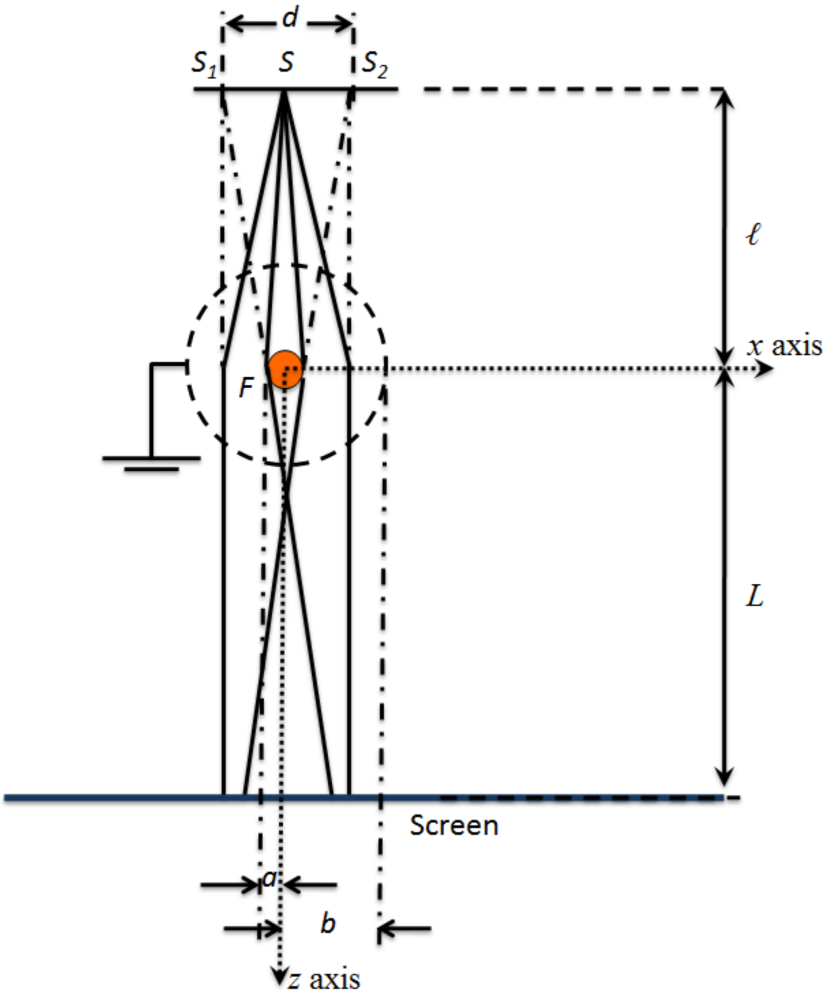
(l) **(1 point)** Which of the following statements best explains why ε is less than the ideal value? Tick the correct answer(s). There can be more than one correct answer.

- (I) We have ignored work done against surface friction.
- (II) Condensation occurs at a temperature lower than the temperature of the heat source.
- (III) There is irreversible evaporation of water at the surface.
- (IV) The ideal efficiency is applicable only when there is no phase change of water.

Solution: (II) & (III) **(0.5 point each)**
To discourage guessing, minus 0.5 point for each wrong answer.
The minimum points to be awarded in this part is 0.

Question 2

The two-slit electron interference experiment was first performed by Möllenstedt *et al*, Merli-Missiroli and Pozzi in 1974 and Tonomura *et al* in 1989. In the two-slit electron interference experiment, a monochromatic electron point source emits particles at S that first passes through an electron "biprism" before impinging on an observational plane; S_1 and S_2 are virtual sources at distance d. In the diagram, the filament is pointing into the page. Note that it is a very thin filament (not drawn to scale in the diagram).



The electron "biprism" consists of a grounded cylindrical wire mesh with a fine filament F at the center. The distance between the source and the "biprism" is ℓ, and the distance between the distance between the "biprism" and the screen is L.

(a) (**2 points**) Taking the center of the circular cross section of the filament as the origin O, find the electric potential at any point (x,z) very near the filament in terms of V_a, a and b where V_a is the electric potential of the surface of the filament, a is the radius of the filament and b is the distance between the center of the filament and the cylindrical wire mesh. (Ignore mirror charges.)

Writing out $|E| = \frac{\lambda}{2\pi\epsilon_0 r} = -\frac{\partial}{\partial r}V(r)$

$$= -\frac{\partial}{\partial r}\frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{r}$$ (**1 point**)

Note that

$$V(r) = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{r} \text{ (= 0 at the mesh)}$$

Also at the edge of the filament, $V_a = V(r = a)$, so

$$V_a = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$$

Giving together

$$V(r) = V_a\frac{\ln(b/r)}{\ln(b/a)} \text{ where } r = \sqrt{x^2 + z^2}$$

(**1 point** for final expression)

(b) (**4 points**) An incoming electron plane wave with wave vector k_z is deflected by the "biprism" due to the x-component of the force exerted on the electron. Determine k_x the x-component of the wave vector due to the "biprism" in terms of the electron charge, e, v_z, V_a, k_z, a and b, where e and v_z are the charge and the z-component of the velocity of the electrons $(k_x \ll k_z)$. Note that $\vec{k} = \frac{2\pi\vec{p}}{h}$ where h is the Planck constant.

There are several ways to work out the solution:

A charge in an electric field will experience a force and hence a change in momentum. Note that potential energy of the electron (charge = $-e_0$) is $-e_0 V(r)$. Using impulse acting on the electron due to the electric field, (**2 points**)

$$\text{Impulse} = \frac{1}{v_z}\int_{-\infty}^{\infty}(-e_0)\left(-\frac{\partial V(x,z')}{\partial x}\right)dz'|_{x=a}$$

$$= -\frac{1}{v_z}\int_{-\infty}^{\infty}\frac{-e_0 V_a x}{(x^2+z'^2)\ln\frac{b}{a}}dz'|_{x=a}$$

$$= \frac{e_0 V_a \pi}{v_z \ln\frac{b}{a}}$$

$$\Rightarrow \quad k_x = \frac{e_0 V_a \pi}{\hbar v_z \ln\frac{b}{a}}$$

(**2 points** for final expression)

The alternative solution is to write down the equations of motion for the electrons (**2 points**) and determine the deflection of the electron as it passes through the "biprism":

$$\frac{\Delta x}{\Delta z} = \frac{\lambda e}{2\epsilon_0 m v_z^2}$$

Since $V_a = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$,

$$\frac{\Delta x}{\Delta z} = \frac{\pi e V_a}{m v_z^2 \ln\frac{a}{b}}$$

(**2 points** for final expression)

(c) Before the point S, the electrons are emitted from a field emission tip and accelerated through a potential V_0. Determine the wavelength of the electron in terms of the (rest) mass m, charge $-e_0$ and V_0,

(i) (**2 points**) assuming relativistic effects can be ignored.

Equating the kinetic energy to eV_0 (**1 point**)

$$\frac{h}{\lambda} = \sqrt{2m|-e_0|V_0}$$

$$\lambda = \frac{h}{\sqrt{2me_0V_0}}$$

(**1 point** for final expression)

(ii) (**3 points**) taking relativistic effects into consideration.

Consider

$$E^2 = (pc)^2 + (mc^2)^2$$

$$= \left(\frac{h}{\lambda}c\right)^2 + (mc^2)^2$$

$$\frac{h^2c^2}{\lambda^2} = (mc^2 + |-e_0|V_0)^2 - (mc^2)^2$$

$$= 2mc^2e_0V_0\left(1 + \frac{e_0V_0}{2m_0c^2}\right)$$

$$\lambda = \frac{h}{\sqrt{2me_0V_0\left(1 + \frac{e_0V_0}{2m_0c^2}\right)}}$$

(**1 point** for knowing relativitic E − p relation
1 point for manipulating the equations
1 point for final expression)

(d) In Tonomura et al experiment,

v_z = $c/2$,
V_a = 10 V,
V_0 = 50 kV,
a = 0.5 μm,
b = 5 mm,
ℓ = 25 cm,
L = 1.5 m,
h = 6.6×10^{-34} Js,
electron charge, $-e = -1.6 \times 10^{-19}$ C,
mass of electron, $m = 9.1 \times 10^{-31}$ kg,
and the speed of light *in vacuo*, $c = 3 \times 10^{8}$ ms^{-1}

(i) (**2 points**) calculate the value of k_x ,

Previous equation:

$$k_x = \frac{e_0 V_a \pi}{\hbar v_z \ln \frac{b}{a}}$$

Plugging the relevant numbers into the equation gives:

(**1 point** for plugging the correct values)

$$k_x = \frac{\pi}{907} \text{Å}^{-1} \; or \; 3.46 \times 10^{7} m^{-1}$$

(**1 point** for final expression)

(ii) (**2 points**) determine the fringe separation of the interference pattern on the screen,

Fringe separation is given by $\frac{1}{2}\frac{2\pi}{k_x} = 907\text{Å}$

$$\left(\begin{array}{c} \textbf{1 point} \text{ for formula, note the factor } \frac{1}{2} \\ \textbf{1 point} \text{ for final expression with units} \end{array} \right)$$

(iii) (**1 point**) If the electron wave is a spherical wave instead of a plane wave, is the fringe spacing larger, the same or smaller than the fringe spacing calculated in (ii)?

Larger. (**1 point** for the correct answer)

(iv) (**2 points**) In part (c), determine the percentage error in the wavelength of the electron using non-relativistic approximation.

Non-relativistic:

$$\frac{h}{\lambda} = \sqrt{2me_0V_0}$$
$$\lambda_{nonrel} = \frac{h}{\sqrt{2me_0V_0}}$$
$$= 5.4697 \times 10^{-12}m$$

Relativistic:

$$\lambda_{rel} = \frac{h}{\sqrt{2meV_0\left(1 + \frac{eV_0}{2m_0c^2}\right)}}$$
$$= 5.3408 \times 10^{-12}m$$

Percentage error:

$$\text{Error} = \frac{\lambda_{nonrel} - \lambda_{rel}}{\lambda_{rel}}$$
$$= 0.024$$

or 2.4 percent.

(**1 point** for working out non − relativistic and relativistic wavelength
1 point for final expression)

(v) (**2 points**) Calculate the distance *d* between the apparent double slits.

The double slit formula is given by

$$y = \frac{m\lambda(\ell + L)}{d}$$

where *m* is the order and *y* is the distance for maximum intensity from the central fringe.

In this case, since the fringe spacing is 907Å,

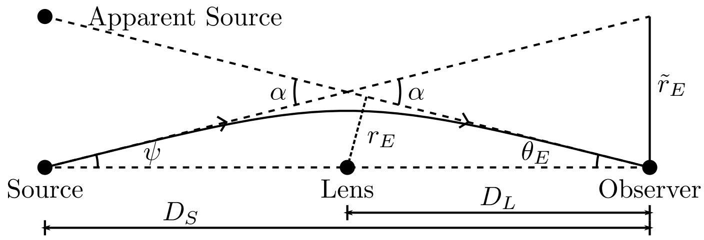
$$d = 1.03 \times 10^{-4}m$$

(**1 point** for formula
1 point for final numerical answer)

Question 3

(a) **(4 points)** Draw a diagram to describe the physical layout of an ideal (observer, lens and point source in a straight line) lensing system. Draw the light path and mark the quantities α and r_E. Also mark the angular Einstein radius θ_E (the angular deflection of the source image as seen from earth), and the other quantities that an observer on earth can measure.

Solution:



Other relevant quantities include the distances to the lens and source D_L and D_S. (D_L and D_S need not be equal.)

- 1 point for correct layout
  - Correct answers should show that light is bent
  - Apparent source is not required
  - Accept answers that show the system as a thin lens approximation (sharp deflection angles)
- 1 point for light direction correctly marked
  - Arrows on the light path
- 1 point for θ_E, α and r_E correctly identified
  - 1 correct: 0.4 points
  - 2 correct: 0.7 points
  - 3 correct: 1.0 points
- 1 point for D_L and D_S (observables; may have different notation)
  - 0.5 points each

Notes:

- ψ and $\tilde{r}_E$ need not be identified, but may be useful in a later part.

- r_E should be perpendicular to the projected light path, but in our astronomical system, it makes no difference if it is perpendicular to the source-observer line since θ_E is small. Accept answers that have r_E perpendicular to the source-observer line.

(b) **(2 points)** Sketch the image of the source (such as a star), as seen by an observer on earth, in the case where the source, lensing object and observer are on a straight line.

Solution: The image of the source should be a symmetrical **(1 point)** and circular **(1 point)** ring around the lensing object.

Notes:

- Do not accept solutions that indicate a magnified image of the source. This includes answers which state that the image is a filled in circle.
- 1 point for answers that have 2 source images symmetrically on either side of the lens because the system should be considered in 3 dimensions instead of 2.
- Text answers (without any sketch or diagram) are not accepted. Correct answers must have a sketch (as specified in the question).

(c) **(3 points)** Sketch the image of the source (such as a star), as seen by an observer on earth, in the non-ideal case where the source, lensing object and observer are not in a straight line. Sketch the source-lens system to explain why this is so.

Solution:

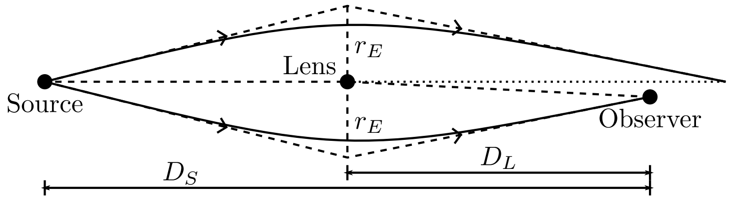
WHAT: (1.5 points) The observer will see light from one side of the lens but not the other side. This means that the Einstein ring should be an arc instead of a complete circle. The ring may be distorted or broken depending on how much deviation from an ideal case. Correct answers should not be a perfect circle or straight line.

Note:

- Solutions that give 2 source images on either side of the lens (with asymmetry) are awarded 1 point instead of 1.5 point because the system should be considered in 3 dimensions instead of 2.
- Text answers (without any sketch or diagram) are not accepted. Correct answers must have a sketch (as specified in the question).

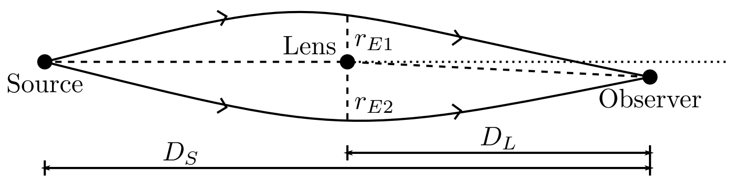
WHY: (1.5 points)

One possible answer:



For slight deviations from the ideal case, accept also the following diagram if r_{E1}

is smaller than r_{E2}.



In general, accept answers which show that the asymmetry in the system will cause the observer to see something asymmetrical.

Notes:
The key concept in this question is *asymmetry*. Correct answers for either part must demonstrate that departures from the ideal case will result in asymmetry in the observed system, and that the asymmetry about the source-observer line is the cause of the asymmetry in the observation.

(d) **(3 points)** The Schwarzschild radius of a black hole defines the point of no return. A correct expression for the Schwarzschild radius can be obtained by taking it to be the radius where the escape speed is equal to the speed of light. This means that something inside the Schwarzschild radius cannot escape the black hole.

Using Newtonian mechanics, derive the formula for the escape speed at a distance r away from a point object of mass M. Hence, derive the Schwarzschild radius for a point object of mass M in terms of the gravitational constant G and the speed of light c. Show your steps and reasoning clearly. (This happens to give the correct expression for the Schwarzschild radius that comes from general relativity.)

Solution: By definition, the gravitational potential energy of a test mass m at a distance r from the mass is **(0.5 point)**

$$\phi = -\frac{GMm}{r}.$$

To escape the gravitational potential, the *total* energy of the test mass needs to be at least 0 so it should have a kinetic energy of **(0.5 point)**

$$K = \frac{GMm}{r} = \frac{1}{2}mv_e^2.$$

Rearranging the above, the escape speed at distance from mass r is **(1 point)**

$$v_e = \sqrt{\frac{2GM}{r}}.$$

Substitute $v_e = c$ and rearrange to get **(1 point)**

$$r_S = \frac{2GM}{c^2}.$$

2 points for deriving escape speed (Any reasonable and physically sound method based on Newtonian mechanics)
1 point for deriving the Schwarzschild radius from the escape speed.

(e) **(1 point)** Using the formula for light deflection, write down an expression for the Schwarzschild radius of a lensing object in the case where the source, lens and observer is in a straight line.

Solution: The Schwarzschild radius is

$$r_S = \frac{2GM}{c^2} \quad \text{so} \quad r_S = \frac{1}{2}\alpha r_E$$

Notes: Full marks for correct working.

(f) **(2 points)** Consider the case where we have a lensing object of the order of a few solar masses ($M \sim$ a few $\times 10^{30}$ kg) in the nearby regions of the galaxy (distance $D_L \sim$ a few $\times 10^{18}$ m away) and a source object somewhat further out ($D_S \sim$ a few $\times D_L$). What can we say about α and θ_E in this case? (Choose your answer on your answer sheet. Points will be deducted for wrong answers.)

- α is large and $\tan\alpha$, $\sin\alpha$, $\cos\alpha$ must be calculated exactly.
- α is small and the small angle approximations to $\tan\alpha$, $\sin\alpha$, $\cos\alpha$ are permissable.
- α is irrelevant and need not be calculated

- θ_E is large and $\tan\theta_E$, $\sin\theta_E$, $\cos\theta_E$ must be calculated exactly.
- θ_E is small and the small angle approximations to $\tan\theta_E$, $\sin\theta_E$, $\cos\theta_E$ are permissable.
- θ_E is irrelevant and need not be calculated

Solution:

- α is small
- θ_E is small

Notes:

- Choices pertaining to α and θ_E are to be marked independently (1 point each).
- The conditions are mutually exclusive so accept only one condition for each quantity (α, θ_E). Answers that select more than one condition for a quantity (α, θ_E) are wrong (no point to be awarded).

Reasoning: Working out the numbers, we can find that the Schwarzschild radius is on the order of 10^4 m. Because α has a maximum of 2π (largest possible angle), this means the physical Einstein radius $r_E \sim 10^4$ m is very small compared to the distance to the lens $D_L \sim 10^{20}$ m. The geometry of the system therefore means that α is actually a very small angle.

Another approximation comes from the geometry of the system which sets bounds on α and θ_E so that (see figure in part (a))

$$\tan\theta_E = \frac{r_E}{D_L} = \frac{2r_S/\alpha}{D_L} \approx \frac{10^{-16}}{\alpha}$$

which suggests that α or θ_E or both should be small.

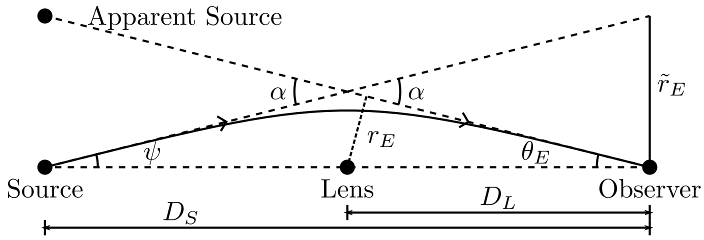
Based on the geometry of the setup and what we have already established (α small), we then have the following cases:

- θ_E large means that D_L is small which is not the case here.
- α small, θ_E small is the only valid outcome here

The result and constraints in the question suggests that α and θ_E are both small Because θ_E is small, the Einstein radius $r_E \sim 10^4$ m is very small compared to the distance to the lensing object $D_L \sim 10^{20}$ m or source D_S. We can therefore take the small angle approximation where α and θ_E is involved.

(g) **(3 points)** Using the conditions in part (f), rewrite your expression in part (e) in terms of measurable quantities (which are θ_E, D_S and D_L) for a lensing object of the order of a few solar masses ($M \sim$ a few $\times 10^{30}$ kg) and in the nearby regions of the galaxy (distance $D_L \sim$ a few $\times 10^{18}$ m away) with a source object somewhat further out ($D_S \sim$ a few $\times D_L$). Show your working.

Solution: Adding up exterior angles, we see that $\alpha = \theta_E + \psi$ so $\theta_E = \alpha - \psi$ where is small (ψ and $\tilde{r}_E$ defined on the following diagram). Also note that r_E is approximately perpendicular to the source-observer system because θ_E is small.



Using the small angle approximation for α and θ_E, we can write

$$\frac{r_E}{D_L} = \tan\theta_E \approx \theta_E \quad \text{and} \quad \frac{\tilde{r}_E}{D_S} = \frac{r_E}{D_S - D_L} = \tan\psi \approx \psi$$

This gives **(1 point)**

$$\alpha = \frac{r_E}{D_L} + \frac{r_E}{D_S - D_L}$$

So that **(1 point)**

$$r_S = \frac{1}{2} r_E \alpha = \frac{1}{2} r_E^2 \left(\frac{D_S}{D_L(D_S - D_L)} \right)$$

To write this in terms of θ_E, D_L and D_S, we use $r_E = \theta_E D_L$ to get **(1 point)**

$$r_S = \frac{1}{2} \theta_E^2 \left(\frac{D_S D_L}{D_S - D_L} \right)$$

Notes:

- 1 point for α
- 1 point for r_S
- 1 point for final equation

(h) **(2 points)** Suppose we have an event where a lensing object of 6.0×10^{30} kg (3.0 solar masses), 2.6×10^{18} m away from earth passes in front of a star 9.2×10^{18} m away from earth. This happens such that the ideal configuration occurs during the event. What is the angular Einstein radius θ_E (as seen from earth) during this event when the source, lens and observer line up?

Solution: The Schwarzschild radius of the lens is

$$r_S = \frac{2 \times (6.673 \times 10^{-11}) \times 6.0 \times 10^{30}}{(3.0 \times 10^8)^2} = 8.9 \times 10^3 \text{ m}$$

From the previous part, the angular Einstein radius is given by

$$\begin{aligned}\theta_E^2 &= 2r_S \times \left(\frac{D_S - D_L}{D_S D_L}\right) \\ &= 2 \times 8.9 \times 10^3 \times \left(\frac{(9.2 - 2.6) \times 10^{18}}{(9.2 \times 10^{18}) \times (2.6 \times 10^{18})}\right) \\ &= 4.9 \times 10^{-15}\end{aligned}$$

Thus the angular Einstein radius is

$$\theta_E = \sqrt{4.9 \times 10^{-15}} = 7.0 \times 10^{-8} \text{ radians} = 0.014 \text{ arcseconds}$$

(**1 point** for correct answer, **1 point** for correct working)
Notes:

- Students are expected to use the formula derived in part (g) to answer this question.
- For the final answer:
  - 0.5 point off for missing units. While angles are mathematically dimensionless, a good student should be cognisant of the fact that there are different physical units for angular measurement, and that units for angles should be specified.
  - 0.5 point off for final answers given to 1 significant figure or less.
  - 1 point off if the final number is incorrect.