Theoretical Question 1: The Shockley-James Paradox

SOLUTION

a. The magnetic field created by the large loop at its center is:

$$B = \frac{\mu_0 I_2}{2R}$$

Since $r \ll R$, this is the field throughout the area of the small loop. Therefore, the flux through the small loop is given by:

$$\Phi_{B1} = \pi r^2 B = \frac{\pi \mu_0 r^2 I_2}{2R}$$

The mutual inductance is then given by:

$$M_{21} = \frac{\pi \mu_0 r^2}{2R}$$

b. Since $M_{12} = M_{21} = M$, we have:

$$\Phi_{B2} = M I_1 = \frac{\pi \mu_0 r^2 I_1}{2R}$$

Taking the derivative with respect to time, this becomes:

$$\varepsilon_2 = \frac{\pi \mu_0 r^2 \dot{I}_1}{2R}$$

c. The EMF is work per unit charge, while the electric field is force per unit charge. Therefore:

$$E = \frac{\varepsilon_2}{2\pi R} = \frac{\mu_0 r^2 \dot{I}_1}{4R^2}$$

d. The electric field from part (c) leads to a force:

$$F = EQ = \frac{\mu_0 r^2 Q \dot{I}}{4R^2}$$

Integrating over dt (and disregarding the sign), we get the impulse:

$$\Delta p = \frac{\mu_0 r^2 I Q}{4R^2}$$

e. The current can be written as:

$$I = nAqv$$

where v is the charge carriers' velocity. We therefore have:

$$v = \frac{I}{nAq}$$

The momentum is then given by:

$$p = \gamma mnAlv = \frac{mnAlv}{\sqrt{1 - v^2/c^2}} = \frac{mIl}{q}\left(1 - \left(\frac{I}{nAqc}\right)^2\right)^{-1/2}$$

where γ is the Lorentz factor associated with v.

f. The hidden momentum is due to the charge carriers in the two vertical sides of the loop. Let m be the mass of the charge carriers, let q be their charge, and let $\Delta U = kQql/R^2$ be the potential energy difference for a charge carrier between the two sides. Denote the longitudinal densities and velocities of the charges in the two sides by λ_1, v_1, λ_2 and v_2. Let γ_1 and γ_2 be the appropriate Lorentz factors. From the constant value of the current, we have:

$$q\lambda_1 v_1 = q\lambda_2 v_2 = I$$

Energy conservation for the charge carriers passing from one side to the other reads:

$$(\gamma_2 - \gamma_1) \cdot mc^2 = \Delta U$$

The total momentum now reads:

$$p_{hid} = p_2 - p_1 = ml(\gamma_2\lambda_2 v_2 - \gamma_1\lambda_1 v_1) = \frac{mIl}{q}(\gamma_2 - \gamma_1) = \frac{Il\Delta U}{qc^2} = \frac{kQIl^2}{R^2c^2}$$

Note that all the microscopic quantities m, q, λ_i and v_i have dropped out.

g) In part (d), the magnetic moment is $\mu = \pi r^2 I$, and we get:

$$\Delta p = \frac{\mu_0 Q\mu}{4\pi R^2}$$

In part (f), the magnetic moment is $\mu = l^2 I$, and we get:

$$p_{hid} = \frac{kQ\mu}{R^2c^2} = \frac{\mu_0 Q\mu}{4\pi R^2}$$

We see that the results are identical.

h) The answer is (A)+(C). (A) is true because ΔU between the near side and the far side of the loop vanishes. (B) cannot be true, because the back-reaction of the induced charges on the external charge is a higher-order effect; for instance, it involves higher powers of Q. Then the conservation of center-of-mass velocity requires that (C) is true.

Theoretical Question 2: Creaking Door

SOLUTION

a1. The motion here is pure sliding under a constant kinetic friction. This is harmonic motion with a displaced equilibrium point. The angular frequency is given by:

$$\omega_0 = \sqrt{k/m}$$

From here, the period is:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

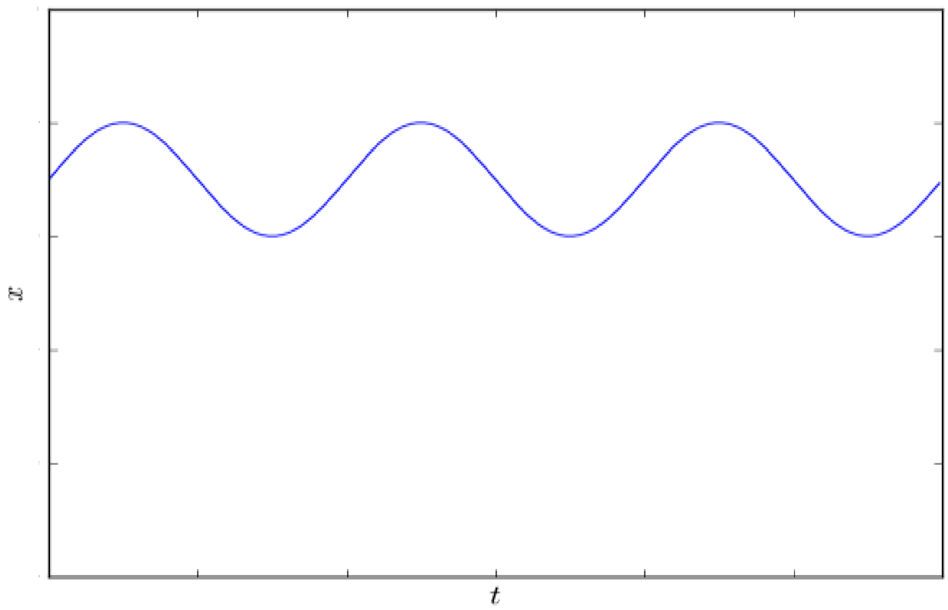
The initial slope is given by:

$$\left(\frac{dx}{dt}\right)_0 = u - v_0$$

Therefore, the amplitude of oscillations is:

$$A = \frac{(dx/dt)_0}{\omega_0} = (u - v_0)\sqrt{\frac{m}{k}}$$

a2. The graph is sinusoidal, as shown below.

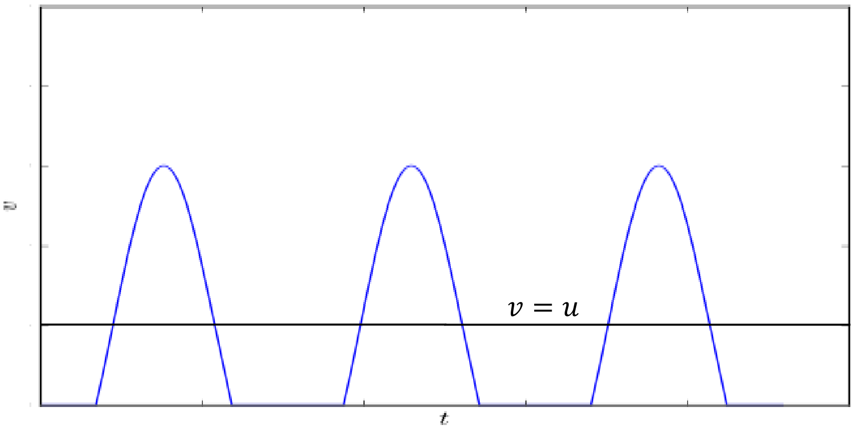


The initial point is given by:

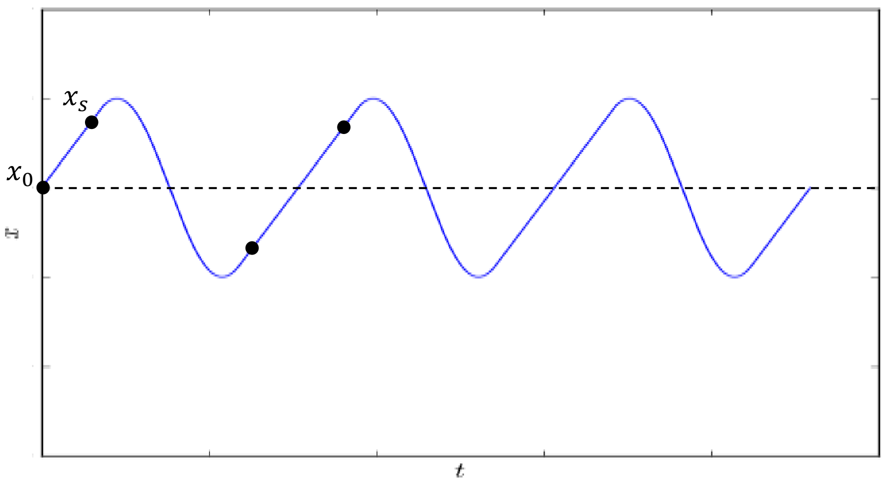
$$x_0 = \frac{\mu_k mg}{k}$$

This is the equilibrium point of the sine function. The students are not required to find this equilibrium point, but they are required to understand that it is positive.

b. This will be a stick-slip graph. The "humps" are sinusoidal, with a non-continuous derivative at their intersections with the horizontal segments. The peaks of the humps are higher than $v = u$, since u must be the *average* velocity of the box. In fact, they are also higher than $v = 2u$, but this is not required from the students.



c. Let's pass into the reference frame of the driven end of the spring. The position of the box is then given by minus the elongation x. The motion is an oscillation around the equilibrium position x_0. The slip phase is sinusoidal as in part (a), while the stick phase corresponds to motion with a constant velocity $-u$. The stick phase ends when the elastic force balances the static friction, i.e. at $x_s = \mu_s mg/k$, and starts again at the symmetric point with respect to x_0.



We see that the average elongation is again the sine's equilibrium point:

$$\bar{x} = x_0 = \frac{\mu_k mg}{k}$$

d. Again, let us pass into the reference frame of the driven end of the spring. During the stick phase, the box traverses a distance of:

$$2(x_s - x_0) = 2(\mu_s - \mu_k)mg/k.$$

Its velocity during this phase is u, so the duration of the stick phase is:

$$t_{stick} = \frac{2(\mu_s - \mu_k)mg}{ku}$$

The slip phase is a sinusoidal motion around x_0 with angular frequency ω_0. The sinusoidal period is missing a phase of 2φ, where φ is given by the ratio of initial position and initial velocity with respect to the equilibrium point:

$$\tan\varphi = \frac{\omega_0(x_s - x_0)}{u} = \frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}}$$

Then the length of the slip phase is:

$$t_{slip} = T_0\left(1 - \frac{\varphi}{\pi}\right) = 2\sqrt{\frac{m}{k}}\left(\pi - \tan^{-1}\left(\frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}}\right)\right)$$

And the total period is:

$$T = t_{stick} + t_{slip} = 2\sqrt{\frac{m}{k}}\left(\frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}} + \pi - \tan^{-1}\left(\frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}}\right)\right)$$

e. Consider again stick-slip motion in the reference frame of the driven end of the spring. During the sinusoidal slip phase, the sine's amplitude will decrease due to the dissipation. At the beginning of the slip phase, the velocity is $-u$, while the sine is at the phase φ, which we found in the solution to the previous part. Thus, the sine's velocity amplitude is $u/\cos\varphi$. For periodic stock-slip to occur, the sine must return to the slope $-u$. Due to the dissipation, this will happen at a phase larger than $2\pi - \varphi$. In other words, dissipation shortens the stick phase. The critical case is when stick phase shortens to zero. This will happen if the sine reaches the slope $-u$ precisely at the equilibrium point, i.e. at the phase 2π. If the slope at 2π is less steep than $-u$, the box will continue its damped sinusoidal motion without ever reaching a stick phase again.

If it is to be killed by weak dissipation, the stick phase must be short to begin with. This corresponds to a large u. The slip phase then takes up almost an entire period of the sine wave. Thus, to a good approximation, the amplitude loss during the slip phase is given by η. The critical point is when the velocity amplitude drops from $u/\cos\varphi$ to u during one period:

$$\eta = \left|\frac{\Delta A}{A}\right| = \left|\frac{u/\cos\varphi - u}{u/\cos\varphi}\right| = 1 - \cos\varphi \approx \frac{\varphi^2}{2}$$

where the LHS is the change in the amplitude due to dissipation over one period. Using the results from (d) in the limit of small φ, we get:

$$\eta = \frac{m(\mu_s - \mu_k)^2 g^2}{2ku_c^2}$$

$$u_c = (\mu_s - \mu_k) g \sqrt{\frac{m}{2k\eta}}$$

Another derivation method based on the same reasoning is to use explicitly the initial amplitude A of the harmonic motion:

$$u_c = \omega_0 A(1-\eta), \qquad A^2 = (x_s - x_0)^2 + (m/k)u_c^2$$

A third method is to consider energy losses $|\Delta E/E| = 2\eta$ in the reference frame of the spring's driven end:

$$2\eta \cdot \frac{1}{2} m u_c^2 = \frac{1}{2} k (x_s - x_0)^2$$

f. For small rotations, the lower edge of the cylinder will remain stuck to the base. When the cylinder is deformed by an angle α, a point on its upper edge shifts by a distance $h\alpha$. This corresponds to a rotation angle $\theta = h\alpha/r$ of the door around the cylinder's axis. The shear force on an area element dA of the base is given by:

$$dF = G\alpha dA = \frac{Gr}{h}\theta dA$$

The corresponding torque is:

$$d\tau = rdF = \frac{Gr^2}{h}\theta dA$$

Summing over the contact area with the base, the total torque is:

$$\tau = \frac{Gr^2}{h}\theta \cdot 2\pi r\Delta r = \frac{2\pi G r^3 \Delta r}{h}\theta$$

Therefore, the torsion coefficient is:

$$\kappa = \frac{2\pi G r^3 \Delta r}{h} \approx 2000\text{Nm}$$

The numerical result is not required from the student. Any expression which reduces to the one above in the limit $\Delta r \ll r$ will be accepted.

g. We neglect the duration of the slip phase. Using the results of section (d) with M instead of m and rotation instead of linear motion, we get:

$$t_{stick} = \frac{2(\mu_s - \mu_k)Mgr}{\kappa\Omega}$$

$$\Omega = \frac{2(\mu_s - \mu_k)Mgr}{\kappa t_{stick}} = \frac{2(\mu_s - \mu_k)Mgrf}{\kappa} = \frac{2(\mu_s - \mu_k)Mghf}{\pi G r^2 \Delta r} = 1.12 \cdot 10^{-2} \mathrm{s}^{-1}$$

Any expression which reduces to the one above in the limit $\Delta r \ll r$ will be accepted. Numerical results from such different expressions may vary significantly, since $\Delta r / r = 0.2$ is not really negligible. Each numerical result should be checked against its expression.

Theoretical Question 3: Birthday Balloon

SOLUTION

a. Solution using forces:

Let the balloon's radius be r, and let P be the pressure of the inside air. Consider the balloon's rear half, and write down the equilibrium of forces on it along the cylinder's axis:

$$\pi r^2(P - P_0) = 2\pi r\sigma_L$$

On the other hand, let us cut the balloon in half with a plane that runs along its axis, and consider a half-cylindrical section of length x. The equilibrium of forces in perpendicular to the cutting plane reads:

$$2rx(P - P_0) = 2x\sigma_t$$

from which we derive $\sigma_L/\sigma_t = 1/2$.

Solution using energies:

If we stretch the balloon longitudinally by length dL, the energy cost is:

$$E_1 = 2\pi r\sigma_L \cdot dL$$

If we inflate the balloon radially with an increment dr, the energy cost is:

$$E_2 = L\sigma_t \cdot 2\pi dr$$

The two deformations can be combined while keeping the volume fixed, if we take $\pi r^2 dL = -Ld(\pi r^2) = -2\pi Lrdr$, i.e. $rdL = -2Ldr$. The equilibrium state is the one where the combined energy cost $E_1 + E_2$ of such a deformation is zero. This gives again the result $\sigma_L/\sigma_t = 1/2$.

b. From part (a), we are reminded of the relation between surface tension and pressure:

$$P = P_0 + \frac{\sigma_t}{r} = P_0 + \frac{k(r - r_0)}{r_0 r} = P_0 + k\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

The volume is related to the radius by:

$$V = \pi r^2 L_0$$

So we get:

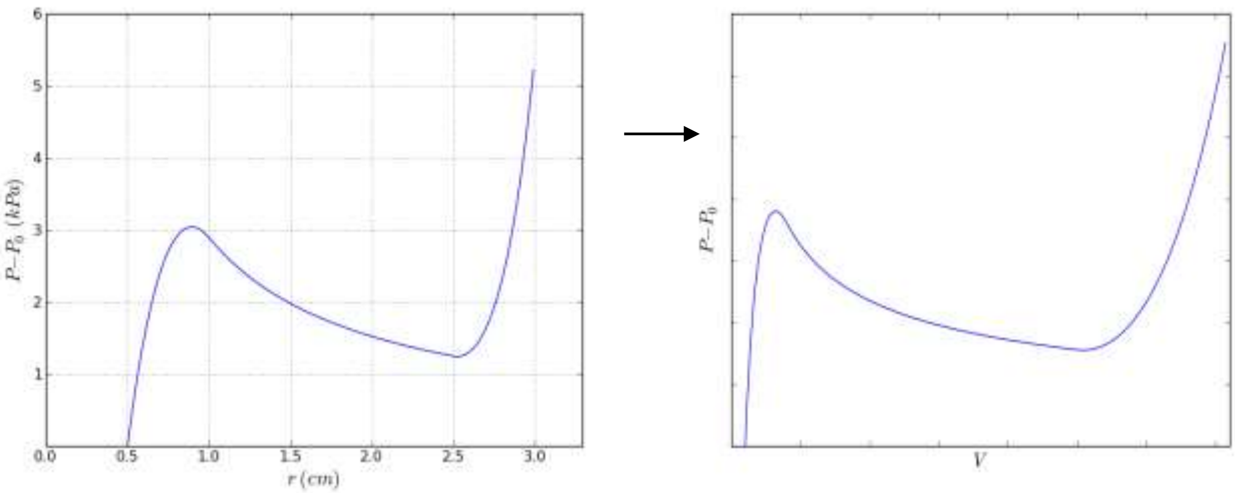
$$P(V) = P_0 + k\left(\frac{1}{r_0} - \sqrt{\frac{\pi L_0}{V}}\right)$$

The graph of $P - P_0$ is a hyperbola-like function increasing from 0 at $V = \pi r_0^2 L_0$ to an asymptotic value of k/r_0 at $V \to \infty$.

The maximal pressure is obtained at $V \to \infty$:

$$P_{max} = P_0 + \frac{k}{r_0}$$

c. The graph of $P - P_0$ as a function of V has the same qualitative form as $P - P_0 = \sigma_t/r$ as a function of r, shown below. The graph rises from zero, then decreases, and then increases again. The points $r = 1\text{cm}$ and $r = 2.5\text{cm}$ lie in the decreasing portion (and not on the local extrema).



The pressures at the two requested points are approximately given by:

$$P - P_0(r = 1\text{cm}) = \frac{\sigma}{r} = \frac{30}{0.01} = 3000\text{Pa}; \qquad P - P_0(r = 2.5\text{cm}) = \frac{30}{0.025} = 1200\text{Pa}$$

d. The work done on the pressure-controlling mechanism during continuous inflation from volume V_i to volume V_f is:

$$W_{mech} = -P(V_f - V_i)$$

The work done on the atmosphere is:

$$W_{surr} = P_0(V_f - V_i)$$

The condition for the jump is:

$$W_{rubber} + W_{surr} + W_{mech} = 0$$

This translates into Maxwell's equal-areas condition:

$$\int_{V_i}^{V_f} (P - P_0)dV = (P - P_0)\left(V_f - V_i\right)$$

Or, equivalently:

$$\int_{V_i}^{V_f} PdV = P(V_f - V_i)$$

The cubic function $P(V)$ is symmetric around the point $V = u$, $P - P_0 = ac$.

The equal-areas condition is therefore satisfied at:

$$P_c = P_0 + ac$$

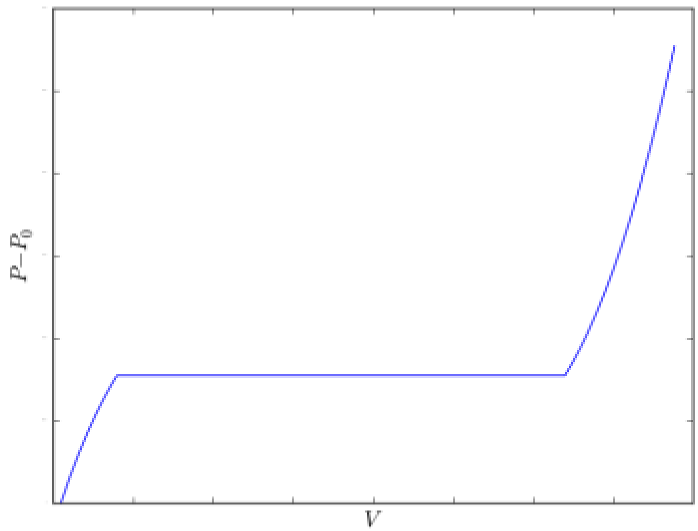
The volumes V_1 and V_2 are given by the points where:

$$(V - u)^3 - b(V - u) = 0$$

This gives:

$$V_{1,2} = u \pm \sqrt{b}$$

e. The range of volumes where a phase separation will occur is $V_1 < V < V_2$. The pressure is constant throughout this range, and equals the transition pressure P_c. The graph of $P - P_0$ as a function of V is monotonous, with a rising piece, a horizontal plateau at $V_1 < V < V_2$, $P = P_c$, followed by another rising piece. At the start and end of the plateau, the slope has a discontinuity, i.e. the graph has a kink.



f. The radii of the two domains correspond to the volumes V_1 and V_2. As the total volume increases from V_1 to V_2, the volume of the thin domain changes linearly from V_1 to 0. We get:

$$V_{thin} = \frac{V_1}{V_2 - V_1}(V_2 - V)$$

Converting this into length, we have:

$$L_{thin} = \frac{V_{thin}}{\pi r_1^2} = \frac{V_1(V_2 - V)}{\pi r_1^2(V_2 - V_1)}$$

g. The increase in the balloon's volume as a result of converting a length L_{thin} into the thick phase is:

$$\Delta V = \frac{V_2 - V_1}{V_1}\Delta V_{thin} = \frac{\pi r_1^2(V_2 - V_1)}{V_1}\Delta L_{thin}$$

The corresponding work is:

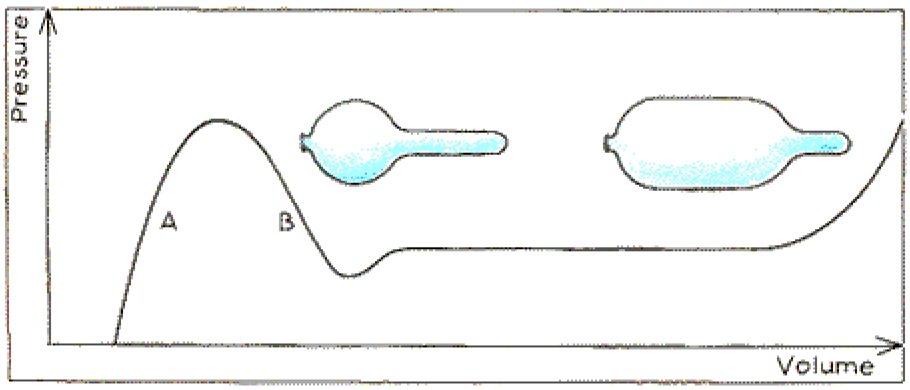
$$\Delta W = P_c \Delta V = \frac{\pi r_1^2 P_c(V_2 - V_1)}{V_1}\Delta L_{thin}$$

Therefore:

$$\frac{\Delta W}{\Delta L_{thin}} = \frac{\pi r_1^2 P_c(V_2 - V_1)}{V_1}$$

Additional discussion (doesn't appear as part of the question):

During a realistic inflation, perturbations are not strong enough to keep the system in global equilibrium at all times. The experimental graph increases up to P_c, continues to increase some way beyond it, reaches a local maximum, then decreases and settles on the plateau at P_c. This over-increase of the pressure is responsible for the fact that inflating a balloon is difficult during the first few puffs. After the plateau, the graph sharply increases as discussed above. The decrease towards the plateau "overshoots" slightly again, reaches a local minimum and rises again to settle on the plateau. This behavior is depicted in the graph below.



The illustration is taken from:
http://www.science-project.com/_members/science-projects/1989/12/1989-12-body.html

Theoretical Question 1: The Shockley-James Paradox

MARKING SCHEME

a) 1.0	Finding B at the center	0.3	
	Writing $\Phi_{B1} = \pi r^2 B$	0.3	
	Final answer	0.4	No credit for internal propagating error
b) 0.8	Understanding that $\Phi_{B2} = MI_1$	0.2	
	Understanding that $\varepsilon_2 = -\dot{\Phi}_{B2}$	0.2	Disregard sign
	Final answer	0.4	No credit for internal propagating error
c) 0.5	Writing $\varepsilon_2 = 2\pi r E$	0.3	Partial credit for $\varepsilon_2 = \oint E dl$ - 0.1
	Final answer	0.2	No credit for internal propagating error
d) 1.0	Writing $F = QE$	0.2	
	Writing F as a function of $\dot{I}_1$	0.2	
	Writing $\Delta p = \int F dt$	0.2	
	Final answer	0.4	
e) 1.1	Understanding that $N = nlA$	0.2	
	Understanding that $v = I/(nAq)$	0.3	
	Understanding that $p = Nmv/\sqrt{1 - v^2/c^2}$ (or γNmv)	0.3	
	Final answer	0.3	No credit for internal propagating error
f) 3.3	Understanding that $I = \lambda q v$ or $I = nAqv$	0.3	
	Understanding that there are separate $v_{1,2}$ and $\lambda_{1,2}$ (or $n_{1,2}$)	0.4	
	Expressing p_{hid} in terms of the charge densities and velocities	0.4	E.g. $p_{hid} = ml(\lambda_2\gamma_2 v_2 - \lambda_1\gamma_1 v_1)$
	Cancelling out the charge densities	0.7	E.g. $p_{hid} = (\gamma_2 - \gamma_1)Ilm/q$
	Understanding that $\Delta E_k = \Delta U$	0.5	
	Finding $\Delta U = kQql/R^2$	0.4	
	Final answer	0.6	If the result was reverse-engineered from part (g), this will be the only credit given. No credit for internal propagating error.
g) 0.8	Writing $\mu = I\pi r^2$ for part (d)	0.1	
	Re-expressing the result of part (d)	0.3	
	Writing $\mu = Il^2$ for part (f)	0.1	
	Re-expressing the result of part (f)	0.3	No credit here if the answer to (f) was reverse-engineered.
h) 1.5	Correct answer (yes/no) for each statement	0.5*3	No credit at all if a statement was decided incorrectly.

Theoretical Question 2: Creaking Door

MARKING SCHEME

a1) 0.6	Understood in T_0, A calculation that the motion is purely harmonic	0.1	
	Result for T_0	0.2	
	Result for A	0.3	Correct amplitude $u - v_0$ of $\dot{x}$ – 0.1 Deducing A (using either direct division by ω or energy conservation in the moving frame) – 0.2
a2) 0.4	Sinusoidal shape with enough periods	0.1	
	Starts at a positive slope	0.1	
	Starts at $x > 0$	0.1	
	Positive mean value of x	0.1	Judge sparingly, penalize only in obvious cases
b) 1.2	Enough periods	0.1	
	Starts at $v = 0$ (stick)	0.1	
	Has finite segments with $v = 0$ (stick phases)	0.3	
	The "humps" (slip phases) are always above the horizontal segments	0.2	Always to the same side – 0.1 Always above – 0.1
	Continuity of $v$between the different segments	0.1	
	Slope (acceleration) discontinuity between the horizontal segments (stick) and the humps (slip)	0.1	
	u is drawn below the maximum of $v(t)$	0.3	
	Penalty for clearly unreasonable shape of the humps (very asymmetric, contain straight lines etc.)	-0.3	
c) 0.5	Correct result	0.5	Wrote the formal integral for $\langle x \rangle$ - 0.1
d) 2.4	Writing $T = t_{stick} + t_{slip}$	0.1	
	Finding the detachment offset $x_1 = (\mu_s - \mu_k)mg/k$ (or finding $2x_1$)	0.3	
	Finding the stick time $t_{stick} = 2x_1/u$	0.2	Correct except for factor-of-2 – 0.1
	Understanding that t_{slip} is part of a harmonic period T_0	0.2	
	Finding the phase corresponding to t_{slip}	1.1	Partial credit for the amplitude of the harmonic motion – 0.3
	Final result for t_{slip}	0.2	Correct except for factor-of-2 – 0.1
	Final result for T	0.3	Correct except for factors-of-2 – 0.2 Otherwise, no credit for propagating errors.
e) 2.4	Understanding that at u_c, the box sticks back to the floor at the equilibrium of the harmonic motion	0.4	
	Understanding that at u_c, $t_{stick} \ll t_{slip}$	0.4	

	Writing correct equations for u_c	1.2	Partial credit for correct equations involving the amplitude A of the harmonic motion or the detachment phase φ, without finding them – 0.4
	Final answer	0.4	
f) 1.0	Relation between τ and α	0.4	
	Relation between α and θ	0.4	
	Final answer	0.2	Any expression which reduces to the official one in the limit $\Delta r \ll r$ will be accepted.
g) 1.5	Understanding that $t_{stick} \gg t_{slip}$	0.2	
	Correct expression for the result	1.0	Any expression which reduces to the official one in the limit $\Delta r \ll r$ will be accepted. Penalty for factor-of-2 (when not propagated) – 0.2 Partial credit for using t_{stick} from part (d) without taking the limit $t_{stick} \gg t_{slip}$ - 0.3
	Correct numerical result	0.3	A numerical result without an expression will not receive credit. If the expression was acceptable but is different from the official one, the result will be graded according to the student's expression.

Theoretical Question 3: Birthday Balloon

MARKING SCHEME

a) 1.8	Relation between $P - P_0$ and σ_t	0.8	
	Relation between $P - P_0$ and σ_L	0.6	
	Final answer	0.4	No credit for internal propagating error
b) 1.0	Finding $P(V)$	0.4	Relation between $P - P_0$ and σ_t - 0.1 Relation between r and V – 0.1 Final answer for $P(V)$ – 0.2
	Graph	0.4	Starts at $V > 0$ – 0.1 Starts at $P - P_0 = 0$ – 0.1 Monotonously rising – 0.1 Convex – 0.1
	Finding P_{max}	0.2	
c) 1.3	Graph	1.1	Starts at $V > 0$ – 0.1 Starts at $P - P_0 = 0$ – 0.1 Rising at the end – 0.1 Decreasing in the middle – 0.2 Maximum marked – 0.1 Minimum marked – 0.1 $r = 0.5$cm marked after the maximum – 0.2 $r = 2.5$cm marked after $r = 0.5$cm and before the minimum – 0.2 Penalty for negative $P - P_0$ - 0.3
	$P - P_0$ value at $r = 0.5$cm	0.1	
	$P - P_0$ value at $r = 2.5$cm	0.1	
d) 2.3	Result for P_c	1.2	Partial credit for writing the equal-areas law – 0.6 Writing the equal-areas law with misplaced P_0 - 0.3
	Equation for $V_{1,2}$	0.5	
	Result for V_1	0.3	
	Result for V_2	0.3	
e) 1.0	Starts at $V > 0$	0.1	
	Starts at $P - P_0 = 0$	0.1	
	Rising at the end	0.1	
	Horizontal in the middle	0.3	
	Slope discontinuity at the ends of the horizontal segment	0.1	
	$P_c - P_0$ coincides with the horizontal segment	0.1	
	V_1 coincides with the beginning of the horizontal segment	0.1	
	V_2 coincides with the end of the horizontal	0.1	

	segment		
	Penalty for negative $P - P_0$	-0.3	
f) 1.4	Finding V_{thin}	1.0	Partial credit for correct equations for V_{thin} – 0.6 Partial credit if there are less equations than unknowns – 0.2 Partial credit for linear relation between V_{thin} and V without correct equations – 0.3
	Relation between V_{thin} and L_{thin}	0.2	
	Final answer	0.2	No credit for internal propagating error
g) 1.2	Writing $\Delta W = P_c \Delta V$	0.3	
	Relation between ΔV and ΔV_{thin}	0.5	Partial credit for understanding that ΔV is not equal but proportional to ΔV_{thin} – 0.2
	Relation between ΔV_{thin} and ΔL_{thin}	0.2	
	Final answer	0.2	No credit for internal propagating error