4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

First Solution: Extend segment DM through M to G such that $FG \parallel CD$.

1 point for constructing G.

Then $MF = MC$ if and only if quadrilateral $CDFG$ is a parallelogram, or, $FD \parallel CG$. Hence $MC = MF$ if and only if $\angle GCD = \angle FDA$, that is, $\angle FDA + \angle CGF = 180^\circ$.

1 point for reducing side information to angle information.

Because quadrilateral $ABED$ is cyclic, $\angle FDA = \angle ABE$. It follows that $MC = MF$ if and only if

$$180^\circ = \angle FDA + \angle CGF = \angle ABE + \angle CGF,$$

that is, quadrilateral $CBFG$ is cyclic, which is equivalent to

$$\angle CBM = \angle CBG = \angle CFG = \angle DCF = \angle DCM.$$

Because $\angle DMC = \angle CMB$, $\angle CBM = \angle DCM$ if and only if triangles BCM and CDM are similar, that is

$$\frac{CM}{BM} = \frac{DM}{CM},$$

or $MB \cdot MD = MC^2$.

5 points for completing the proof.

Remark: The possible marks for this problem are 0, 1, 2, 7. This is not a very hard geometry problem. If a student knows many facts but cannot make final connections between the facts, he can only get at most 2 points.

Second Solution:

We first assume that $MB \cdot MD = MC^2$. Because $\frac{MC}{MD} = \frac{MB}{MC}$ and $\angle CMD = \angle BMC$, triangles CMD and BMC are similar. Consequently, $\angle MCD = \angle MBC$.

1 point for proving this fact.

Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle FCA = \angle MCD = \angle MBC = \angle DBE = \angle DAE = \angle CAE,$$

implying that $AE \parallel CF$, so $\angle AEF = \angle CFE$. Because quadrilateral $ABED$ is cyclic, $\angle ABD = \angle AED$. Hence

$$\angle FBM = \angle ABD = \angle AED = \angle AEF = \angle CFE = \angle MFD.$$

Because $\angle FBM = \angle DFM$ and $\angle FMB = \angle DMF$, triangles BFM and FDM are similar. Consequently, $\frac{FM}{DM} = \frac{BM}{FM}$, or $FM^2 = BM \cdot DM = CM^2$. Therefore $MC^2 = MB \cdot MD$ implies $MC = MF$.

2 points for proving this part.

Now we assume that $MC = MF$. Applying **Ceva's Theorem** to triangle BCF and **cevians** BM, CA, FE gives

$$\frac{BA}{AF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EB} = 1,$$

implying that $\frac{BA}{AF} = \frac{BE}{EC}$, so $AE \parallel CF$.

2 points for proving this fact.

Consequently, $\angle DCM = \angle DAE$. Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle DCM = \angle DAE = \angle DBE = \angle CBM.$$

Because $\angle CBM = \angle DCM$ and $\angle CMB = \angle DMC$, triangles BCM and CDM are similar. Consequently, $\frac{CM}{DM} = \frac{BM}{CM}$, or $CM^2 = BM \cdot DM$.

Combining the above, we conclude that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

2 points for proving this part.

Remark: 3 points for proving $MB \cdot MD = MC^2$ implying $MF = MC$; 4 points for $MF = MC$ implying $MB \cdot MD = MC^2$. Two partial credits from different parts are not additive. A partial credits in one part and a full mark in the other are not additive. Possible marks are 0, 1, 2, 3 (only for completing the first part), 4 (only for completing the second part), 7.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

First Solution: By multiplying a, b, and c by a suitable factor, we reduce the problem to the case when $a+b+c=3$. The desired inequality reads

$$\frac{(a+3)^2}{2a^2+(3-a)^2} + \frac{(b+3)^2}{2b^2+(3-b)^2} + \frac{(c+3)^2}{2c^2+(3-c)^2} \le 8.$$

1 point for homogeneous approach and expressing $b+c$ in terms of a.

Set

$$f(x) = \frac{(x+3)^2}{2x^2+(3-x)^2}$$

It suffices to prove that $f(a)+f(b)+f(c) \le 8$. Note that

$$\begin{aligned} f(x) &= \frac{x^2+6x+9}{3(x^2-2x+3)} = \frac{1}{3} \cdot \frac{x^2+6x+9}{x^2-2x+3} \\ &= \frac{1}{3}\left(1+\frac{8x+6}{x^2-2x+3}\right) = \frac{1}{3}\left(1+\frac{8x+6}{(x-1)^2+2}\right) \\ &\le \frac{1}{3}\left(1+\frac{8x+6}{2}\right) = \frac{1}{3}(4x+4). \end{aligned}$$

Hence,

$$f(a)+f(b)+f(c) \le \frac{1}{3}(4a+4+4b+4+4c+4) = 8,$$

as desired.

6 points for completing the proof. No partial credits given in this part.

Remark: The possible marks for this approach is 0, 1, 7.

Second Solution: Note that

$$\begin{aligned} (2x+y)^2 + 2(x-y)^2 &= 4x^2+4xy+y^2+2x^2-4xy+2y^2 \\ &= 3(2x^2+y^2). \end{aligned}$$

Setting $x=a$ and $y=b+c$ yields

$$(2a+b+c)^2 + 2(a-b-c)^2 = 3(2a^2+(b+c)^2).$$

Thus, we have

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} = \frac{3(2a^2+(b+c)^2) - 2(a-b-c)^2}{2a^2+(b+c)^2} = 3 - \frac{2(a-b-c)^2}{2a^2+(b+c)^2}.$$

and its analogous forms. Thus, the desired inequality is equivalent to

$$\frac{(a-b-c)^2}{2a^2+(b+c)^2} + \frac{(b-a-c)^2}{2b^2+(c+a)^2} + \frac{(c-a-b)^2}{2c^2+(a+b)^2} \ge \frac{1}{2}.$$

3 points for transforming into this formation. Serious but unsuccessful attempt to use a and $b+c$ as two variables will be awarded 1 point.

Because $(b+c)^2 \le 2(b^2+c^2)$, we have $2a^2+(b+c)^2 \le 2(a^2+b^2+c^2)$ and its analogous forms. It suffices to show that

$$\frac{(a-b-c)^2}{2(a^2+b^2+c^2)} + \frac{(b-a-c)^2}{2(a^2+b^2+c^2)} + \frac{(c-a-b)^2}{2(a^2+b^2+c^2)} \ge \frac{1}{2},$$

or,

$$(a-b-c)^2 + (b-a-c)^2 + (c-a-b)^2 \ge a^2+b^2+c^2. \tag{1}$$

3 points for reducing to this inequality.

Multiplying this out the left-hand side of the last inequality gives $3(a^2+b^2+c^2)-2(ab+bc+ca)$. Therefore the inequality (1) is equivalent to $2[a^2+b^2+c^2-(ab+bc+ca)] \ge 0$, which is evident because

$$2[a^2+b^2+c^2-(ab+bc+ca)] = (a-b)^2+(b-c)^2+(c-a)^2.$$

Equalities hold if $(b+c)^2 = 2(b^2+c^2)$ and $(c+a)^2 = 2(c^2+a^2)$, that is, $a=b=c$.

1 point for completing the proof.

Remark: Because the last step is only meaningful with previous steps, the final 1 point will not be awarded to students if no evidence why it is useful was provided. One the other hand, any serious attempt to use a and $b+c$ as two variables will be awarded 1 point. The possible marks for this approach is 0, 1, 3, 6, 7.

Third Solution: Given a function f of three variables, define the cyclic sum

$$\sum_{\text{cyc}} f(p,q,r) = f(p,q,r) + f(q,r,p) + f(r,p,q).$$

We first convert the inequality into

$$\frac{2a(a+2b+2c)}{2a^2+(b+c)^2} + \frac{2b(b+2c+2a)}{2b^2+(c+a)^2} + \frac{2c(c+2a+2b)}{2c^2+(a+b)^2} \le 5.$$

Splitting the 5 among the three terms yields the equivalent form

$$\sum_{\text{cyc}} \frac{4a^2-12a(b+c)+5(b+c)^2}{3[2a^2+(b+c)^2]} \ge 0. \tag{2}$$

1 points for transforming into this formation.

The numerator of the term shown factors as $(2a-x)(2a-5x)$, where $x=b+c$. We will show that

$$\frac{(2a-x)(2a-5x)}{3(2a^2+x^2)} \geq -\frac{4(2a-x)}{3(a+x)}. \tag{3}$$

Indeed, (3) is equivalent to

$$(2a-x)[(2a-5x)(a+x)+4(2a^2+x^2)] \geq 0,$$

which reduces to

$$(2a-x)(10a^2-3ax-x^2) = (2a-x)^2(5a+x) \geq 0,$$

evident. We proved that

$$\frac{4a^2-12a(b+c)+5(b+c)^2}{3[2a^2+(b+c)^2]} \geq -\frac{4(2a-b-c)}{3(a+b+c)},$$

hence (2) follows. Equality holds if and only if $2a=b+c$, $2b=c+a$, $2c=a+b$, i.e., when $a=b=c$.

6 points for transform into this formation.

Remark: The possible marks of this approach are 0, 1 and 7.

Fourth Solution: Given a function f of three variables, we define the symmetric sum

$$\sum_{\text{sym}} f(x_1,\ldots,x_n) = \sum_{\sigma} f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

where σ runs over all permutations of $1,\ldots,n$ (for a total of $n!$ terms). For example, if $n=3$, and we write x,y,z for x_1,x_2,x_3,

$$\sum_{\text{sym}} x^3 = 2x^3+2y^3+2z^3$$

$$\sum_{\text{sym}} x^2y = x^2y+y^2z+z^2x+x^2z+y^2x+z^2y$$

$$\sum_{\text{sym}} xyz = 6xyz.$$

We combine the terms in the desired inequality over a common denominator and use symmetric sum notation to simplify the algebra. The numerator of the difference between the two sides is

$$\sum_{\text{sym}} 8a^6+8a^5b+2a^4b^2+10a^4bc+10a^3b^3-52a^3b^2c+14a^2b^2c^2.$$

1 point for multiplying out correctly.

Recalling **Schur's Inequality**, we have

$$\begin{aligned} & a^3+b^3+c^3+3abc-(a^2b+b^2c+c^a+ab^2+bc^2+ca^2) \\ = \; & a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b) \geq 0, \end{aligned}$$

or

$$\sum_{\text{sym}} a^3-2a^2b+abc \geq 0.$$

Hence,

$$0 \leq 14abc\sum_{\text{sym}} a^3-2a^2b+abc = 14\sum_{\text{sym}} a^4bc-28a^3b^2c+14a^2b^2c^2$$

3 points for proving this inequality.

and by repeated **AM-GM Inequality**,

$$0 \leq \sum_{\text{sym}} 4a^6-4a^4bc$$

(because $a^46+a^6+a^6+a^6+b^6+c^6 \geq 6a^4bc$ and its analogous forms)

1 point for proving this inequality.

and

$$0 \leq \sum_{\text{sym}} 4a^6+8a^5b+2a^4b^2+10a^3b^3-24a^3b^2c.$$

2 points for proving this inequality.

Adding these three inequalities yields the desired result.

Remark: In this approach, we have $1+1=2$ and the other partial credits are not additive. (Indeed, because the last two inequalities are in very artificial forms, it is almost impossible to state and prove them with the third to last inequality.) The possible marks for this approaches are 0, 1, 2, 3, 7.

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Note: Let

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

denote a position, where A, B, C, D, E, F denote the numbers written on the vertices of the hexagon. We write

$$A \begin{matrix} B & C \\ F & E \end{matrix} D \pmod 2$$

if we consider the numbers written modulo 2.

Solution: Define the *sum* and *maximum* of a position to be the sum and maximum of the six numbers at the vertices. We will show that from any position in which the sum is odd, it is possible to reach the all-zero position.

1 point for making this claim.

Our strategy alternates between two steps:

(a) from a position with odd sum, move to a position with exactly one odd number;

(b) from a position with exactly one odd number, move to a position with odd sum and strictly smaller maximum, or to the all-zero position.

Note that no move will ever increase the maximum, so this strategy is guaranteed to terminate, because each step of type (b) decreases the maximum by at least one, and it can only terminate at the all-zero position. It suffices to show how each step can be carried out.

2 points for making this claim.

First, consider a position

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

with odd sum. Then either $A+C+E$ or $B+D+F$ is odd; assume without loss of generality that $A+C+E$ is odd. If exactly one of A, C and E is odd, say A is odd, we can make the sequence of moves

$$1 \begin{matrix} B & 0 \\ F & 0 \end{matrix} D \to 1 \begin{matrix} \mathbf{1} & 0 \\ \mathbf{1} & 0 \end{matrix} \mathbf{0} \to \mathbf{0} \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} 0 \to 0 \begin{matrix} 1 & 0 \\ \mathbf{0} & 0 \end{matrix} 0 \pmod 2,$$

where a letter or number in boldface represents a move at that vertex, and moves that do not affect each other have been written as a single move for brevity. Hence we can reach a position with exactly one odd number. Similarly, if A, C, E are all odd, then the sequence of moves

$$1 \begin{matrix} B & 1 \\ F & 1 \end{matrix} D \to 1 \begin{matrix} \mathbf{0} & 1 \\ \mathbf{0} & 1 \end{matrix} \mathbf{0} \to 1 \begin{matrix} 0 & \mathbf{0} \\ 0 & \mathbf{0} \end{matrix} 0 \pmod 2,$$

brings us to a position with exactly one odd number. Thus we have shown how to carry out step (a).

2 points for proving this part.

Now assume that we have a position

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

with A odd and all other numbers even. We want to reach a position with smaller maximum. Let M be the maximum. There are two cases, depending on the parity of M.

- In this case, M is even, so one of B, C, D, E, F is the maximum. In particular, $A < M$. We claim after making moves at B, C, D, E, and F in that order, the sum is odd and the maximum is less than M. Indeed, the following sequence

$$1 \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} \mathbf{1} & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} 1 & \mathbf{1} \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \mathbf{1} \to 1 \begin{matrix} 1 & 1 \\ 0 & \mathbf{1} \end{matrix} 1 \to 1 \begin{matrix} 1 & 1 \\ \mathbf{0} & 1 \end{matrix} 1 \pmod 2.$$

  shows how the numbers change in parity with each move. Call this new position $A' \begin{matrix} B' & C' \\ F' & E' \end{matrix} D'$. The sum is odd, since there are five odd numbers. The numbers A', B', C', D', E' are all less than M, since they are odd and M is even, and the maximum can never increase. Also, $F' = |A' - E'| \le \max\{A', E'\} < M$. So the maximum has been decreased.

- In this case, M is odd, so $M = A$ and the other numbers are all less than M.

  If $C > 0$, then we make moves at B, F, A, and F, in that order. The sequence of positions is

$$1 \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} \mathbf{1} & 0 \\ 0 & 0 \end{matrix} 0 \to 1 \begin{matrix} 1 & 0 \\ \mathbf{1} & 0 \end{matrix} 0 \to \mathbf{0} \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} 0 \to 0 \begin{matrix} 1 & 0 \\ \mathbf{0} & 0 \end{matrix} 0 \pmod 2.$$

  Call this new position $A' \begin{matrix} B' & C' \\ F' & E' \end{matrix} D'$. The sum is odd, since there is exactly one odd number. As before, the only way the maximum could not decrease is if $B' = A$; but this is impossible, since $B' = |A - C| < A$ because $0 < C < M = A$. Hence we have reached a position with odd sum and lower maximum.

  If $E > 0$, then we apply a similar argument, interchanging B with F and C with E.

  If $C = E = 0$, then we can reach the all-zero position by the following sequence of moves:

$$A \begin{matrix} B & 0 \\ F & 0 \end{matrix} D \to A \begin{matrix} \boldsymbol{A} & 0 \\ \boldsymbol{A} & 0 \end{matrix} \mathbf{0} \to \mathbf{0} \begin{matrix} A & 0 \\ A & 0 \end{matrix} 0 \to 0 \begin{matrix} \mathbf{0} & 0 \\ \mathbf{0} & 0 \end{matrix} 0.$$

  (Here 0 represents zero, not any even number.)

Hence we have shown how to carry out a step of type (b), proving the desired result. The problem statement follows since 2003 is odd.

2 points for proving this part.

Note: Observe that from positions of the form

$$0 \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} 0 \pmod 2 \qquad \text{or rotations}$$

it is impossible to reach the all-zero position, because a move at any vertex leaves the same value modulo 2. Dividing out the greatest common divisor of the six original numbers does not affect whether we can reach the all-zero position, so we may assume that the numbers in the original position are not all even. Then by a more complete analysis in step (a), one can show from any position not of the above form, it is possible to reach a position with exactly one odd number, and thus the all-zero position. This gives a complete characterization of positions from which it is possible to reach the all-zero position.

There are many ways to carry out the case analysis in this problem; the one used here is fairly economical. The important idea is the formulation of a strategy that decreases the maximum value while avoiding the "bad" positions described above.

> **Remark:** Partial credits are not additive. 1 point will be rewarded for somewhat applying a maximum value argument. 3 points will be awarded for stating the strategies clearly and carrying out some significant progress in proving the strategies can indeed be realized, in other words, $2 + 2 = 3$. Possible marks for this approach are 0, 1, 2, 3, 6/7. (Because this problem requires strong combinatorial argument skill, 6 points can be rewarded to solutions with minimum errors. On the other hand, a score of 5 points shall be very extremely special, if not possible.)

Second Solution: We will show that if there is a pair of opposite vertices with odd sum (which of course is true if the sum of all the vertices is odd), then we can reduce to a position of all zeros.

> **1 point for making this claim.**

Focus on such a pair (a, d) with smallest possible $\max(a, d)$. We will show we can always reduce this smallest maximum of a pair of opposite vertices with odd sum or reduce to the all-zero position. Because the smallest maximum takes nonnegative integer values, we must be able to achieve the all-zero position.

> **1 point for making this claim.**

To see this assume without loss of generality that $a \geq d$ and consider an arc (a, x, y, d) of the position

$$a \overset{x\ y}{\underset{*\ *}{}} d$$

Consider updating x and y alternately, starting with x. If $\max(x, y) > a$, then in at most two updates we reduce $\max(x, y)$. Thus, we can repeat this *alternate updating* process and we must eventually reach a point when $\max(x, y) \leq a$, and hence this will be true from then on.

> **1 point for applying this process.**

Under this alternate updating process, the arc of the hexagon will eventually enter an unique cycle of length four modulo 2 in at most one update. Indeed, we have

$$1 \overset{0\ 0}{\underset{*\ *}{}} 0 \to 1 \overset{\mathbf{1}\ 0}{\underset{*\ *}{}} 0 \to 1 \overset{1\ \mathbf{1}}{\underset{*\ *}{}} 0 \to 1 \overset{\mathbf{0}\ 1}{\underset{*\ *}{}} 0 \to 1 \overset{0\ \mathbf{0}}{\underset{*\ *}{}} 0 \pmod 2$$

and

$$\begin{array}{cccc}&0&0&\\1&&&0\\&*&*&\end{array}\to\begin{array}{cccc}&0&\mathbf{0}&\\1&&&0\\&*&*&\end{array}\pmod 2;\quad \begin{array}{cccc}&1&0&\\1&&&0\\&*&*&\end{array}\to\begin{array}{cccc}&\mathbf{1}&0&\\1&&&0\\&*&*&\end{array}\pmod 2$$

$$\begin{array}{cccc}&1&1&\\1&&&0\\&*&*&\end{array}\to\begin{array}{cccc}&1&\mathbf{1}&\\1&&&0\\&*&*&\end{array}\pmod 2;\quad \begin{array}{cccc}&0&1&\\1&&&0\\&*&*&\end{array}\to\begin{array}{cccc}&\mathbf{0}&1&\\1&&&0\\&*&*&\end{array}\pmod 2,$$

or

$$\begin{array}{cccc}&0&1&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&\mathbf{1}&1&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&1&\mathbf{0}&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&\mathbf{0}&0&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&0&\mathbf{1}&\\0&&&1\\&*&*&\end{array}\pmod 2$$

and

$$\begin{array}{cccc}&0&0&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&\mathbf{0}&0&\\0&&&1\\&*&*&\end{array}\pmod 2;\quad \begin{array}{cccc}&0&1&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&0&\mathbf{1}&\\0&&&1\\&*&*&\end{array}\pmod 2$$

$$\begin{array}{cccc}&1&1&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&\mathbf{1}&0&\\0&&&1\\&*&*&\end{array}\pmod 2;\quad \begin{array}{cccc}&1&0&\\0&&&1\\&*&*&\end{array}\to\begin{array}{cccc}&1&\mathbf{0}&\\0&&&1\\&*&*&\end{array}\pmod 2.$$

Further note that each possible parity for x and y will occur equally often.

2 points for proving this part.

Applying this alternate updating process to both arcs (a, b, c, d) and (a, e, f, d) of

$$\begin{array}{cccc}&b&c&\\a&&&d,\\&f&e&\end{array}$$

we can make the other four entries be at most a and control their parity. Thus we can create a position

$$\begin{array}{cccc}&x_1&x_2&\\a&&&d\\&x_5&x_4&\end{array}$$

with $x_i + x_{i+3}$ $(i = 1, 2)$ odd and $M_i = \max(x_i, x_{i+3}) \le a$. In fact, we can have $m = \min(M_1, M_2) < a$, as claimed, unless both arcs enter a cycle modulo 2 where the values congruent to a modulo 2 are always exactly a. More precisely, because the sum of x_i and x_{i+3} is odd, one of them is not congruent to a and so has its value strictly less than a. Thus both arcs must pass through the state (a, a, a, d) (modulo 2, this is either $(0, 0, 0, 1)$ or $(1, 1, 1, 0)$) in a cycle of length four. It is easy to check that for this to happen, $d = 0$. Therefore, we can achieve the position

$$\begin{array}{cccc}&a&a&\\a&&&0.\\&a&a&\end{array}$$

From this position, the sequence of moves

$$\begin{array}{cccc}&a&a&\\a&&&0\\&a&a&\end{array}\to\begin{array}{cccc}&\mathbf{0}&a&\\a&&&0\\&\mathbf{0}&a&\end{array}\to\begin{array}{cccc}&0&\mathbf{0}&\\\mathbf{0}&&&0\\&0&\mathbf{0}&\end{array}$$

completes the task.

2 points for proving this part.

Remark: For this approach, we have the following addition rule: $1+1 = 2$ (as both of the first two claims are quiet insightful), $1+1+1 = 2$, $1+2 = 2$ (which seems hard to be realized), $1+1+2 = 1+1+1+2 = 3$. The possible marks for this approach are 0, 1, 2, 3, 6/7. (Because this problem requires strong combinatorial argument skill, 6 points can be rewarded to solutions with minimum errors. On the other hand, a score of 5 points shall be very extremely special, if not possible.)