Remark: The general philosophy of this marking scheme follows that of IMO 2002. This scheme encourages *complete solutions*. Partial credits will be given under more strict circumstances. Each solution by students shall be graded from one of the two approaches: (1) from 7 going down (a complete solution with possible minor errors); (2) from 0 going up (a solution missing at least one critical idea.) Most partial credits are not additive. Because there are many results need to be proved progressively in problem 3, most partial credits in this problem are accumulative. Many problems have different approaches. Graders are encouraged to choose the approach that most favorable to students. But the partial credits from different approaches are not additive.

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

   Solution: We proceed by induction. The property is clearly true for $n = 1$. Assume that $N = a_1 a_2 \ldots a_n$ is divisible by 5^n and has only odd digits. Consider the numbers

   $$N_1 = 1a_1 a_2 \ldots a_n = 1 \cdot 10^n + 5^n M = 5^n (1 \cdot 2^n + M),$$
   $$N_2 = 3a_1 a_2 \ldots a_n = 3 \cdot 10^n + 5^n M = 5^n (3 \cdot 2^n + M),$$
   $$N_3 = 5a_1 a_2 \ldots a_n = 5 \cdot 10^n + 5^n M = 5^n (5 \cdot 2^n + M),$$
   $$N_4 = 7a_1 a_2 \ldots a_n = 7 \cdot 10^n + 5^n M = 5^n (7 \cdot 2^n + M),$$
   $$N_5 = 9a_1 a_2 \ldots a_n = 9 \cdot 10^n + 5^n M = 5^n (9 \cdot 2^n + M).$$

   > **2 points for setting up clear induction steps and making attempts to discuss numbers N_1, N_2, N_3, N_4, N_5.**

   The numbers $1 \cdot 2^n + M, 3 \cdot 2^n + M, 5 \cdot 2^n + M, 7 \cdot 2^n + M, 9 \cdot 2^n + M$ give distinct remainders when divided by 5. Otherwise the difference of some two of them would be a multiple of 5, which is impossible, because 2^n is not a multiple of 5, nor is the difference of any two of the numbers $1, 3, 5, 7, 9$. It follows that one of the numbers N_1, N_2, N_3, N_4, N_5 is divisible by $5^n \cdot 5$, and the induction is complete.

   > **5 points for completing the proof.**

   > **Remark:** No points should be given for just mentioning induction. If there is any unclear arguments in the proof of the inducting step, 2 points should be deducted. The possible marks for this problem are 0, 2, 5, 7.

2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Solution: Let $\mathcal{P} = A_1 A_2 \ldots A_n$, where n is an integer with $n \geq 3$. The problem is trivial for $n = 3$ because there are no diagonals and thus no dissections. We assume that $n \geq 4$. Our proof is based on the following Lemma.

Lemma 1.   *Let $ABCD$ be a convex quadrilateral such that all its sides and diagonals have rational lengths. If segments AC and BD meet at P, then segments AP, BP, CP, DP all have rational lengths.*

It is clear by Lemma 1 that the desired result holds when $\mathcal{P}$ is a convex quadrilateral. Let $A_i A_j$ $(1 \leq i < j \leq n)$ be a diagonal of $\mathcal{P}$. Assume that C_1, C_2, $\ldots$, C_m are the consecutive division points on diagonal $A_i A_j$ (where point C_1 is the closest to vertex A_i and C_m is the closest to A_j). Then the segments $C_\ell C_{\ell+1}$, $1 \leq \ell \leq m-1$, are the sides of all polygons in the dissection. Let C_ℓ be the point where diagonal $A_i A_j$ meets diagonal $A_s A_t$. Then quadrilateral $A_i A_s A_j A_t$ satisfies the conditions of Lemma 1. Consequently, segments $A_i C_\ell$ and $C_\ell A_j$ have rational lengths. Therefore, segments $A_i C_1, A_i C_2, \ldots, A_j C_m$ all have rational lengths. Thus, $C_\ell C_{\ell+1} = AC_{\ell+1} - AC_\ell$ is rational. Because i, j, ℓ are arbitrarily chosen, we proved that all sides of all polygons in the dissection are also rational numbers.

> **1 point for reducing this problem to a problem on convex quadrilateral.**

Now we present four proofs of Lemma 1 to finish our proof.

- *First approach*   We show only that segment AP is rational, the others being similar. Introduce Cartesian coordinates with $A = (0,0)$ and $C = (c, 0)$. Put $B = (a, b)$ and $D = (d, e)$. Then by hypothesis, the numbers
$$AB = \sqrt{a^2 + b^2}, \quad AC = c, \quad AD = \sqrt{d^2 + e^2},$$
$$BC = \sqrt{(a-c)^2 + b^2}, \quad BD = \sqrt{(a-d)^2 + (b-e)^2}, \quad CD = \sqrt{(d-c)^2 + e^2},$$
  are rational. In particular,
$$BC^2 - AB^2 - AC^2 = (a-c)^2 + b^2 - (a^2 + b^2) - c^2 = 2ac$$
  is rational. Because $c \neq 0$, a is rational. Likewise d is rational.

> **2 points for setting up coordinates and verifying the rationality of at least one non-trivial component of a vertex.**

  Now we have that $b^2 = AB^2 - a^2$, $e^2 = AD^2 - d^2$, $(b-e)^2 = BD^2 - (a-d)^2$ are rational, and so that $2be = b^2 + e^2 - (b-e)^2$ is rational. Because quadrilateral $ABCD$ is convex, b and e are nonzero and have opposite sign. Hence $\frac{b}{e} = \frac{2be}{2b^2}$ is rational.
  We now calculate
$$P = \left(\frac{bd - ae}{b - e}, 0 \right),$$
  so
$$AP = \frac{\frac{b}{e} \cdot d - a}{\frac{b}{e} - 1}$$
  is rational.   ∎

- *Second approach*

Note that, for an angle α, if $\cos\alpha$ is rational, then $\sin\alpha = r_\alpha\sqrt{m_\alpha}$ for some rational r and square-free positive integer m (and this expression is unique when r is written in the lowest term). We say two angles α and β with rational cosine are *equivalent* if $m_\alpha = m_\beta$, that is, if $\sin\alpha/\sin\beta$ is rational. We establish the following lemma.

Lemma 2.   *Let α and β be two angles.*

(a) *If α, β and $\alpha+\beta$ all have rational cosines, then all three are equivalent.*

(b) *If α and β have rational cosine values and are equivalent, then $\alpha+\beta$ has rational cosine value (and is equivalent to the other two).*

(c) *If α, β and γ are the angles of a triangle with rational sides, then all three have rational cosine values and are equivalent.*

Proof: Assume that $\cos\alpha = s$ and $\cos\beta = t$.

(a) Assume that s and t are rational. By the **Addition formula**, we have

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \tag{$*$}$$

or, $\sin\alpha\sin\beta = st - \cos(\alpha+\beta)$, which is rational by the given conditions. Hence α and β are equivalent. Thus $\sin\alpha = r_a\sqrt{m}$ and $\sin\beta = r_b\sqrt{m}$ for some rational numbers r_a and r_b and some positive square free integer m. By the Addition formula, we have

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = (tr_a + sr_b)\sqrt{m},$$

implying that $\alpha+\beta$ is equivalent to both α and β.

(b) By $(*)$, $\cos(\alpha+\beta)$ is rational if s, t are rational and α and β are equivalent. Then by (a), α, β, $\alpha+\beta$ are equivalent.

(c) Applying the **Law of Cosine** to triangle ABC shows that $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are all rational. Note that $\cos\gamma = \cos(180° - \alpha - \beta) = -\cos(\alpha+\beta)$. The desired conclusions follow from (a). ∎

We say a triangle *rational* if all its sides are rational. By Lemma 2 (c), all the angles in a rational triangle have rational cosine values and are equivalent to each other. To prove Lemma 1, we set $\angle DAC = A_1$, $\angle CAB = A_2$, $\angle ABD = B_1$, $\angle DBC = B_2$, $\angle BCA = C_1$, $\angle ACD = C_2$, $\angle CDB = D_1$, $\angle BDA = D_2$. Because triangles ABC, ABD, ADC are rational, angles $A_2, A_1 + A_2, A_1$ all have rational cosine values. By Lemma 2 (a), A_1 and A_2 are equivalent. Similarly, we can show that B_1 and B_2, C_1 and C_2, D_1 and D_2 are equivalent. Because triangle ABC is rational, angles A_2 and C_1 are equivalent. There all angles $A_1, A_2, B_1, \ldots, D_2$ have rational cosine values and are equivalent.

Because angles A_2 and B_1 are equivalent, angle $A_2 + B_1$ has rational values and is equivalent to A_2 and B_1. Thus, $\angle APB = 180° - (A_2 + B_1)$ has rational cosine value and is equivalent to A_2 and B_1. Apply the **Law of Sine** to triangle ABP gives

$$\frac{AB}{\sin \angle APB} = \frac{AP}{\sin \angle B_1} = \frac{BP}{\sin \angle A_2},$$

implying that both AP and BP have rational length. Similarly, we can show that both CP and DP has rational length, proving Lemma 1.

- *Third approach* This approach applies the techniques used in the first approach into the second approach. To prove Lemma 1, we set $\angle DAP = A_1$ and $\angle BAP = A_2$. Applying the Law of Cosine to triangle ABC, ABC, ADC shows that angles $A_1, A_2, A_1 + A_2$ all has rational cosine values. By the Addition formula, we have

$$\sin A_1 \sin A_2 = \cos A_1 \cos A_2 - \cos(A_1 + A_2),$$

implying that $\sin A_1 \sin A_2$ is rational.

Thus,

$$\frac{\sin A_2}{\sin A_1} = \frac{\sin A_2 \sin A_1}{\sin^2 A_1} = \frac{\sin A_2 \sin A_1}{1 - \cos^2 A_1}$$

is rational.

Note that the ratio between areas of triangle ADP and ABP is equal to $\frac{PD}{BP}$. Therefore,

$$\frac{BP}{PD} = \frac{[ABP]}{[ADP]} = \frac{\frac{1}{2} AB \cdot AP \cdot \sin A_2}{\frac{1}{2} AD \cdot AP \cdot \sin A_1} = \frac{AB}{AD} \cdot \frac{\sin A_2}{\sin A_1},$$

implying that $\frac{PD}{BP}$ is rational. Because $BP + PD = BD$ is rational, both BP and PD are rational. Similarly, AP and PC are rational, proving Lemma 1.

- *Fourth approach* This approach is based on the following lemma.

Lemma 3. *Let ABC be a triangle, D be a point on side AC, $\phi_1 = \angle DAB$, $\phi_2 = \angle DBA$, $\phi_3 = \angle DBC$, $\phi_4 = \angle DCB$, $AB = c$, $BC = a$, $AD = x$, and $DC = y$. If the numbers a, c, and $\cos \phi_i$ $(1 \le i \le 4)$ are all rational, then numbers x and y are also rational.*

Proof: Note that $x + y = AC = c \cos \phi_1 + a \cos \phi_4$ is rational. Hence x is rational if and only if y is rational. Let $BD = z$. Projecting point D onto the lines AB and BC yields

$$\begin{cases} x \cos \phi_1 + z \cos \phi_2 = c, \\ y \cos \phi_4 + z \cos \phi_3 = a, \end{cases}$$

or, denoting $c_i = \cos \phi_i$ for $i = 1, 2, 3, 4$,

$$\begin{cases} c_1 x + c_2 z = c, \\ c_4 y + c_3 z = a. \end{cases}$$

Eliminating z, we get $(c_1 c_3)x - (c_2 c_4)y = c_3 c - c_2 a$, which is rational. Hence there exist rational numbers, r_1 and r_2, such that

$$\begin{cases} (c_1 c_3)x - (c_2 c_4)y = r_1, \\ x + y = r_2. \end{cases}$$

---

2 points for reducing to this linear system.

We consider two cases.

- In this case, we assume that the determinant of the above system, $c_1 c_3 + c_2 c_4$, is not equal to 0, then this system has a unique solution (x, y) in rational numbers.
- In this case, we assume that the determinant $c_1 c_3 + c_2 c_4 = 0$, or

$$\cos \phi_1 \cos \phi_3 = - \cos \phi_2 \cos \phi_4.$$

Let's denote $\theta = \angle BDC$, then $\phi_2 = \theta - \phi_1$ and $\phi_3 = 180° - (\theta + \phi_4)$. Then the above equation becomes

$$\cos \phi_1 \cos(\theta + \phi_4) = \cos \phi_4 \cos(\theta - \phi_1).$$

by the **Product-to-sum formulas**, we have

$$\cos(\theta + \phi_1 + \phi_4) + \cos(\theta + \phi_4 - \phi_1) = \cos(\theta + \phi_4 - \phi_1) + \cos(\theta - \phi_1 - \phi_4),$$

or

$$\cos(\theta + \phi_1 + \phi_4) = \cos(\theta - \phi_1 - \phi_4).$$

It is possible only if $[\theta + \phi_1 + \phi_4] \pm [\theta - \phi_1 - \phi_4] = 360°$, that is, either $\theta = 180°$ or $\phi_1 + \phi_4 = 180°$, which is impossible because they are angles of triangles.

Thus, the determinant $c_1c_3 + c_2c_4$ is not equal to 0 and x and y are both rational numbers. ∎

---

3 points for solving this system. No points for discussing non-degenerate case.

Now we are ready to prove Lemma 1. Applying the Law of Cosine to triangles ABC, ACD, ABD shows that $\cos \angle BAC$, $\cos \angle CAD$, $\cos \angle ABD$, $\cos \angle ADB$ are all rational. Applying Lemma 1 to triangle ABD shows that both of the segments BP and DP have rational lengths. In exactly the same way, we can show that both of the segments AP and CP have rational lengths.

---

1 point for this final finishing touch.

Remark: In grading this approach, we have some new rules in addition: $1 + 1 = 1$, $1 + 2 = 2$, $1 + 1 + 2 = 2$. Basically, without a complete proof of Lemma 2, there will be at most 2 points given. Partial credits are additive with complete proof of Lemma 3. The possible marks following this approach are 0, 1, 2, 5, 6, 7 (even though a score of 5 is very unlikely).

Note: It's interesting how easy it is to get a gap in the proof of the Lemma 1 by using the core idea of the proof of Lemma 3. Here is an example.

Let us project the intersection point of the diagonals, O, onto the four lines of all sides of the quadrilateral. We get the following 4×4 system of linear equations:

$$\begin{cases} \cos \phi_1 \ x + \cos \phi_2 y = a, \\ \cos \phi_3 y + \cos \phi_4 z = b, \\ \cos \phi_5 z + \cos \phi_6 t = c, \\ \cos \phi_7 t + \cos \phi_8 x = d. \end{cases}$$

Using the **Kramer's Rule**, we conclude that all x, y, z, and t must be rational numbers, for all the corresponding determinants are rational. However, this logic works only if the determinant of the system is different from 0.

Unfortunately, there are many geometric configurations for which the determinant of the system vanishes (for example, this occurs for rectangles), and you cannot make a conclusion of rationality of the segments x, y, z, and t. That's why Lemma 2 plays the central role in the solution to this problem.

3. Let $n \neq 0$. For every sequence of integers

$$A = a_0, a_1, a_2, \ldots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \ldots, n$, define another sequence

$$t(A) = t(a_0), t(a_1), t(a_2), \ldots, t(a_n)$$

by setting $t(a_i)$ to be the number of terms in the sequence A that precede the term a_i and are different from a_i. Show that, starting from any sequence A as above, fewer than n applications of the transformation t lead to a sequence B such that $t(B) = B$.

Solution: Note first that the transformed sequence $t(A)$ also satisfies the inequalities $0 \leq t(a_i) \leq i$, for $i = 0, \ldots, n$. Call any integer sequence that satisfies these inequalities an *index bounded sequence*.

We prove now that that $a_i \leq t(a_i)$, for $i = 0, \ldots, n$. Indeed, this is clear if $a_i = 0$. Otherwise, let $x = a_i > 0$ and $y = t(a_i)$. None of the first x consecutive terms $a_0, a_1, \ldots, a_{x-1}$ is greater than $x - 1$ so they are all different from x and precede x (see the diagram below). Thus $y \geq x$, that is, $a_i \leq t(a_i)$, for $i = 0, \ldots, n$.

index	0	1	...	$x-1$	...	i
A	a_0	a_1	...	a_{x-1}	...	x
$t(A)$	$t(a_0)$	$t(a_1)$	...	$t(a_{x-1})$	...	y

> **1 point for stating and proving this fact.**

This already shows that the sequences stabilize after finitely many applications of the transformation t, because the value of the index i term in index bounded sequences cannot exceed i. Next we prove that if $a_i = t(a_i)$, for some $i = 0, \ldots, n$, then no further applications of t will ever change the index i term. We consider two cases.

- In this case, we assume that $a_i = t(a_i) = 0$. This means that no term on the left of a_i is different from 0, that is, they are all 0. Therefore the first i terms in $t(A)$ will also be 0 and this repeats (see the diagram below).

index	0	1	...	i
A	0	0	...	0
$t(A)$	0	0	...	0

- In this case, we assume that $a_i = t(a_i) = x > 0$. The first x terms are all different from x. Because $t(a_i) = x$, the terms $a_x, a_{x+1}, \ldots, a_{i-1}$ must then all be equal to x. Consequently, $t(a_j) = x$ for $j = x, \ldots, i-1$ and further applications of t cannot change the index i term (see the diagram below).

index	0	1	...	$x-1$	x	$x+1$	...	i
A	a_0	a_1	...	a_{x-1}	x	x	...	x
$t(A)$	$t(a_0)$	$t(a_1)$	...	$t(a_{x-1})$	x	x	...	x

For $0 \leq i \leq n$, the index i entry satisfies the following properties: (i) it takes integer values; (ii) it is bounded above by i; (iii) its value does not decrease under transformation t; and (iv) once it stabilizes under transformation t, it never changes again. This shows that no more than n applications of t lead to a sequence that is stable under the transformation t.

Finally, we need to show that no more than $n - 1$ applications of t is needed to obtain a fixed sequence from an initial $n + 1$-term index bounded sequence $A = (a_0, a_1, \ldots, a_n)$. We induct on n.

For $n = 1$, the two possible index bounded sequences $(a_0, a_1) = (0, 0)$ and $(a_0, a_1) = (0, 1)$ are already fixed by t so we need zero applications of t.

Assume that any index bounded sequences $(a_0, a_1, \ldots, a_n)$ reach a fixed sequence after no more than $n - 1$ applications of t. Consider an index bounded sequence $A = (a_0, a_1, \ldots, a_{n+1})$. It suffices to show that A will be stabilized in no more than n applications of t. We approach indirectly by assume on the contrary that $n + 1$ applications of transformations are needed. This can happen only if $a_{n+1} = 0$ and each application of t increased the index $n + 1$ term by exactly 1. Under transformation t, the resulting value of index term i will not the effected by index term j for $i < j$. Hence by the induction hypothesis, the subsequence $A' = (a_0, a_1, \ldots, a_n)$ will be stabilized in no more than $n - 1$ applications of t. Because index n term is stabilized at value $x \le n$ after no more than $\min\{x, n - 1\}$ applications of t and index $n + 1$ term obtains value x after x exactly applications of t under our current assumptions. We conclude that the index $n + 1$ term would become equal to the index n term after no more than $n - 1$ applications of t. However, once two consecutive terms in a sequence are equal they stay equal and stabilize together. Because the index n term needs no more than $n - 1$ transformations to be stabilized, A can be stabilized in no more than $n - 1$ applications of t, which contradicts our assumption of $n + 1$ applications needed. Thus our assumption was wrong and we need at most n applications of transformation t to stabilize an $(n + 1)$-term index bounded sequence. This completes our inductive proof.

Remark: It seems that it is difficult to reaching later stages of the proof without previous results. Possible marks for this problem are 0, 1, 4, 7.