N2. Let $n \geqslant 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution. To solve the problem it suffices to find three squares and three cards with numbers a, b, c on them such that pairwise sums $a+b, b+c, a+c$ are equal to the chosen squares. By choosing the three consecutive squares $(2k-1)^2, (2k)^2, (2k+1)^2$ we arrive at the triple

$$(a,b,c) = \left(2k^2 - 4k, \quad 2k^2+1, \quad 2k^2+4k\right).$$

We need a value for k such that

$$n \leqslant 2k^2 - 4k, \quad \text{and} \quad 2k^2 + 4k \leqslant 2n.$$

A concrete k is suitable for all n with

$$n \in \left[k^2 + 2k, 2k^2 - 4k + 1\right] =: I_k.$$

For $k \geqslant 9$ the intervals I_k and I_{k+1} overlap because

$$(k+1)^2 + 2(k+1) \leqslant 2k^2 - 4k + 1.$$

Hence $I_9 \cup I_{10} \cup \ldots = [99, \infty)$, which proves the statement for $n \geqslant 99$.

A4. Show that for all real numbers $x_1, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}.$$

Solution 1. If we add t to all the variables then the left-hand side remains constant and the right-hand side becomes

$$H(t) := \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2t|}.$$

Let T be large enough such that both $H(-T)$ and $H(T)$ are larger than the value L of the left-hand side of the inequality we want to prove. Not necessarily distinct points $p_{i,j} := -(x_i + x_j)/2$ together with T and $-T$ split the real line into segments and two rays such that on each of these segments and rays the function $H(t)$ is concave since $f(t) := \sqrt{|\ell + 2t|}$ is concave on both intervals $(-\infty, -\ell/2]$ and $[-\ell/2, +\infty)$. Let $[a, b]$ be the segment containing zero. Then concavity implies $H(0) \geqslant \min\{H(a), H(b)\}$ and, since $H(\pm T) > L$, it suffices to prove the inequalities $H(-(x_i + x_j)/2) \geqslant L$, that is to prove the original inequality in the case when all numbers are shifted in such a way that two variables x_i and x_j add up to zero. In the following we denote the shifted variables still by x_i.

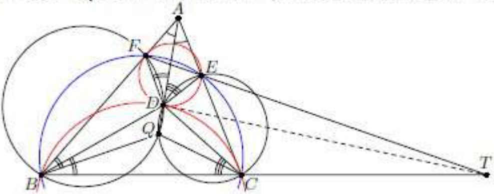
If $i = j$, i.e. $x_i = 0$ for some index i, then we can remove x_i which will decrease both sides by $2\sum_k \sqrt{|x_k|}$. Similarly, if $x_i + x_j = 0$ for distinct i and j we can remove both x_i and x_j which decreases both sides by

$$2\sqrt{2|x_i|} + 2 \cdot \sum_{k \neq i,j}\left(\sqrt{|x_k + x_i|} + \sqrt{|x_k + x_j|}\right).$$

In either case we reduced our inequality to the case of smaller n. It remains to note that for $n = 0$ and $n = 1$ the inequality is trivial.

G7. A point D is chosen inside an acute-angled triangle ABC with $AB > AC$ so that $\angle BAD = \angle DAC$. A point E is constructed on the segment AC so that $\angle ADE = \angle DCB$. Similarly, a point F is constructed on the segment AB so that $\angle ADF = \angle DBC$. A point X is chosen on the line AC so that $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and DXE. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Common remarks. Let Q be the isogonal conjugate of D with respect to the triangle ABC. Since $\angle BAD = \angle DAC$, the point Q lies on AD. Then $\angle QBA = \angle DBC = \angle FDA$, so the points Q, D, F, and B are concyclic. Analogously, the points Q, D, E, and C are concyclic. Thus $AF \cdot AB = AD \cdot AQ = AE \cdot AC$ and so the points B, F, E, and C are also concyclic.



Let T be the intersection of BC and FE.

Claim. $TD^2 = TB \cdot TC = TF \cdot TE$.

Proof. We will prove that the circles (DEF) and (BDC) are tangent to each other. Indeed, using the above arguments, we get

$$\angle BDF = \angle AFD - \angle ABD = (180^\circ - \angle FAD - \angle FDA) - (\angle ABC - \angle DBC)$$
$$= 180^\circ - \angle FAD - \angle ABC = 180^\circ - \angle DAE - \angle FEA = \angle FED + \angle ADE = \angle FED + \angle DCB,$$

which implies the desired tangency.

Since the points B, C, E, and F are concyclic, the powers of the point T with respect to the circles (BDC) and (EDF) are equal. So their radical axis, which coincides with the common tangent at D, passes through T, and hence $TD^2 = TE \cdot TF = TB \cdot TC$. □

Solution 1. Let TA intersect the circle (ABC) again at M. Due to the circles $(BCEF)$ and $(AMCB)$, and using the above Claim, we get $TM \cdot TA = TF \cdot TE = TB \cdot TC = TD^2$; in particular, the points A, M, E, and F are concyclic.

Under the inversion with centre T and radius TD, the point M maps to A, and B maps to C, which implies that the circle (MBD) maps to the circle (ADC). Their common point D lies on the circle of the inversion, so the second intersection point K also lies on that circle, which means $TK = TD$. It follows that the point T and the centres of the circles (KDE) and (ADC) lie on the perpendicular bisector of KD.

Since the center of (ADC) is O_1, it suffices to show now that the points D, K, E, and X are concyclic (the center of the corresponding circle will be O_2).

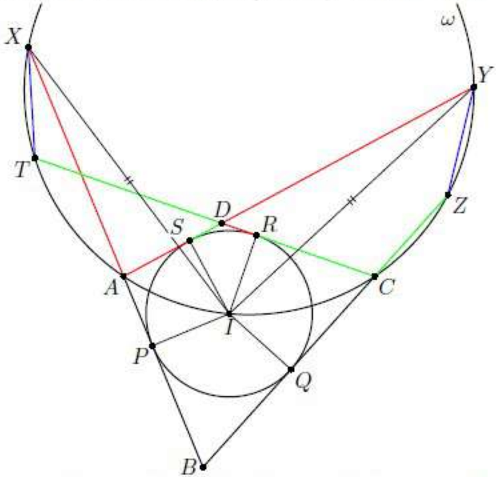
The lines BM, DK, and AC are the pairwise radical axes of the circles $(ABCM)$, $(ACDK)$ and $(BMDK)$, so they are concurrent at some point P. Also, M lies on the circle (AEF), thus

$$\measuredangle(EX, XB) = \measuredangle(CX, XB) = \measuredangle(XC, BC) + \measuredangle(BC, BX) = 2\measuredangle(AC, CB)$$
$$= \measuredangle(AC, CB) + \measuredangle(EF, FA) = \measuredangle(AM, BM) + \measuredangle(EM, MA) = \measuredangle(EM, BM),$$

so the points M, E, X, and B are concyclic. Therefore, $PE \cdot PX = PM \cdot PB = PK \cdot PD$, so the points E, K, D, and X are concyclic, as desired.

G2. Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Prove that the perimeters of the (possibly self-intersecting) quadrilaterals $ADTX$ and $CDYZ$ are equal.

Solution. The point I is the intersection of the external bisector of the angle TCZ with the circumcircle ω of the triangle TCZ, so I is the midpoint of the arc TCZ and $IT = IZ$. Similarly, I is the midpoint of the arc YAX and $IX = IY$. Let O be the centre of ω. Then X and T are the reflections of Y and Z in IO, respectively. So $XT = YZ$.



Let the incircle of $ABCD$ touch AB, BC, CD, and DA at points P, Q, R, and S, respectively.

The right triangles IXP and IYS are congruent, since $IP = IS$ and $IX = IY$. Similarly, the right triangles IRT and IQZ are congruent. Therefore, $XP = YS$ and $RT = QZ$.

Denote the perimeters of $ADTX$ and $CDYZ$ by P_{ADTX} and P_{CDYZ} respectively. Since $AS = AP$, $CQ = RC$, and $SD = DR$, we obtain

$$P_{ADTX} = XT + XA + AS + SD + DT = XT + XP + RT$$
$$= YZ + YS + QZ = YZ + YD + DR + RC + CZ = P_{CDYZ},$$

as required.

C3. A thimblerigger has 2021 thimbles numbered from 1 through 2021. The thimbles are arranged in a circle in arbitrary order. The thimblerigger performs a sequence of 2021 moves; in the k^{th} move, he swaps the positions of the two thimbles adjacent to thimble k.

Prove that there exists a value of k such that, in the k^{th} move, the thimblerigger swaps some thimbles a and b such that $a < k < b$.

Solution. Assume the contrary. Say that the k^{th} thimble is the *central thimble* of the k^{th} move, and its position on that move is the *central position* of the move.

Step 1: Black and white colouring.

Before the moves start, let us paint all thimbles in white. Then, after each move, we repaint its central thimble in black. This way, at the end of the process all thimbles have become black.

By our assumption, in every move k, the two swapped thimbles have the same colour (as their numbers are either both larger or both smaller than k). At every moment, assign the colours of the thimbles to their current positions; then the only position which changes its colour in a move is its central position. In particular, each position is central for exactly one move (when it is being repainted to black).

Step 2: Red and green colouring.

Now we introduce a colouring of the *positions*. If in the k^{th} move, the numbers of the two swapped thimbles are both less than k, then we paint the central position of the move in red; otherwise we paint that position in green. This way, each position has been painted in red or green exactly once. We claim that among any two adjacent positions, one becomes green and the other one becomes red; this will provide the desired contradiction since 2021 is odd.

Consider two adjacent positions A and B, which are central in the a^{th} and in the b^{th} moves, respectively, with $a < b$. Then in the a^{th} move the thimble at position B is white, and therefore has a number greater than a. After the a^{th} move, position A is green and the thimble at position A is black. By the arguments from Step 1, position A contains only black thimbles after the a^{th} step. Therefore, on the b^{th} move, position A contains a black thimble whose number is therefore less than b, while thimble b is at position B. So position B becomes red, and hence A and B have different colours.

A6. Let A be a finite set of (not necessarily positive) integers, and let $m \geqslant 2$ be an integer. Assume that there exist non-empty subsets $B_1, B_2, B_3, \ldots, B_m$ of A whose elements add up to the sums $m^1, m^2, m^3, \ldots, m^m$, respectively. Prove that A contains at least $m/2$ elements.

Solution. Let $A = \{a_1, \ldots, a_k\}$. Assume that, on the contrary, $k = |A| < m/2$. Let

$$s_i := \sum_{j: a_j \in B_i} a_j$$

be the sum of elements of B_i. We are given that $s_i = m^i$ for $i = 1, \ldots, m$.

Now consider all m^m expressions of the form

$$f(c_1, \ldots, c_m) := c_1 s_1 + c_2 s_2 + \ldots + c_m s_m, \ c_i \in \{0, 1, \ldots, m-1\} \text{ for all } i = 1, 2, \ldots, m.$$

Note that every number $f(c_1, \ldots, c_m)$ has the form

$$\alpha_1 a_1 + \ldots + \alpha_k a_k, \ \alpha_i \in \{0, 1, \ldots, m(m-1)\}.$$

Hence, there are at most $(m(m-1)+1)^k < m^{2k} < m^m$ distinct values of our expressions; therefore, at least two of them coincide.

Since $s_i = m^i$, this contradicts the uniqueness of representation of positive integers in the base-m system.