

Language: English Day: 1

Tuesday, July 8, 2014

**Problem 1.** Let  $a_0 < a_1 < a_2 < \cdots$  be an infinite sequence of positive integers. Prove that there exists a unique integer  $n \ge 1$  such that

$$a_n < \frac{a_0 + a_1 + \dots + a_n}{n} \le a_{n+1}.$$

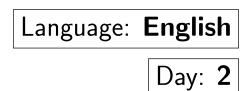
**Problem 2.** Let  $n \ge 2$  be an integer. Consider an  $n \times n$  chessboard consisting of  $n^2$  unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a  $k \times k$  square which does not contain a rook on any of its  $k^2$  unit squares.

**Problem 3.** Convex quadrilateral ABCD has  $\angle ABC = \angle CDA = 90^{\circ}$ . Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^{\circ}, \quad \angle THC - \angle DTC = 90^{\circ}.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.





Wednesday, July 9, 2014

**Problem 4.** Points P and Q lie on side BC of acute-angled triangle ABC so that  $\angle PAB = \angle BCA$  and  $\angle CAQ = \angle ABC$ . Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

**Problem 5.** For each positive integer n, the Bank of Cape Town issues coins of denomination  $\frac{1}{n}$ . Given a finite collection of such coins (of not necessarily different denominations) with total value at most  $99 + \frac{1}{2}$ , prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

**Problem 6.** A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least  $\sqrt{n}$  of the lines blue in such a way that none of its finite regions has a completely blue boundary.

*Note:* Results with  $\sqrt{n}$  replaced by  $c\sqrt{n}$  will be awarded points depending on the value of the constant c.