EGMO 2023 – Problems and solutions

Problems

Problem 1. There are $n \geqslant 3$ positive real numbers $a_1, a_2, \dots, a_n$. For each $1 \leqslant i \leqslant n$ we let $b_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ (here we define a_0 to be a_n and a_{n+1} to be a_1). Assume that for all i and j in the range 1 to n, we have $a_i \leqslant a_j$ if and only if $b_i \leqslant b_j$.

Prove that $a_1 = a_2 = \cdots = a_n$.

Problem 2. We are given an acute triangle ABC. Let D be the point on its circumcircle such that AD is a diameter. Suppose that points K and L lie on segments AB and AC, respectively, and that DK and DL are tangent to circle AKL.

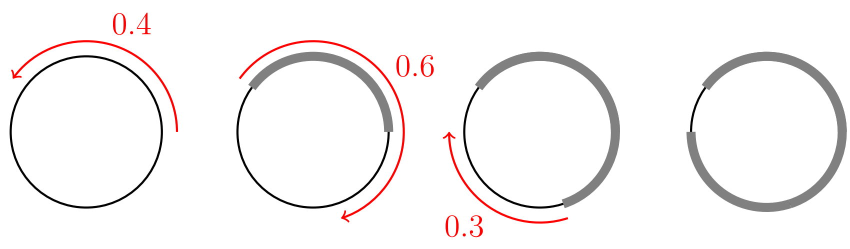
Show that line KL passes through the orthocentre of ABC.

The altitudes of a triangle meet at its orthocentre.

Problem 3. Let k be a positive integer. Lexi has a dictionary $\mathcal{D}$ consisting of some k-letter strings containing only the letters A and B. Lexi would like to write either the letter A or the letter B in each cell of a $k \times k$ grid so that each column contains a string from $\mathcal{D}$ when read from top-to-bottom and each row contains a string from $\mathcal{D}$ when read from left-to-right.

What is the smallest integer m such that if $\mathcal{D}$ contains at least m different strings, then Lexi can fill her grid in this manner, no matter what strings are in $\mathcal{D}$?

Problem 4. Turbo the snail sits on a point on a circle with circumference 1. Given an infinite sequence of positive real numbers $c_1, c_2, c_3, \dots$, Turbo successively crawls distances $c_1, c_2, c_3, \dots$ around the circle, each time choosing to crawl either clockwise or counter-clockwise.
For example, if the sequence $c_1, c_2, c_3, \dots$ is $0.4, 0.6, 0.3, \dots$, then Turbo may start crawling as follows:



Determine the largest constant $C > 0$ with the following property: for every sequence of positive real numbers $c_1, c_2, c_3, \dots$ with $c_i < C$ for all i, Turbo can (after studying the sequence) ensure that there is some point on the circle that it will never visit or crawl across.

Problem 5. We are given a positive integer $s \geqslant 2$. For each positive integer k, we define its *twist* k' as follows: write k as $as + b$, where a, b are non-negative integers and $b < s$, then $k' = bs + a$. For the positive integer n, consider the infinite sequence $d_1, d_2, \dots$ where $d_1 = n$ and d_{i+1} is the twist of d_i for each positive integer i.

Prove that this sequence contains 1 if and only if the remainder when n is divided by $s^2 - 1$ is either 1 or s.

Problem 6. Let ABC be a triangle with circumcircle Ω. Let S_b and S_c respectively denote the midpoints of the arcs AC and AB that do not contain the third vertex. Let N_a denote the midpoint of arc BAC (the arc BC containing A). Let I be the incentre of ABC. Let ω_b be the circle that is tangent to AB and internally tangent to Ω at S_b, and let ω_c be the circle that is tangent to AC and internally tangent to Ω at S_c. Show that the line IN_a, and the line through the intersections of ω_b and ω_c, meet on Ω.

The incentre of a triangle is the centre of its incircle, the circle inside the triangle that is tangent to all three sides.

Solutions

Problem 1. There are $n \geqslant 3$ positive real numbers $a_1, a_2, \ldots, a_n$. For each $1 \leqslant i \leqslant n$ we let $b_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ (here we define a_0 to be a_n and a_{n+1} to be a_1). Assume that for all i and j in the range 1 to n, we have $a_i \leqslant a_j$ if and only if $b_i \leqslant b_j$.

Prove that $a_1 = a_2 = \cdots = a_n$.

Solution 1. Suppose that not all a_i are equal. Consider an index i such that a_i is maximal and $a_{i+1} < a_i$. Then

$$b_i = \frac{a_{i-1} + a_{i+1}}{a_i} < \frac{2a_i}{a_i} = 2.$$

But since a_i is maximal, b_i is also maximal, so we must have $b_j < 2$ for all $j \in \{1, 2, \ldots, n\}$.

However, consider the product $b_1 b_2 \ldots b_n$. We have

$$\begin{aligned} b_1 b_2 \ldots b_n &= \frac{a_n + a_2}{a_1} \cdot \frac{a_1 + a_3}{a_2} \cdot \ldots \cdot \frac{a_{n-1} + a_1}{a_n} \\ &\geqslant 2^n \frac{\sqrt{a_n a_2}\sqrt{a_1 a_3} \ldots \sqrt{a_{n-1} a_1}}{a_1 a_2 \ldots a_n} \\ &= 2^n, \end{aligned}$$

where we used the inequality $x + y \geqslant 2\sqrt{xy}$ for $x = a_{i-1}, y = a_{i+1}$ for all $i \in \{1, 2, \ldots, n\}$ in the second row.

Since the product of all b_i is at least 2^n, at least one of them must be greater than 2, which is a contradiction with the previous conclusion.

Thus, all a_i must be equal. □

Solution 2. This is a version of Solution 1 without use of proof by contradiction.

Taking a_i such that it is maximal among $a_1, \ldots, a_n$, we obtain $b_i \leqslant 2$. Thus $b_i \leqslant 2$ for all $j \in \{1, 2, \ldots, n\}$.

The second part of Solution 1 then gives $2^n \geqslant b_1 \cdots b_n \geqslant 2^n$, which together with $b_i \leqslant 2$ for all $j \in \{1, 2, \ldots, n\}$ implies that $b_j = 2$ for all $j \in \{1, 2, \ldots, n\}$. Since we have $b_1 = b_2 = \cdots = b_n$, the condition that $a_i \leqslant a_j \iff b_i \leqslant b_j$ gives that $a_1 = a_2 = \cdots = a_n$. □

Solution 3. We first show that $b_j \leqslant 2$ for all j as in Solution 2. Then

$$\begin{aligned} 2n \geqslant & b_1 + \cdots + b_n = \frac{a_n}{a_1} + \frac{a_2}{a_1} + \frac{a_1}{a_2} + \frac{a_3}{a_2} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_1}{a_n} \\ \geqslant & 2n \sqrt[n]{\frac{a_n}{a_1} \cdot \frac{a_2}{a_1} \cdot \frac{a_1}{a_2} \cdot \frac{a_3}{a_2} \cdots \frac{a_{n-1}}{a_n} \cdot \frac{a_1}{a_n}} = 2n \cdot 1 = 2n, \end{aligned}$$

where we used the AM-GM inequality.

It follows that all b_j's are equal which as in Solution 2 gives $a_1 = a_2 = \cdots = a_n$. □

Solution 4. By assumption $a_i b_i = a_{i-1} + a_{i+1}$ for $i = \{1, 2, \ldots, n\}$, hence,

$$\sum_{i=1}^{n} a_i b_i = 2 \sum_{i=1}^{n} a_i.$$

Since $a_i \leqslant a_j$ if and only if $b_i \leqslant b_j$, the Chebyshev's inequality implies

$$\left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} b_i\right) \leqslant n \cdot \sum_{i=1}^{n} a_i b_i = 2n \cdot \sum_{i=1}^{n} a_i$$

and so $\sum_{i=1}^{n} b_i \leqslant 2n$. On the other hand, we have

$$\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} \frac{a_{i-1}}{a_i} + \sum_{i=1}^{n} \frac{a_{i+1}}{a_i} = \sum_{i=1}^{n} \frac{a_{i-1}}{a_i} + \sum_{i=1}^{n} \frac{a_i}{a_{i-1}} = \sum_{i=1}^{n} \left(\frac{a_{i-1}}{a_i} + \frac{a_i}{a_{i-1}}\right),$$

so we can use the AM-GM inequality to estimate

$$\sum_{i=1}^{n} b_i \geqslant \sum_{i=1}^{n} 2\sqrt{\frac{a_{i-1}}{a_i} \cdot \frac{a_i}{a_{i-1}}} = 2n.$$

We conclude that we must have equalities in all the above, which implies $\frac{a_{i-1}}{a_i} = \frac{a_i}{a_{i-1}}$ and consequently $a_i = a_{i-1}$ for all positive integers i. Hence, all a's are equal. □

Solution 5. As in Solution 4 we show that $\sum_{i=1}^{n} b_i \leqslant 2n$ and as in Solution 1 we show that $\prod_{i=1}^{n} b_i \geqslant 2^n$. We now use the AM-GM inequality and the first inequality to get

$$\prod_{i=1}^{n} b_i \leqslant \left(\frac{1}{n} \sum_{i=1}^{n} b_i\right)^n \leqslant \left(\frac{1}{n} \cdot 2n\right)^n = 2^n.$$

This implies that we must have equalities in all the above. In particular, we have equality in the AM-GM inequality, so all b's are equal and as in Solution 2 then all a's are equal. □

Solution 6. Let a_i to be minimal and a_j maximal among all a's. Then

$$b_j = \frac{a_{j-1} + a_{j+1}}{a_j} \leqslant \frac{2a_j}{a_j} = 2 = \frac{2a_i}{a_i} \leqslant \frac{a_{i-1} + a_{i+1}}{a_i} = b_i$$

and by assumption $b_i \leqslant b_j$. Hence, we have equalities in the above so $b_j = 2$ so $a_{j-1} + a_{j+1} = 2a_j$ and therefore $a_{j-1} = a_j = a_{j+1}$. We have thus shown that the two neighbors of a maximal a are also maximal. By an inductive argument all a's are maximal, hence equal. □

Solution 7. Choose an arbitrary index i and assume without loss of generality that $a_i \leqslant a_{i+1}$. (If the opposite inequality holds, reverse all the inequalities below.) By induction we will show that for each $k \in \mathbb{N}_0$ the following two inequalities hold

$$a_{i+1+k} \geqslant a_{i-k} \tag{1}$$

$$a_{i+1+k} a_{i+1-k} \geqslant a_{i-k} a_{i+k} \tag{2}$$

(where all indices are cyclic modulo n). Both inequalities trivially hold for $k = 0$.

Assume now that both inequalities hold for some $k \geqslant 0$. The inequality $a_{i+1+k} \geqslant a_{i-k}$ implies $b_{i+1+k} \geqslant b_{i-k}$, so

$$\frac{a_{i+k} + a_{i+2+k}}{a_{i+1+k}} \geqslant \frac{a_{i-1-k} + a_{i+1-k}}{a_{i-k}}.$$

We may rearrange this inequality by making a_{i+2+k} the subject so

$$a_{i+2+k} \geqslant \frac{a_{i+1+k}a_{i-1-k}}{a_{i-k}} + \frac{a_{i+1-k}a_{i+1+k} - a_{i+k}a_{i-k}}{a_{i-k}} \geqslant \frac{a_{i+1+k}a_{i-1-k}}{a_{i-k}},$$

where the last inequality holds by (2). It follows that

$$a_{(i+1)+(k+1)}a_{(i+1)-(k+1)} \geqslant a_{i+(k+1)}a_{i-(k+1)},$$

i.e. the inequality (2) holds also for $k+1$. Using (1) we now get

$$a_{(i+1)+(k+1)} \geqslant \frac{a_{i+k+1}}{a_{i-k}} a_{i-(k+1)} \geqslant a_{i-(k+1)},$$

i.e. (1) holds for $k+1$.

Now we use the inequality (1) for $k = n-1$. We get $a_i \geqslant a_{i+1}$, and since at the beginning we assumed $a_i \leqslant a_{i+1}$, we get that any two consecutive a's are equal, so all of them are equal. $\square$

Solution 8. We first prove the following claim by induction:

Claim 1: If $a_k a_{k+2} < a_{k+1}^2$ and $a_k < a_{k+1}$, then $a_j a_{j+2} < a_{j+1}^2$ and $a_j < a_{j+1}$ for all j.

We assume that $a_i a_{i+2} < a_{i+1}^2$ and $a_i < a_{i+1}$, and then show that $a_{i-1}a_{i+1} < a_i^2$ and $a_{i-1} < a_i$.

Since $a_i \leq a_{i+1}$ we have that $b_i \leq b_{i+1}$. By plugging in the definition of b_i and b_{i+1} we have that

$$a_{i+1}a_{i-1} + a_{i+1}^2 \leq a_i^2 + a_{i+2}a_i. \tag{3}$$

Using $a_i a_{i+2} < a_{i+1}^2$ we get that

$$a_{i+1}a_{i-1} < a_i^2. \tag{4}$$

Since $a_i < a_{i+1}$ we have that $a_{i-1} < a_i$, which concludes the induction step and hence proves the claim.

We cannot have that $a_j < a_{j+1}$ for all indices j. Similar as in the above claim, one can prove that if $a_k a_{k+2} < a_{k+1}^2$ and $a_{k+2} < a_{k+1}$, then $a_{j+1} < a_j$ for all j, which also cannot be the case. Thus we have that $a_k a_{k+2} \geq a_{k+1}^2$ for all indices k.

Next observe (e.g. by taking the product over all indices) that this implies $a_k a_{k+2} = a_{k+1}^2$ for all indices k, which is equivalent to $b_k = b_{k+1}$ for all k and hence $a_{k+1} = a_k$ for all k. $\square$

Solution 9. Define $c_i := \frac{a_i}{a_{i+1}}$, then $b_i = c_{i-1} + 1/c_i$. Assume that not all c_i are equal to 1. Since, $\prod_{i=1}^n c_i = 1$ there exists a k such that $c_k \geqslant 1$. From the condition given in the problem statement for $(i,j) = (k,k+1)$ we have

$$c_k \geqslant 1 \iff c_{k-1} + \frac{1}{c_k} \geqslant c_k + \frac{1}{c_{k+1}} \iff c_{k-1}c_kc_{k+1} + c_{k+1} \geqslant c_k^2 c_{k+1} + c_k. \tag{5}$$

Now since $c_{k+1} \leqslant c_k^2 c_{k+1}$, it follows that

$$c_{i-1}c_ic_{i+1} \geqslant c_i \implies (c_{i-1} \geqslant 1 \text{ or } c_{i+1} \geqslant 1)\,. \tag{6}$$

So there exist a set of at least 2 consecutive integers, such that the corresponding c_i are greater or equal to one. By the innitial assumption there must exist an index ℓ, such that $c_{\ell-1}, c_\ell \geqslant 1$ and $c_{\ell+1} < 1$. We distinguish two cases:

Case 1: $c_\ell > c_{\ell-1} \geqslant 1$
From $c_{\ell-1}c_\ell c_{\ell+1} < c_\ell^2 c_{\ell+1}$ and the inequality (5), we get that $c_{\ell+1} > c_\ell \geqslant 1$, which is a contradiction to our choice of ℓ.

Case 2: $c_{\ell-1} \geqslant c_\ell \geqslant 1$
Once again looking at the inequality (5) we can find that

$$c_{\ell-2}c_{\ell-1}c_\ell \geqslant c_{\ell-1}^2 c_\ell \implies c_{\ell-2} \geqslant c_{\ell-1}. \tag{7}$$

Note that we only needed $c_{\ell-1} \geqslant c_\ell \geqslant 1$ to show $c_{\ell-2} \geqslant c_{\ell-1} \geqslant 1$. So using induction we can easily show $c_{\ell-s-1} \geqslant c_{\ell-s}$ for all s.
So

$$c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n \leqslant c_1 \tag{8}$$

a contradiction to our innitial assumption.

So our innitial assumtion must have been wrong, which implies that all the a_i must have been equal from the start. □

Problem 2. We are given an acute triangle ABC. Let D be the point on its circumcircle such that AD is a diameter. Suppose that points K and L lie on segments AB and AC, respectively, and that DK and DL are tangent to circle AKL.

Show that line KL passes through the orthocentre of ABC.

The altitudes of a triangle meet at its orthocentre.

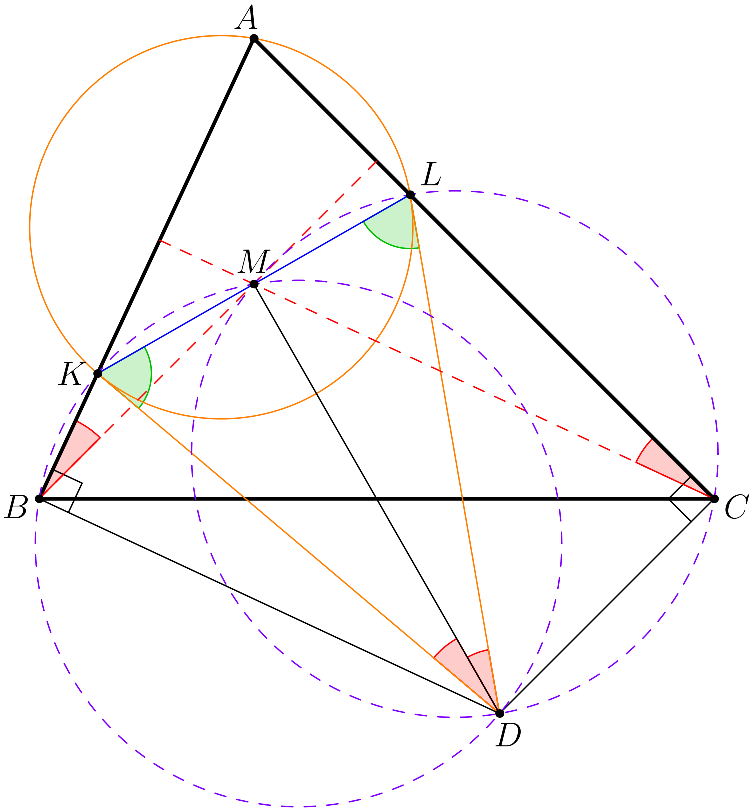


Figure 1: Diagram to solution 1

Solution 1. Let M be the midpoint of KL. We will prove that M is the orthocentre of ABC. Since DK and DL are tangent to the same circle, $|DK| = |DL|$ and hence $DM \perp KL$. The theorem of Thales in circle ABC also gives $DB \perp BA$ and $DC \perp CA$. The right angles then give that quadrilaterals $BDMK$ and $DMLC$ are cyclic.

If $\angle BAC = \alpha$, then clearly $\angle DKM = \angle MLD = \alpha$ by angle in the alternate segment of circle AKL, and so $\angle MDK = \angle LDM = \frac{\pi}{2} - \alpha$, which thanks to cyclic quadrilaterals gives $\angle MBK = \angle LCM = \frac{\pi}{2} - \alpha$. From this, we have $BM \perp AC$ and $CM \perp AB$, and so M indeed is the orthocentre of ABC. □

Solution 2. Preliminaries

Let ABC be a triangle with circumcircle Γ. Let X be a point in the plane. The Simson line (Wallace-Simson line) is defined via the following theorem. Drop perpendiculars from X to each of the three side lines of ABC. The feet of these perpendiculars are collinear (on the Simson line of X) if and only if X lies of Γ. The Simson line of X in the circumcircle bisects the line segment XH where H is the orthocentre of triangle ABC. See Figure 2

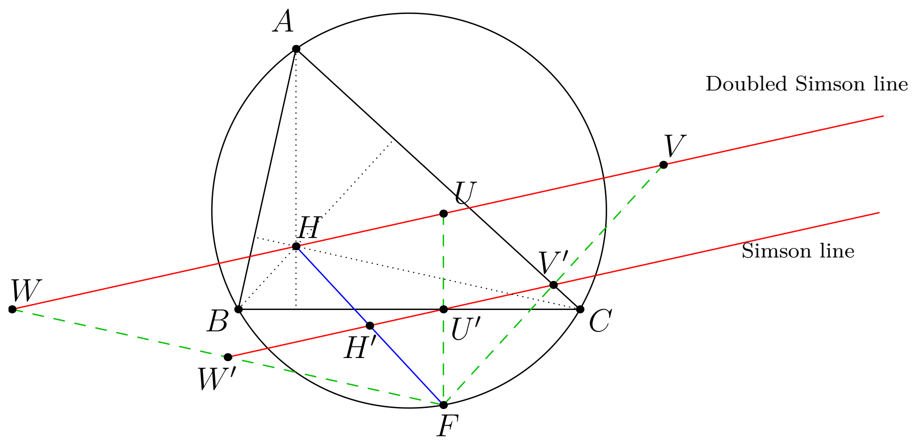


Figure 2: The Wallace-Simson configuration

When X is on Γ, we can enlarge from X with scale factor 2 (a homothety) to take the Simson line to the *doubled* Simson line which passes through the orthocentre H and contains the reflections of X in each of the three sides of ABC.

Solution of the problem

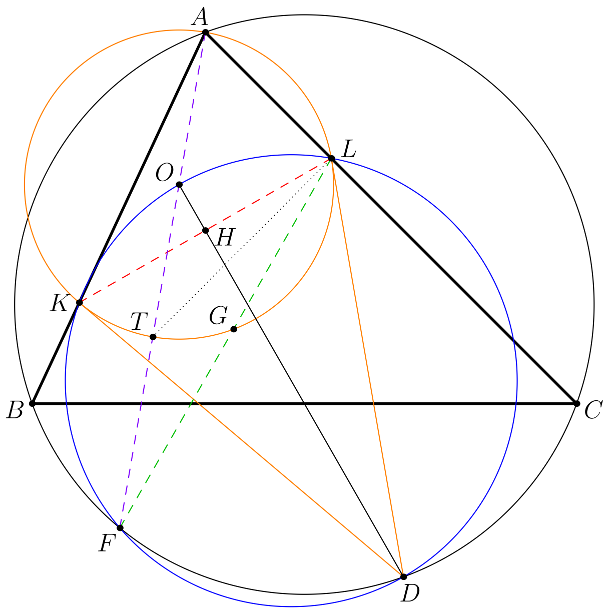


Figure 3: Three circles do the work

Let Γ be the circle ABC, Σ be the circle AKL with centre O, and Ω be the circle on diameter OD so K and L are on this circle by converse of Thales. Let Ω and Γ meet at D and F. By Thales in both circles, $\angle AFD$ and $\angle OFD$ are both right angles so AOF is a line. Let AF meet Σ again at T so AT (containing O) is a diameter of this circle and by Thales, $TL \perp AC$.

Let G (on Σ) be the reflection of K in AF. Now AT is the internal angle bisector of $\angle GAK$ so, by an upmarket use of angles in the same segment (of Σ), TL is the internal angle bisector of $\angle GLK$. Thus the line GL is the reflection of the line KL in TL, and so also the reflection of KL in the line AC (internal and external angle bisectors).

Our next project is to show that LGF are collinear. Well $\angle FLK = \angle FOK$ (angles in the same segment of Ω) and $\angle GLK = \angle GAK$ (angles in the same segment of Σ) $= 2\angle OAK$ (AKG is isosceles with apex A) $= \angle TOK$ (since OAK is isosceles with apex O, and this is an external angle at O). The point T lies in the interior of the line segment FO so $\angle TOK = \angle FOK$. Therefore $\angle FLK = \angle GLK$ so LGF is a line.

Now from the second paragraph, F is on the reflection of KL in AC. By symmetry, F is also on the reflection of KL in AB. Therefore the reflections of F in AB and AC are both on KL which must therefore be the doubled Wallace-Simson line of F. Therefore the orthocentre of ABC lies on KL. □

Solution 3. Let H be the orthocentre of triangle ABC and Σ the circumcircle of AKL with centre O. Let Ω be the circle with diameter OD, which contains K and L by Thales, and let Γ be the circumcircle of ABC containing D. Denote the second intersection of Ω and Γ by F. Since OD and AD are diameters of Ω and Γ we have $\angle OFD = \frac{\pi}{2} = \angle AFD$, so the points A, O, F are collinear. Let M and N be the second intersections of CH and BH with Γ, respectively. It is well-known that M and N are the reflections of H in AB and AC, respectively (because $\angle NCA = \angle NBA = \angle ACM = \angle ABM$). By collinearity of A, O, F and the angles in Γ we have

$$\angle NFO = \angle NFA = \angle NBA = \frac{\pi}{2} - \angle BAC = \frac{\pi}{2} - \angle KAL.$$

Since DL is tangent to Σ we obtain

$$\angle NFO = \frac{\pi}{2} - \angle KLD = \angle LDO,$$

where the last equality follows from the fact that OD is bisector of $\angle LDK$ since LD

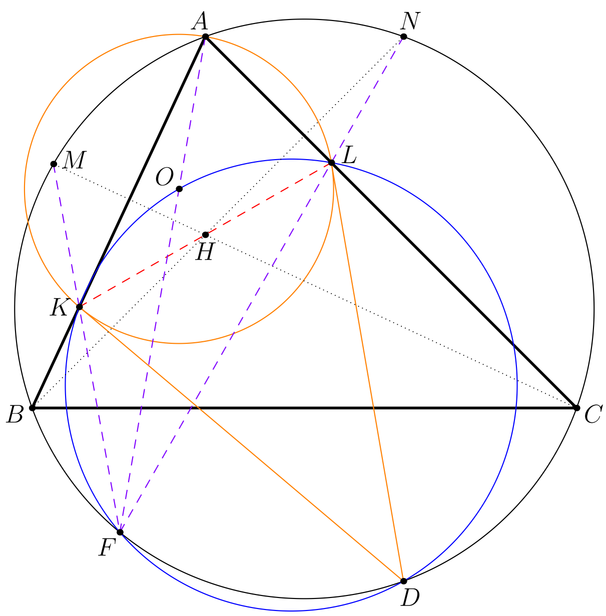


Figure 4: Diagram to Solution 3

and KD are tangent to Σ. Furthermore, $\angle LDO = \angle LFO$ since these are angles in Ω. Hence, $\angle NFO = \angle LFO$, which implies that points N, L, F are collinear. Similarly points M, K, F are collinear. Since N and M are reflections of H in AC and AB we have

$$\angle LHN = \angle HNL = \angle BNF = \angle BMF = \angle BMK = \angle KHB.$$

Hence,

$$\angle LHK = \angle LHN + \angle NHK = \angle KHB + \angle NHK = \pi$$

and the points L, H, K are collinear. □

Solution 4. As in Solution 3 let M and N be the reflections of the orthocentre in AB and AC. Let $\angle BAC = \alpha$. Then $\angle NDM = \pi - \angle MAN = \pi - 2\alpha$.

Let MK and NL intersect at F. See Figure 3.

Claim. $\angle NFM = \pi - 2\alpha$, so F lies on the circumcircle.

Proof. Since KD and LD are tangents to circle AKL, we have $|DK| = |DL|$ and $\angle DKL = \angle KLD = \alpha$, so $\angle LDK = \pi - 2\alpha$.

By definition of M, N and D, $\angle MND = \angle AND - \angle ANM = \frac{\pi}{2} - (\frac{\pi}{2} - \alpha) = \alpha$ and analogously $\angle DMN = \alpha$. Hence $|DM| = |DN|$.

From $\angle NDM = \angle LDK = \pi - 2\alpha$ if follows that $\angle LDN = \angle KDM$. Since $|DK| = |DL|$ and $|DM| = |DN|$, triangles MDK and NDL are related by a rotation about D through angle $\pi - 2\alpha$, and hence the angle between MK and NL is $\pi - 2\alpha$, which proved the claim. □

We now finish as in Solution 3:

$$\angle MHK = \angle KMH = \angle FMC = \angle FAC,$$

$$\angle LHN = \angle HNL = \angle BNF = \angle BAF.$$

As $\angle BAF + \angle FAC = \alpha$, we have $\angle LHK = \alpha + \angle NHM = \alpha + \pi - \alpha = \pi$, so H lies on KL. □

Solution 5. Since AD is a diameter, it is well known that $DBHC$ is a parallelogram (indeed, both BD and CH are perpendicular to AB, hence parallel, and similarly for $DC \parallel BH$). Let B', C' be the reflections of D in lines AKB and ALC, respectively; since ABD and ACD are right angles, these are also the factor-2 homotheties of B and C with respect to D, hence H is the midpoint of $B'C'$. We will prove that $B'KC'L$ is a parallelogram: it will then follow that the midpoint of $B'C'$, which is H, is also the midpoint of KL, and in particular is on the line, as we wanted to show.

We will prove $B'KC'L$ is a parallelogram by showing that $B'K$ and $C'L$ are the same length and direction. Indeed, for lengths we have $KB' = KD = LD = LC'$, where the first and last equalities arise from the reflections defining B' and C', and the middle one is equality of tangents. For directions, let α, β, γ denote the angles of triangle AKL. Immediate angle chasing in the circle AKL, and the properties of the reflections, yield

$$\angle C'LC = \angle CLD = \angle AKL = \beta$$
$$\angle BKB' = \angle DKB = \angle KLA = \gamma$$
$$\angle LDK = 2\alpha - \pi$$

and therefore in directed angles (mod 2π) we have

$$\angle(C'L, B'K) = \angle C'LC + \angle CLD + \angle LDK + \angle DKB + \angle BKB' = 2\alpha + 2\beta + 2\gamma - \pi = \pi$$

and hence $C'L$ and $B'K$ are parallel and in opposite directions, i.e. $C'L$ and KB' are in the same direction, as claimed. □

Comment. While not necessary for the final solution, the following related observation motivates how the fact that H is the midpoint of KL (and therefore $B'KC'L$ is a parallelogram) was first conjectured. We have $AB' = AD = AC'$ by the reflections, i.e. $B'AC'$ is an isosceles triangle with H being the midpoint of the base. Thus AH is the median, altitude and angle bisector in $B'AC'$, thus $\angle B'AK + \angle KAH = \angle HAL + \angle LAC'$. Since from the reflections we also have $\angle B'AK = \angle KAD$ and $\angle DAL = \angle LAC'$ it follows that $\angle HAL = \angle KAD$ and $\angle KAH = \angle DAL$. Since D is the symmedian point in AKL, the angle conjugation implies AH is the median line of KL. Thus, if H is indeed on KL (as the problem assures us), it can only be the midpoint of KL.

Solution 6. There are a number of "phantom point" arguments which define K' and L' in terms of angles and then deduce that these points are actually K and L.

Note: In these solutions it is necessary to show that K and L are uniquely determined by the conditions of the problem. One example of doing this is the following:

To prove uniqueness of K and L, let us consider that there exist two other points K' and L' that satisfy the same properties (K' on AB and L' on AC such that DK' and DL' are tangent to the circle $AK'L'$).
Then, we have that $DK = DL$ and $DK' = DL'$. We also have that $\angle KDL = \angle K'DL' = \pi - 2\angle A$. Hence, we deduce $\angle KDK' = \angle KDL - \angle K'DL = \angle K'DL' - \angle K'DL = \angle LDL'$ Thus we have that $\triangle KDK' \equiv \triangle LDL'$, so we deduce $\angle DKA = \angle DKK' = \angle DLL' = \pi - \angle ALD$. This implies that $AKDL$ is concyclic, which is clearly a contradiction since $\angle KAL + \angle KDL = \pi - \angle BAC$. □

Solution 7. We will use the usual complex number notation, where we will use a capital letter (like Z) to denote the point associated to a complex number (like z). Consider $\triangle AKL$ on the unit circle. So, we have $a \cdot \bar{a} = k \cdot \bar{k} = l \cdot \bar{l} = 1$ As point D is the intersection of the tangents to the unit circle at K and L, we have that

$$d = \frac{2kl}{k+l} \text{ and } \bar{d} = \frac{2}{k+l}$$

Defining B as the foot of the perpendicular from D on the line AK, and C as the foot of the perpendicular from D on the line AL , we have the formulas:

$$b = \frac{1}{2}\left(d + \frac{(a-k)\bar{d} + \bar{a}k - a\bar{k}}{\bar{a} - \bar{k}}\right)$$

$$c = \frac{1}{2}\left(d + \frac{(a-l)\bar{d} + \bar{a}l - a\bar{l}}{\bar{a} - \bar{l}}\right)$$

Simplyfing these formulas, we get:

$$b = \frac{1}{2}\left(d + \frac{(a-k)\frac{2}{k+l} + \frac{k}{a} - \frac{a}{k}}{\frac{1}{a} - \frac{1}{k}}\right) = \frac{1}{2}\left(d + \frac{\frac{2(a-k)}{k+l} + \frac{k^2-a^2}{ak}}{\frac{k-a}{ak}}\right)$$

$$b = \frac{1}{2}\left(\frac{2kl}{k+l} - \frac{2ak}{k+l} + (a+k)\right) = \frac{k(l-a)}{k+l} + \frac{1}{2}(k+a)$$

$$c = \frac{1}{2}\left(d + \frac{(a-l)\frac{2}{k+l} + \frac{l}{a} - \frac{a}{l}}{\frac{1}{a} - \frac{1}{l}}\right) = \frac{1}{2}\left(d + \frac{\frac{2(a-l)}{k+l} + \frac{l^2-a^2}{al}}{\frac{l-a}{al}}\right)$$

$$c = \frac{1}{2}\left(\frac{2kl}{k+l} - \frac{2al}{k+l} + (a+l)\right) = \frac{l(k-a)}{k+l} + \frac{1}{2}(l+a)$$

Let O be the the circumcenter of triangle $\triangle ABC$. As AD is the diameter of this circle, we have that:

$$o = \frac{a+d}{2}$$

Defining H as the orthocentre of the $\triangle ABC$, we get that:

$$h = a + b + c - 2 \cdot o = a + \left(\frac{k(l-a)}{k+l} + \frac{1}{2}(k+a)\right) + \left(\frac{l(k-a)}{k+l} + \frac{1}{2}(l+a)\right) - (a+d)$$

$$h = a + \frac{2kl}{k+l} - \frac{a(k+l)}{k+l} + \frac{1}{2}k + +\frac{1}{2}l + +a - \left(a + \frac{2kl}{k+l}\right)$$

$$h = \frac{1}{2}(k+l)$$

Hence, we conclude that H is the midpoint of KL, so H, K, L are collinear. □

Solution 8. Let us employ the barycentric coordinates. Set $A(1,0,0)$, $K(0,1,0)$, $L(0,0,1)$. The tangent at K of (AKL) is $a^2z + c^2x = 0$, and the tangent of of L at (AKL) is $a^2y + b^2x = 0$. Their intersection is

$$D(-a^2 : b^2 : c^2).$$

Since $B \in AK$, we can let $B(1-t, t, 0)$. Solving for $\overrightarrow{AB} \cdot \overrightarrow{BD} = 0$ gives

$$t = \frac{3b^2 + c^2 - a^2}{2(b^2 + c^2 - a^2)} \implies B = \left(\frac{-a^2 - b^2 + c^2}{2(b^2 + c^2 - a^2)}, \frac{-a^2 + 3b^2 + c^2}{2(b^2 + c^2 - a^2)}, 0\right).$$

Likewise, C has the coordinate

$$C = \left(\frac{-a^2 + b^2 - c^2}{2(b^2 + c^2 - a^2)}, 0, \frac{-a^2 + b^2 + 3c^2}{2(b^2 + c^2 - a^2)}\right).$$

The altitude from B for triangle ABC is

$$-b^2\left(x-z-\frac{-a^2-b^2+c^2}{2(b^2+c^2-a^2)}\right)+(c^2-a^2)\left(y-\frac{-a^2+3b^2+c^2}{2(b^2+c^2-a^2)}\right)=0.$$

Also the altitude from C for triangle ABC is

$$-c^2\left(x-y-\frac{-a^2+b^2-c^2}{2(b^2+c^2-a^2)}\right)+(a^2-b^2)\left(z-\frac{-a^2+b^2+3c^2}{2(b^2+c^2-a^2)}\right)=0.$$

The intersection of these two altitudes, which is the orthocenter of triangle ABC, has the barycentric coordinate

$$H=(0,1/2,1/2),$$

which is the midpoint of the segment KL. □

Problem 3. Let k be a positive integer. Lexi has a dictionary $\mathcal{D}$ consisting of some k-letter strings containing only the letters A and B. Lexi would like to write either the letter A or the letter B in each cell of a $k \times k$ grid so that each column contains a string from $\mathcal{D}$ when read from top-to-bottom and each row contains a string from $\mathcal{D}$ when read from left-to-right.

What is the smallest integer m such that if $\mathcal{D}$ contains at least m different strings, then Lexi can fill her grid in this manner, no matter what strings are in $\mathcal{D}$?

Solution. We claim the minimum value of m is 2^{k-1}.

Firstly, we provide a set $\mathcal{S}$ of size $2^{k-1} - 1$ for which Lexi cannot fill her grid. Consider the set of all length-k strings containing only As and Bs which end with a B, and remove the string consisting of k Bs. Clearly there are 2 independent choices for each of the first $k-1$ letters and 1 for the last letter, and since exactly one string is excluded, there must be exactly $2^{k-1} - 1$ strings in this set.

Suppose Lexi tries to fill her grid. For each row to have a valid string, it must end in a B. But then the right column would necessarily contain k Bs, and not be in our set. Thus, Lexi cannot fill her grid with our set, and we must have $m \geqslant 2^{k-1}$.

Now, consider any set $\mathcal{S}$ with at least 2^{k-1} strings. Clearly, if $\mathcal{S}$ contained either the uniform string with k As or the string with k Bs, then Lexi could fill her grid with all of the relevant letters and each row and column would contain that string.

Consider the case where $\mathcal{S}$ contains neither of those strings. Among all 2^k possible length-k strings with As and Bs, each has a complement which corresponds to the string with Bs in every position where first string had As and vice-versa. Clearly, the string with all As is paired with the string with all Bs. We may assume that we do not take the two uniform strings and thus applying the pigeonhole principle to the remaining set of strings, we must have two strings which are complementary.

Let this pair of strings be $\ell, \ell' \in \mathcal{S}$ in some order. Define the set of indices $\mathcal{J}$ corresponding to the As in ℓ and thus the Bs in ℓ', and all other indices (not in $\mathcal{J}$) correspond to Bs in ℓ (and thus As in ℓ'). Then, we claim that Lexi puts an A in the cell in row r, column c if $r, c \in \mathcal{J}$ or $r, c \notin \mathcal{J}$, and a B otherwise, each row and column contains a string in $\mathcal{S}$.

We illustrate this with a simple example: If $k = 6$ and we have that $AAABAB$ and $BBBABA$ are both in the dictionary, then Lexi could fill the table as follows:

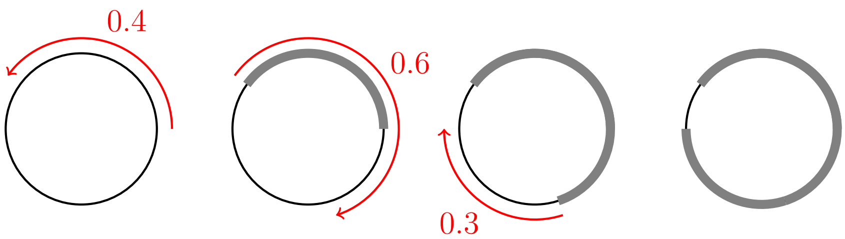
A	A	A	B	A	B
A	A	A	B	A	B
A	A	A	B	A	B
B	B	B	A	B	A
A	A	A	B	A	B
B	B	B	A	B	A

Suppose we are looking at row i or column i for $i \in \mathcal{J}$. Then by construction the string in this row/column contains As at indices k with $k \in \mathcal{J}$ and Bs elsewhere, and thus is precisely ℓ. Suppose instead we are looking at row i or column i for $i \notin \mathcal{J}$. Then again by construction the string in this row/column contains As at indices k with $k \notin \mathcal{J}$ and Bs elsewhere, and thus is precisely ℓ'. So each row and column indeed contains a string in $\mathcal{S}$.

Thus, for any $\mathcal{S}$ with $|\mathcal{S}| \geqslant 2^{k-1}$, Lexi can definitely fill the grid appropriately. Since we know $m \geqslant 2^{k-1}$, 2^{k-1} is the minimum possible value of m as claimed. $\square$

Problem 4. Turbo the snail sits on a point on a circle with circumference 1. Given an infinite sequence of positive real numbers $c_1, c_2, c_3, \ldots$, Turbo successively crawls distances $c_1, c_2, c_3, \ldots$ around the circle, each time choosing to crawl either clockwise or counterclockwise.

For example, if the sequence $c_1, c_2, c_3, \ldots$ is $0.4, 0.6, 0.3, \ldots$, then Turbo may start crawling as follows:



Determine the largest constant $C > 0$ with the following property: for every sequence of positive real numbers $c_1, c_2, c_3, \ldots$ with $c_i < C$ for all i, Turbo can (after studying the sequence) ensure that there is some point on the circle that it will never visit or crawl across.

Solution 1. The largest possible C is $C = \frac{1}{2}$.

For $0 < C \leqslant \frac{1}{2}$, Turbo can simply choose an arbitrary point P (different from its starting point) to avoid. When Turbo is at an arbitrary point A different from P, the two arcs AP have total length 1; therefore, the larger of the two the arcs (or either arc in case A is diametrically opposite to P) must have length $\geqslant \frac{1}{2}$. By always choosing this larger arc (or either arc in case A is diametrically opposite to P), Turbo will manage to avoid the point P forever.

For $C > \frac{1}{2}$, we write $C = \frac{1}{2} + a$ with $a > 0$, and we choose the sequence

$$\frac{1}{2}, \quad \frac{1+a}{2}, \quad \frac{1}{2}, \quad \frac{1+a}{2}, \quad \frac{1}{2}, \quad \ldots$$

In other words, $c_i = \frac{1}{2}$ if i is odd and $c_i = \frac{1+a}{2} < C$ when i is even. We claim Turbo must eventually visit all points on the circle. This is clear when it crawls in the same direction two times in a row; after all, we have $c_i + c_{i+1} > 1$ for all i. Therefore, we are left with the case that Turbo alternates crawling clockwise and crawling counterclockwise. If it, without loss of generality, starts by going clockwise, then it will always crawl a distance $\frac{1}{2}$ clockwise followed by a distance $\frac{1+a}{2}$ counterclockwise. The net effect is that it crawls a distance $\frac{a}{2}$ counterclockwise. Because $\frac{a}{2}$ is positive, there exists a positive integer N such that $\frac{a}{2} \cdot N > 1$. After $2N$ crawls, Turbo will have crawled a distance $\frac{a}{2}$ counterclockwise N times, therefore having covered a total distance of $\frac{a}{2} \cdot N > 1$, meaning that it must have crawled over all points on the circle. □

Note: Every sequence of the form $c_i = x$ if i is odd, and $c_i = y$ if i is even, where $0 < x, y < C$, such that $x + y \geqslant 1$, and $x \neq y$ satisfies the conditions with the same argument. There might be even more possible examples.

Solution 2. Alternative solution (to show that $C \leqslant \frac{1}{2}$)

We consider the following related problem:

We assume instead that the snail Chet is moving left and right on the real line. Find the size M of the smallest (closed) interval, that we cannot force Chet out of, using a sequence of real numbers d_i with $0 < d_i < 1$ for all i.

Then $C = 1/M$. Indeed if for every sequence $c_1, c_2, \ldots,$ with $c_i < C$ there exists a point that Turbo can avoid, then the circle can be cut open at the avoided point and mapped to an interval of size M such that Chet can stay inside this interval for any sequence of the from $c_1/C, c_2/C, \ldots,$ see Figure 5. However, all sequences $d_1, d_2, \ldots$ with $d_i < 1$ can be written in this form. Similarly if for every sequence $d_1, d_2, \ldots,$ there exists an interval of length smaller or equal M that we cannot force Chet out of, this projects to a subset of the circle, that we cannot force Turbo out of using any sequence of the form $d_1/M, d_2/M, \ldots$. These are again exactly all the sequences with elements in $[0, C)$.

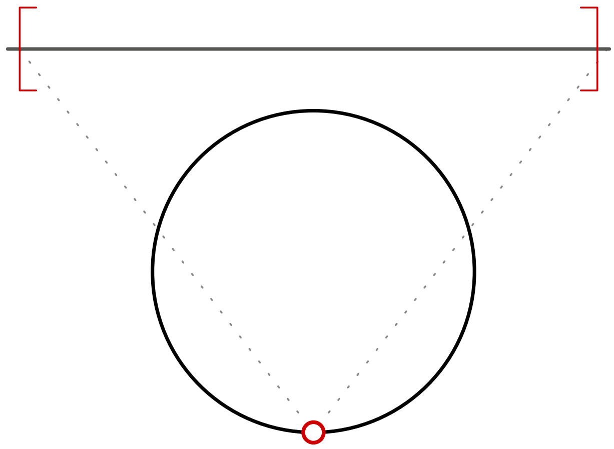


Figure 5: Chet and Turbo equivalence

Claim: $M \geqslant 2$.

Proof. Suppose not, so $M < 2$. Say $M = 2 - 2\varepsilon$ for some $\varepsilon > 0$ and let $[-1+\varepsilon, 1-\varepsilon]$ be a minimal interval, that Chet cannot be forced out of. Then we can force Chet arbitrarily close to $\pm(1-\varepsilon)$. In partiular, we can force Chet out of $[-1+\frac{4}{3}\varepsilon, 1-\frac{4}{3}\varepsilon]$ by minimality of M. This means that there exists a sequence $d_1, d_2, \ldots$ for which Chet has to leave $[-1+\frac{4}{3}\varepsilon, 1-\frac{4}{3}\varepsilon]$, which means he ends up either in the interval $[-1+\varepsilon, -1+\frac{4}{3}\varepsilon)$ or in the interval $(1-\frac{4}{3}\varepsilon, 1-\varepsilon]$.

Now consider the sequence,

$$d_1,\ 1-\frac{7}{6}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{7}{6}\varepsilon,\ d_2,\ 1-\frac{7}{6}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{7}{6}\varepsilon,\ d_3,\ \ldots$$

obtained by adding the sequence $1-\frac{7}{6}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{7}{6}\varepsilon$ in between every two steps. We claim that this sequence forces Chet to leave the larger interval $[-1+\varepsilon, 1-\varepsilon]$. Indeed no two consecutive elements in the sequence $1-\frac{7}{6}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{2}{3}\varepsilon,\ 1-\frac{7}{6}\varepsilon$ can have the same sign, because the sum of any two consecutive terms is larger than $2 - 2\varepsilon$ and Chet would leave the interval $[-1+\varepsilon, 1-\varepsilon]$. It follows that the $(1-\frac{7}{6}\varepsilon)$'s and the $(1-\frac{2}{3}\varepsilon)$'s cancel out, so the position after d_k is the same as before d_{k+1}. Hence, the positions after each d_k remain the same as in the original sequence. Thus, Chet is also forced to the boundary in the new sequence.

If Chet is outside the interval $[-1+\frac{4}{3}\varepsilon, 1-\frac{4}{3}\varepsilon]$, then Chet has to move $1-\frac{7}{6}\varepsilon$ towards 0, and ends in $[-\frac{1}{6}\varepsilon, \frac{1}{6}\varepsilon]$. Chet then has to move by $1-\frac{2}{3}\varepsilon$, which means that he has to leave the interval $[-1+\varepsilon, 1-\varepsilon]$. Indeed the absolute value of the final position is at least $1-\frac{5}{6}\varepsilon$. This contradicts the assumption, that we cannot force Chet out of $[-1+\varepsilon, 1-\varepsilon]$. Hence $M \geqslant 2$ as needed. □

Problem 5. We are given a positive integer $s \geqslant 2$. For each positive integer k, we define its *twist* k' as follows: write k as $as + b$, where a, b are non-negative integers and $b < s$, then $k' = bs + a$. For the positive integer n, consider the infinite sequence $d_1, d_2, \dots$ where $d_1 = n$ and d_{i+1} is the twist of d_i for each positive integer i.

Prove that this sequence contains 1 if and only if the remainder when n is divided by $s^2 - 1$ is either 1 or s.

Solution 1. First, we consider the difference $k - k''$. If $k = as + b$ as in the problem statement, then $k' = bs + a$. We write $a = ls + m$ with m, l non-negative numbers and $m \leq s - 1$. This gives $k'' = ms + (b + l)$ and hence $k - k'' = (a - m)s - l = l(s^2 - 1)$.

We conclude

Fact 1.1. $k \geq k''$ for every every $k \geq 1$

Fact 1.2. $s^2 - 1$ divides the difference $k - k''$.

Fact 1.2 implies that the sequences $d_1, d_3, d_5, \dots$ and $d_2, d_4, d_6, \dots$ are constant modulo $s^2 - 1$. Moreover, Fact 1.1 says that the sequences are (weakly) decreasing and hence eventually constant. In other words, the sequence $d_1, d_2, d_3, \dots$ is 2-periodic modulo $s^2 - 1$ (from the start) and is eventually 2-periodic.

Now, assume that some term in the sequence is equal to 1. The next term is equal to $1' = s$ and since the sequence is 2-periodic from the start modulo $s^2 - 1$, we conclude that d_1 is either equal to 1 or s modulo $s^2 - 1$. This proves the first implication.

To prove the other direction, assume that d_1 is congruent to 1 or s modulo $s^2 - 1$. We need the observation that once one of the sequences $d_1, d_3, d_5, \dots$ or $d_2, d_4, d_6, \dots$ stabilises, then their value is less than s^2. This is implied by the following fact.

Fact 1.3. If $k = k''$, then $k = k'' < s^2$.

Proof. We use the expression for $k - k''$ found before. If $k = k''$, then $l = 0$, and so $k'' = ms + b$. Both m and b are reminders after division by s, so they are both $\leq s - 1$. This gives $k'' \leq (s - 1)s + (s - 1) < s^2$. □

Using Fact 1.2, it follows that the sequence $d_1, d_3, d_5, \dots$ is constant to 1 or s modulo $s^2 - 1$ and stabilises to 1 or s by Fact 1.3. Since $s' = 1$, we conclude that the sequence contains a 1. □

Solution 2. We make a number of initial observations. Let k be a positive integer.

Fact 2.1. If $k \geq s^2$, then $k' < k$.

Proof. Write $k = as + b$, as in the problem statement. If $k \geq s^2$, then $a \geq s$ because $b < s$. So, $k' = bs + a \leq (s - 1)s + a \leq as \leq as + b = k$. Moreover, we cannot have equality since that would imply $s - 1 = b = 0$. □

Fact 2.2. If $k \leq s^2 - 1$, then $k' \leq s^2 - 1$ and $k'' = k$.

Proof. Write $k = as + b$, as in the problem statement. If $k < s^2$, then it must hold $1 \leq a, b < s$, hence $k' = bs + a < s^2$ and $k'' = (bs + a)' = as + b = k$. □

Fact 2.3. We have $k' \equiv sk \pmod{s^2 - 1}$ (or equivalently $k \equiv sk' \pmod{s^2 - 1}$).

Proof. We write $k = as + b$, as in the problem statement. Now,

$$sk - k' = s(as + b) - (bs + a) = a(s^2 - 1) \equiv 0 \pmod{s^2 - 1},$$

as desired. □

Combining Facts 2.1 and 2.2, we find that the sequence $d_1, d_2, d_3 \ldots$ is eventually periodic with period 2, starting at the first value less than s^2. From Fact 2.3, it follows that

$$k'' \equiv sk' \equiv s^2 k \equiv k \pmod{s^2 - 1}$$

and hence the sequence is periodic modulo $s^2 - 1$ from the start with period 2.

Now, if the sequence contains 1, the sequence eventually alternates between 1 and s since the twist of 1 is s and vice versa. Using periodicity modulo $s^2 - 1$, we must have $n \equiv 1, s \pmod{s^2 - 1}$. Conversely, if $n \equiv 1, s \pmod{s^2 - 1}$ then the eventual period must contain at least one value congruent to either 1 or s modulo $s^2 - 1$. Since these values must be less than s^2, this implies that the sequence eventually alternates between 1 and s, showing that it contains a 1. □

Solution 3. We give an alternate proof of the direct implication: if the sequence contains a 1, then the first term is 1 or s modulo $s^2 - 1$. We prove the following fact, which is a combination of Facts 2.1 and 2.3.

Fact 3.1. For all $k \geq s^2$, we have $(k - s^2 + 1)' \in \{k', k' - s^2 + 1\}$.

Proof. We write $k = as + b$, as in the problem statement. Since $k \geq s^2$, we have $a \geq s$. If $b < s - 1$, then

$$(k - s^2 + 1)' = \Big((a - s)s + (b + 1)\Big)' = (b + 1)s + (a - s) = bs + a = k'.$$

On the other hand, if $b = s - 1$, then

$$(k - s^2 + 1)' = \Big((a - s + 1)s + 0\Big)' = 0s + (a - s + 1) = a - s + 1 = k' - s^2 + 1.$$

□

Now assume $n \geq s^2$ and the sequence $d_1, d_2, \ldots$ contains a 1. Denote by $e_1, e_2, \ldots$ the sequence constructed as in the problem statement, but with initial value $e_1 = n - s^2 + 1$. Using the above fact, we deduce that $e_i \equiv d_i \pmod{s^2 - 1}$ and $e_i \leq d_i$ for all $i \geq 1$ by induction on i. Hence, the sequence $e_1, e_2, \ldots$ also contains a 1.

Since the conclusion we are trying to reach only depends on the residue of d_1 modulo $s^2 - 1$, we conclude that without loss of generality we can assume $n < s^2$.

Using Fact 2.2, it now follows that the sequence $d_1, d_2, \dots$ is periodic with period two. Since 1 and s are twists of each other, it follows that if this sequence contains a 1, it must be alternating between 1 and s. Hence, $d_1 \equiv 1, s \pmod{s^2-1}$ as desired.

For the other direction we can make a similar argument, observing that the second of the two cases in the proof of Fact 3.1 can only apply to finitely many terms of the sequence $d_1, d_2, d_3, \dots$, allowing us to also go the other way. □

Solution 4. First assume that $d_k = 1$ for some k. Let k be the smallest such index. If $k = 1$ then $n = 1$, so we may assume $k \geqslant 2$.

Then $d_{k-1} = as + b$ for some non-negative integers a, b satisfying $b < s$ and $bs + a = 1$. The only solution is $b = 0, a = 1$, so $d_{k-1} = s$. So, if $k = 2$, then $n = s$, so we may assume $k \geqslant 3$.

Then there exist non-negative integers c, d satisfying $d_{k-2} = cs + d$, $d < s$ and $ds + c = s$. We have two solutions: $d = 0, c = s$ and $d = 1, c = 0$. However, in the second case we get $d_{k-2} = 1$, which contradicts the minimality of k. Hence, $d_{k-2} = s^2$. If $k = 3$, then $n = d_1 = s^2$, which gives remainder 1 when divided by $s^2 - 1$.

Assume now that $k \geqslant 4$. We will show that for each $m \in \{3, 4, \dots, k-1\}$ there exist $b_1, b_2, \dots, b_{m-2} \in \{0, 1, \dots, s-1\}$ such that

$$d_{k-m} = s^m - \sum_{i=1}^{m-2} b_i \left(s^{m-i} - s^{m-i-2}\right). \tag{9}$$

We will prove this equality by induction on m. If $m = 3$, then $d_{k-3} = a_1 s + b_1$ for some non-negative integers a_1, b_1 satisfying $b_1 < s$ and $b_1 s + a_1 = d_{k-2} = s^2$. Then $a_1 = s^2 - b_1 s$, so $d_{k-3} = s^3 - b_1(s^2 - 1)$, which proves (9) for $m = 3$.

Assume that (9) holds for some m and consider $d_{k-(m+1)}$. There exist non-negative integers a_{m-1}, b_{m-1} such that $d_{k-(m+1)} = a_{m-1}s + b_{m-1}$, $b_{m-1} < s$ and $d_{k-m} = b_{m-1}s + a_{m-1}$. Using the inductive assumption we get

$$a_{m-1} = d_{k-m} - b_{m-1}s = s^m - \sum_{i=1}^{m-2} b_i \left(s^{m-i} - s^{m-i-2}\right) - b_{m-1}s,$$

therefore

$$\begin{aligned} d_{k-(m+1)} =& a_{m-1}s + b_{m-1} = s^{m+1} - \sum_{i=1}^{m-2} b_i \left(s^{m-i} - s^{m-i-2}\right) s - b_{m-1}s^2 + b_{m-1} \\ =& s^{m+1} - \sum_{i=1}^{m-1} b_i \left(s^{m+1-i} - s^{m-i-1}\right), \end{aligned}$$

which completes the proof of (9). In particular, for $m = k-1$ we get

$$d_1 = s^{k-1} - \sum_{i=1}^{k-3} b_i \left(s^{k-i-1} - s^{k-i-3}\right).$$

The above sum is clearly divisible by $s^2 - 1$, and it is clear that the remainder of s^{k-1} when divided by $s^2 - 1$ is 1 when k is odd, and s when k is even. It follows that the remainder when $n = d_1$ is divided by $s^2 - 1$ is either 1 or s.

To prove the other implication, assume that n gives remainder 1 or s when divided by s^2-1. If $n \in \{1, s, s^2\}$, then one of the numbers d_1, d_2 and d_3 is 1. We therefore assume that $n > s^2$. Since the reminder when a power of s is divided by s^2-1 is either 1 or s, there exists a positive integer m such that $s^m - n$ is non-negative and divisible by s^2-1. By our assumption $m \geqslant 3$. We also take the smallest such m, so that $n > s^{m-2}$. The quotient $\frac{s^m-n}{s^2-1}$ is therefore smaller than s^{m-2}, so there exist $b_1, \ldots, b_{m-2} \in \{0, 1, \ldots, s-1\}$ such that $\frac{s^m-n}{s^2-1} = \sum_{i=1}^{m-2} b_i s^{i-1}$. It follows that

$$n = s^m - \sum_{i=1}^{m-2} b_i \left(s^{i+1} - s^{i-1}\right).$$

We now show that

$$d_j = s^{m+1-j} - \sum_{i=1}^{m-1-j} b_i \left(s^{i+1} - s^{i-1}\right) \tag{10}$$

for $j = 1, 2, \ldots, m-2$ by induction on j. For $j = 1$ this follows from $d_1 = n$. Assume now that (10) holds for some $j < m-2$. Then

$$d_j = \left(s^{m-j} - \sum_{i=2}^{m-1-j} b_i \left(s^i - s^{i-2}\right) - b_1 s\right) s + b_1.$$

As d_j is positive and $b_1 \in \{0, 1, \ldots, s-1\}$, the expression $s^{m-j} - \sum_{i=2}^{m-1-j} b_i \left(s^i - s^{i-2}\right) - b_1 s$ has to be non-negative, so we can compute the twist of d_j as

$$d_{j+1} = b_1 s + s^{m-j} - \sum_{i=2}^{m-1-j} b_i \left(s^i - s^{i-2}\right) - b_1 s = s^{m-j} - \sum_{i=1}^{m-2-j} b_i \left(s^{i+1} - s^{i-1}\right),$$

which finishes the induction.

Now we use (10) for $j = m-2$ and get $d_{m-2} = s^3 - b_1(s^2-1) = (s^2 - b_1 s) + b_1$. Then $d_{m-1} = b_1 s + s^2 - b_1 s = s^2 = s \cdot s + 0$, $d_m = 0 \cdot s + s = s = 1 \cdot s + 0$ and $d_{m+1} = 0 \cdot s + 1 = 1$. □

Problem 6. Let ABC be a triangle with circumcircle Ω. Let S_b and S_c respectively denote the midpoints of the arcs AC and AB that do not contain the third vertex. Let N_a denote the midpoint of arc BAC (the arc BC containing A). Let I be the incentre of ABC. Let ω_b be the circle that is tangent to AB and internally tangent to Ω at S_b, and let ω_c be the circle that is tangent to AC and internally tangent to Ω at S_c. Show that the line IN_a, and the line through the intersections of ω_b and ω_c, meet on Ω.

The incentre of a triangle is the centre of its incircle, the circle inside the triangle that is tangent to all three sides.

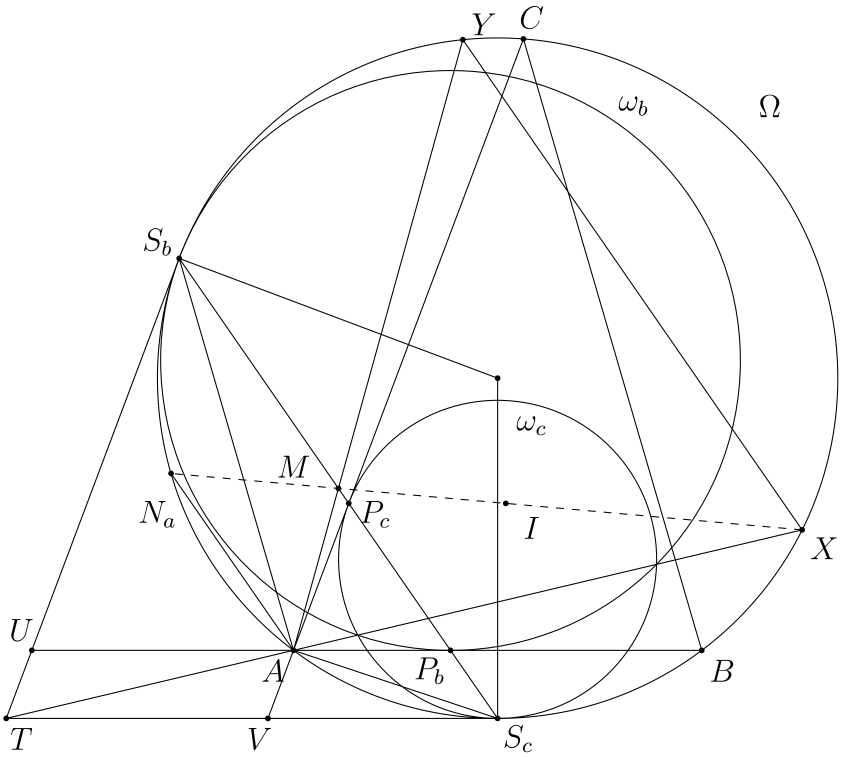


Figure 6: Diagram to Solution 1

Solution 1. Part I: First we show that A lies on the radical axis of ω_b and ω_c.

We first note that the line through the intersections of two circles is the radical line of the two circles. Let the tangents to Ω at S_b and S_c intersect at T. Clearly T is on the radical axis of ω_b and ω_c (and in fact is the radical centre of ω_b, ω_c and Ω).

We next show that A lies on the radical axis of ω_b and ω_c. Let P_b denote the point of tangency of ω_b and AB, and let P_c denote the point of tangency of ω_c and AC. Furthermore, let U be the intersection of the tangent to Ω at S_b with the line AB, and let V be the intersection of the tangent to Ω at S_c with the line AC. Then $TVAU$ is parallelogram. Morover, due to equality of tangent segments we have $|US_b| = |UP_b|$, $|VP_c| = |VS_c|$ and $|TS_b| = |TS_c|$. It follows that

$$\begin{aligned} |AP_b| &= |UP_b| - |UA| = |US_b| - |TV| = |TS_b| - |TU| - |TV| \qquad (11) \\ &= |TS_s| - |TV| - |TU| = |VS_c| - |AV| = |VP_c| - |VA| = |AP_c|. \end{aligned}$$

But $|AP_b|$, $|AP_c|$ are exactly the square roots of powers of A with respect to ω_b and ω_c, hence A is indeed on their radical axis.

Thus, the radical axis of ω_b, ω_c is AT.

Part II: Consider the triangle AS_bS_c. Note that since T is the intersection of the tangents at S_b and S_c to the circumcircle of AS_bS_c, it follows that AT is the symmedian of A in this triangle. Let X denote the second intersection of the symmedian AT with Ω. We wish to show that X is also on IN_a.

Note that AN_a is the external angle bisector of angle A, and therefore it is parallel to S_bS_c. Let M denote the midpoint of S_bS_c, and let Y be the second intersection of AM with Ω. Since in AS_bS_c, AXT is the symmedian and AMY is the median, it follows that XY is also parallel to S_bS_c. Thus, reflecting in the perpendicular bisector of S_bS_c sends the line AMY to line N_aMX.

Next, consider the quadrilateral AS_bIS_c. From the trillium theorem we have $|S_bA| = |S_bI|$ and $|S_cA| = |S_cI|$, thus the quadrilateral is a kite, from which it follows that the reflection of the line AM in S_bS_c is the line IM. But previously we have seen that this is also the line N_aMX. Thus M, I, N_a and X are collinear, as we wanted to show. □

Solution 2. This is a variation of Solution 1 which avoids the theory of the symmedian point.

We begin by showing that the radical axis of ω_b, ω_c is AT as in Solution 1.

Part II: We introduce the point S_a with the obvious meaning. Observe that the incentre I of ABC is the orthocentre of $S_aS_bS_c$ either because this is well-known, or because of an angle argument that A reflects in S_bS_c to I (and similar results by cyclic change of letters). Therefore AS_a is perpendicular to S_bS_c.

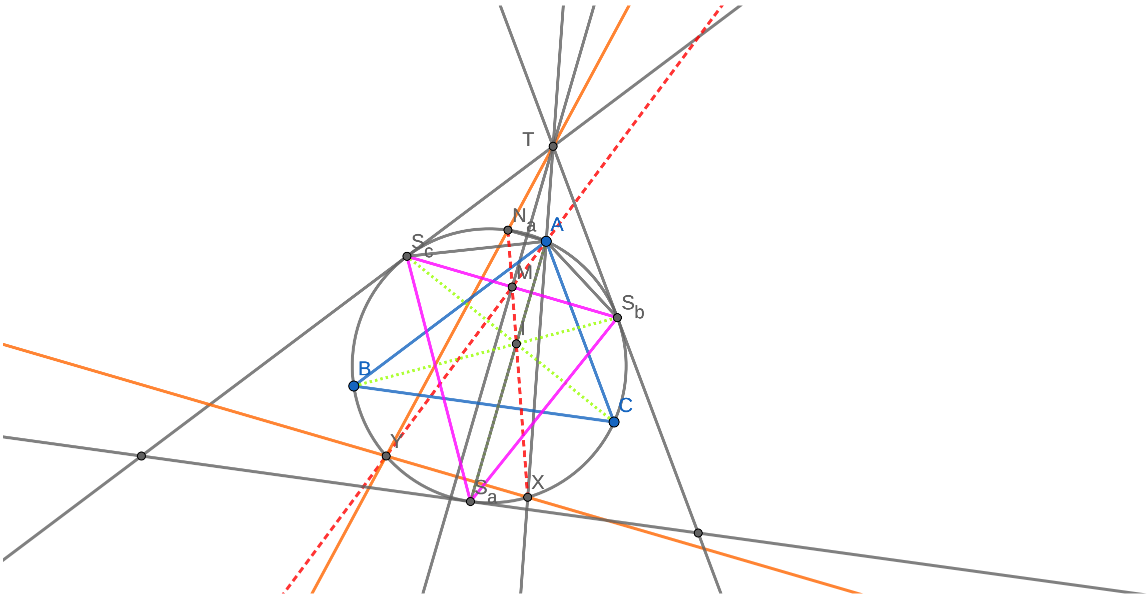


Figure 7: A reflections argument for Solution 2

Let M denote the midpoint of S_bS_c. Then A is the reflection of S_a in the diameter parallel to S_bS_c, so the reflection of A in the diameter perpendicular to S_bS_c is N_a, the antipode of S_a. Let the reflection of X in TM be Y, so TY passes through N_a and is the reflection of TX in TM.

Now S_bS_c is the polar line of T with respect to Ω, so AY and N_aX meet on this line, and by symmetry at its midpoint M. The line N_aMX is therefore the reflection of the line YMA in S_bS_c, and so N_aMX passes through I (the reflection of A in S_bS_c). □

The triangle AS_cS_b can be taken as generic, and from the argument above we can extract the fact that the symmedian point and the centroid are isogonal conjugates in that triangle.

Solution 3. Assume the notation from Solution 1, part I of Solution 1, and let O be the centre of Ω.

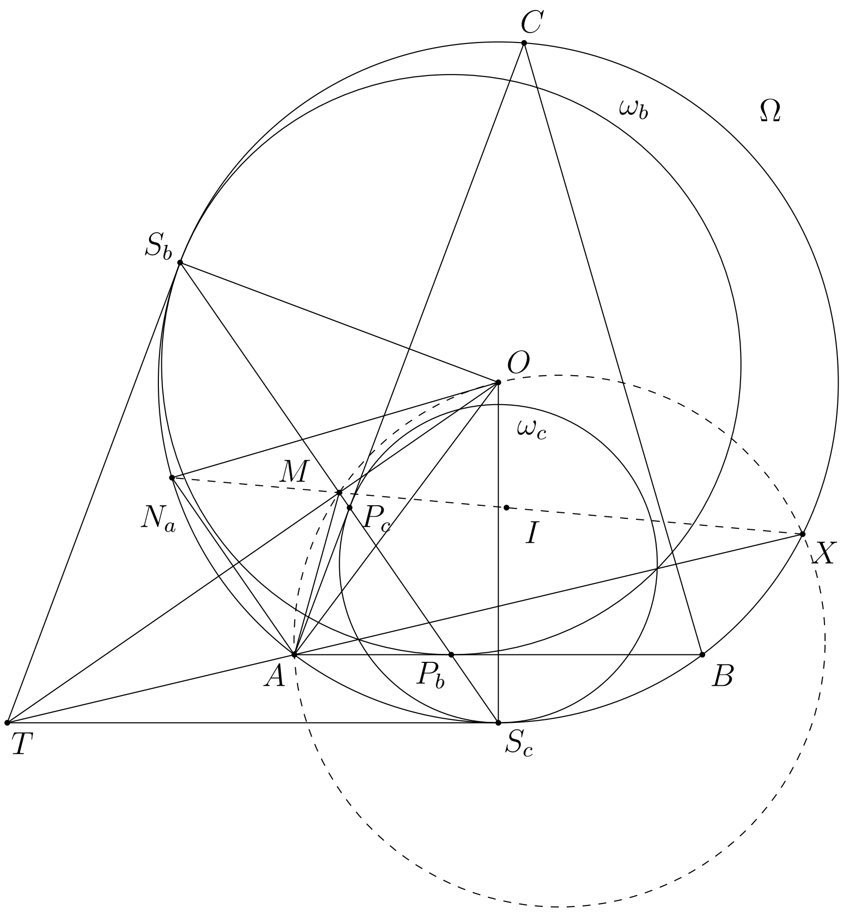


Figure 8: Diagram to Solution 3

Part II: As in Solution 1, by the trillelium theorem, S_cS_b bisects AI, and since $N_aA \parallel S_bS_c$, then OT is a bisector of AN_a. This implies $|MN_a| = |MA| = |MI|$, since M is the midpoint of S_cS_b and lies also on OT. Hence, M is the circumcentre of triangle IAN_a. But this triangle has a right angle at A (since AI and AN_a are the inner and outer angle bisector at A), hence M lies on IN_a.

Again, let X be the second intersection of TA and Ω. By the above, it suffices to prove that X lies on the line N_aM. From the power of point T with respect to Ω we get $|TA| \cdot |TX| = |TS_c|^2$. Since M is the foot of the altitude of right triangle TS_cO, we obtain

$|TS_c|^2 = |TM| \cdot |TO|$. Hence, $|TA| \cdot |TX| = |TM| \cdot |TO|$ so the points O, M, A, X are concyclic. It follows that $\angle MXA = \angle MOA = \frac{1}{2}\angle N_aOA = \angle N_aXA$. Hence, X lies on the line N_aM.

Remark. To show that $OMAX$ is cyclic, one can also invert the line TAX in the circumcircle of the triangle ABC. □

Solution 4. Part I is done as in solution 1.

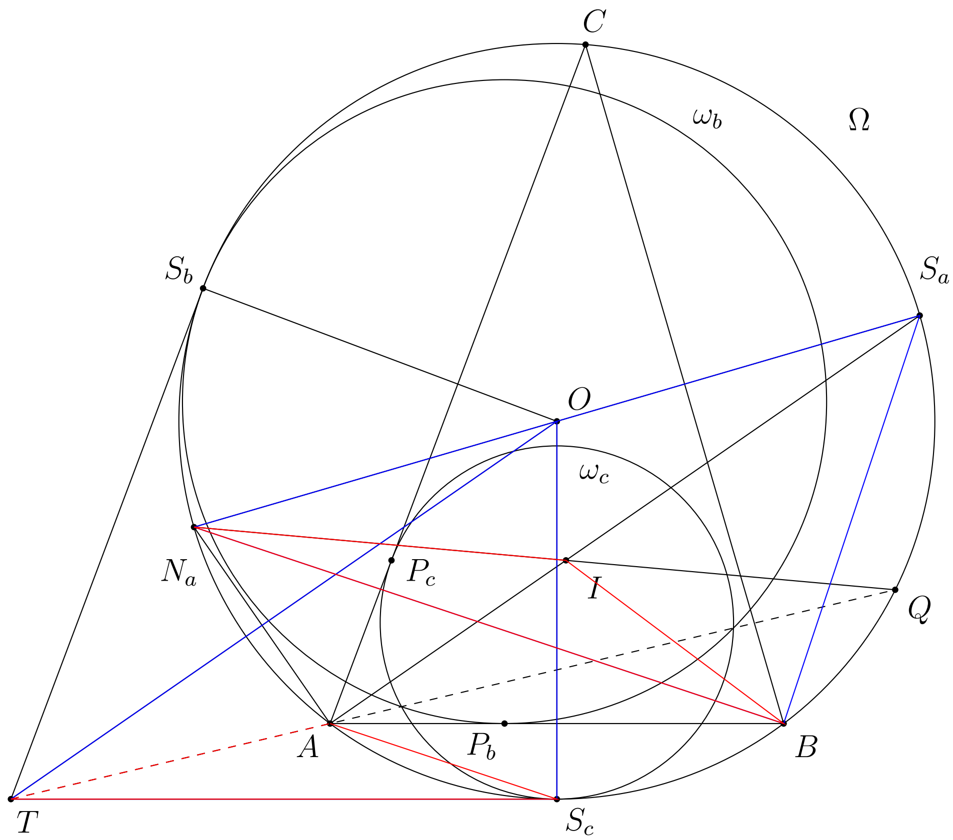


Figure 9: Diagram to Solution 4

Part II: as in Solution 1 we show that $AN_a || S_bS_c$. In particular, $\angle N_aOT = \angle TOA$. The conclusion of the problem trivially holds if $|AB| = |AC|$, therefore we assume without loss of generality that $|AC| > |AB|$. Let S_a be the midpoint of the arc BC which does not contain A. Then N_aS_a is a diameter, so $\angle S_aBN_a = \frac{\pi}{2} = \angle OS_cT$. We also compute $\angle BN_aS_a = \angle BAS_a = \frac{1}{2}\angle BAC = \frac{1}{2}\angle S_cTS_b = \angle S_cTO$. It follows that the triangles TS_cO and N_aBS_a are similar. In particular,

$$\frac{|N_aB|}{|TS_c|} = \frac{|S_aB|}{|OS_c|}. \tag{12}$$

Next we compute

$$\angle IS_aB = \angle N_aS_aB - \angle N_aS_aI = \angle TOS_c - \frac{1}{2}\angle N_aOA = \angle TOS_c - \angle TOA = \angle AOS_c \tag{13}$$

and

$$\angle IBN_a = \angle CBN_a - \angle CBI = \frac{1}{2}(\pi - \angle BN_aC) - \frac{1}{2}\angle CBA = \frac{1}{2}\angle ACB = \angle ACS_c = \angle AS_cT, \quad (14)$$

hence

$$\angle S_aBI = \frac{\pi}{2} - \angle IBN_a = \frac{\pi}{2} - \angle AS_cT = \angle OS_cA.$$

Together with (13) it follows that the triangles IBS_a and AS_cO are similar, so $\frac{|S_aB|}{|OS_c|} = \frac{|IB|}{|AS_c|}$, and (12) implies $\frac{|N_aB|}{|TS_c|} = \frac{|IB|}{|AS_c|}$. Consequently, by (14) the triangles TS_cA and N_aBI are similar and therefore $\angle S_cTA = \angle BN_aI$. Now let Q be the second intersection of N_aI with Ω. Then $\angle BN_aI = \angle BN_aQ = \angle BAQ$, so $\angle S_cTA = \angle BAQ$. Since AB is parallel to TS_c, we get $AQ||TA$, i.e. A, T, Q are collinear. □

Remark. After proving similarity of triangles TS_cO and N_aBS_a one can use spiral symmetry to show similarity of triangles TS_cA and N_aBI.

Solution 5. Part I: First we show that A lies on the radical axis between ω_b and ω_c.

Let T be the radical center of the circumcircle, ω_b and ω_c; then TS_b and TS_c are common tangents of the circles, as shown in Figure 5a. Moreover, let $P_b = AB \cap S_bS_c$ and $P_c = AC \cap S_bS_c$. The triangle TS_cS_b is isosceles: $AB \parallel TS_c$ and $AC \parallel TS_b$ so

$$\angle AP_bP_c = \angle TS_cS_b = \angle S_cS_bT = \angle P_bP_cA.$$

From these angles we can see that ω_b passes through P_b, ω_c passes through P_c, and finally AP_b and AP_c are equal tangents to ω_b and ω_c, so A lies on the radical axis.

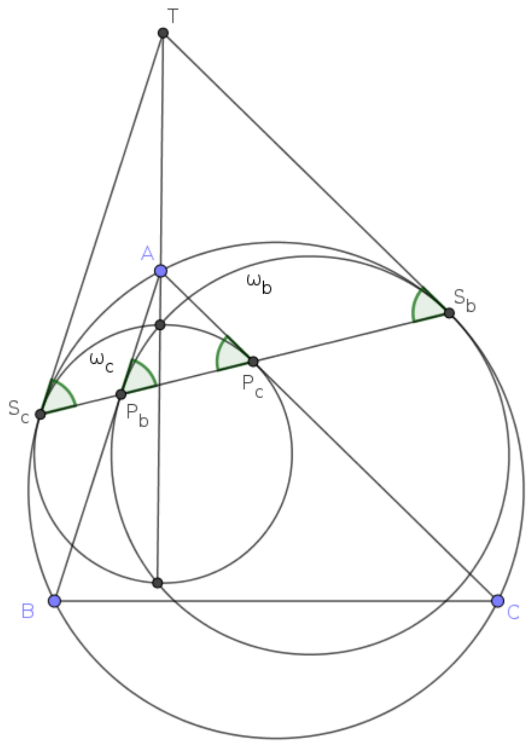


Figure 5a

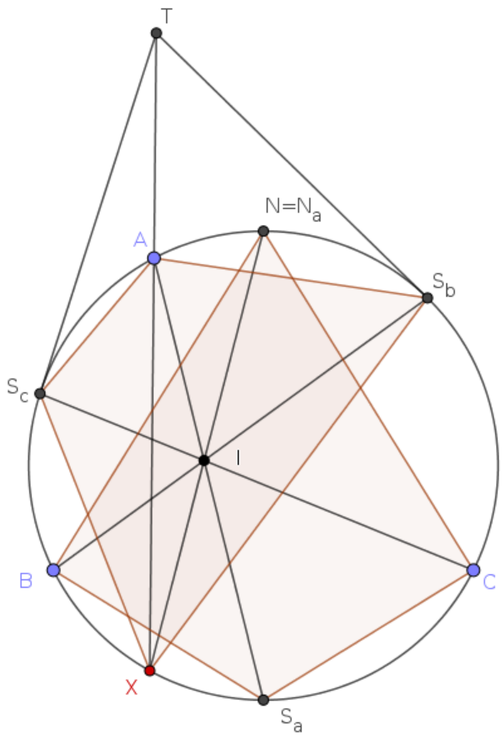


Figure 5b

Part II. Let the radical axis TA meet the circumcircle again at X, let S_a be the midpoint of the arc BC opposite to A, and let XI meet the circumcirlce again at N. (See Figure 2.) For solving the problem, we have prove that $N_a = N$.

The triples of points A, I, S_a; B, I, S_b and C, I, S_c are collinear because they lie on the angle bisectors of the triangle ABC.

Notice that the quadrilateral AS_cXS_b is harmonic, because the tangents at S_b and S_c, and the line AX are concurrent at T. This quadrilateral can be projected (or inverted) to the quadrilateral S_aCNB through I. So, S_aCNB also is a harmonic quadrilateral. Due to $S_aB = S_aC$, this implies $NB = NC$, so $N = N_a$. Done.

Remark. Instead of mentioning inversion and harmonic quadrilaterals, from the similar triangles $\triangle TS_cA \sim \triangle TXS_c$ and $\triangle TAS_b \sim \triangle TS_bX$ we can get

$$\frac{AS_c}{S_cX} = \frac{AS_b}{S_bX}.$$

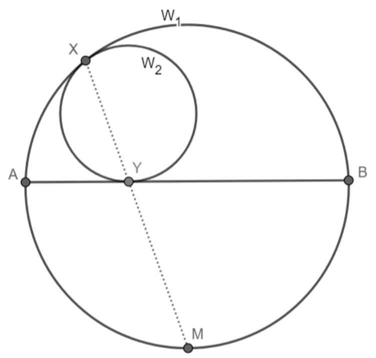
Then, we can apply the trigonometric form of Ceva's theorem to triangle BCX

$$\frac{\sin\angle BXN_a}{\sin\angle N_aXC} \cdot \frac{\sin\angle CBS_b}{\sin\angle S_bBX} \cdot \frac{\sin\angle XCS_c}{\sin\angle S_cCB} = \frac{BN_a}{N_aC} \cdot \frac{-CS_b}{S_bX} \cdot \frac{XS_c}{-S_cB} = 1 \cdot \frac{S_bA}{S_bN_a} \cdot \frac{N_aS_c}{S_cB} = 1,$$

so the Cevians BS_b, CS_c and XN_a are concurrent. □

Solution 6. Part I: First let's show that this is equivalent to proving that TA and N_aI intersect in Ω.

Lemma: Let's recall that if we have two circles ω_1 and ω_2 which are internally tangent at point X and if we have a line AB tangent to ω_2 at Y. Let M be the midpoint of the arc AB not containing Z. We have that Z, Y, M are collinear.



Solution 6: Lemma

Let $P_b = AB \cap \omega_b$ and $P_c = AC \cap \omega_c$. We can notice by the lemma that S_b, P_b and S_c are collinear, and similarly S_c, P_c and S_b are also collinear. Therefore S_c, P_b, P_c, and S_b are collinear, and since $\angle AP_bP_c = \frac{\angle ABC}{2} + \frac{\angle ACB}{2} = \angle AP_cP_b$ then $AP_b = AP_c$ so A is on the radical axis of ω_b and ω_c. Let T be the intersection of the tangent lines of Ω through S_c and S_b. Since $TS_c = TS_b$ then AT is the radical axis between ω_b and ω_c.

Part II: TA and N_aI intersect in Ω.

Let ω_a the A-mixtilinear incircle (that is, the circle internally tangent to Ω, and tangent to both AB and AC), and let $X = \Omega \cap \omega_a$. It is known that N_a, I, X are collinear.

Let M_c and M_b be the tangent points of ω_A to AB and AC respectively, then by the lemma X, M_c, S_c are collinear and X, M_b, S_b are collinear. We can see that S_cTS_b and M_cAM_b are homothetic with respect to X; therefore T and A are homothetic with respect to X, impying that T, A, X are collinear. $\square$

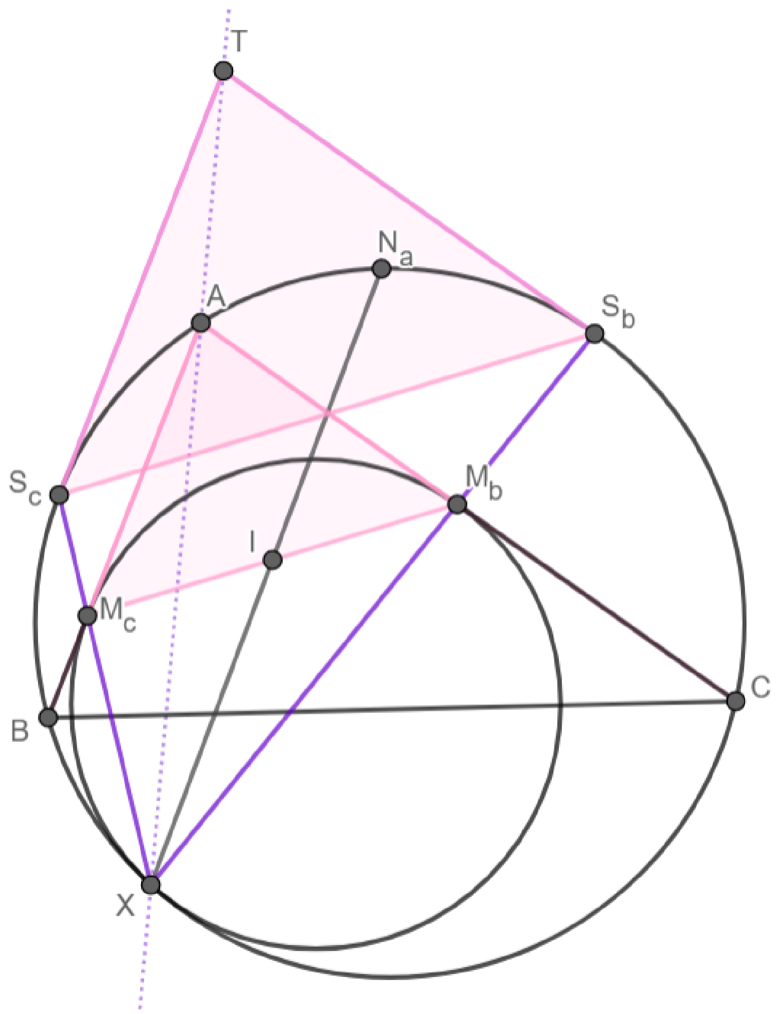


Figure 10: Diagram to Solution 6