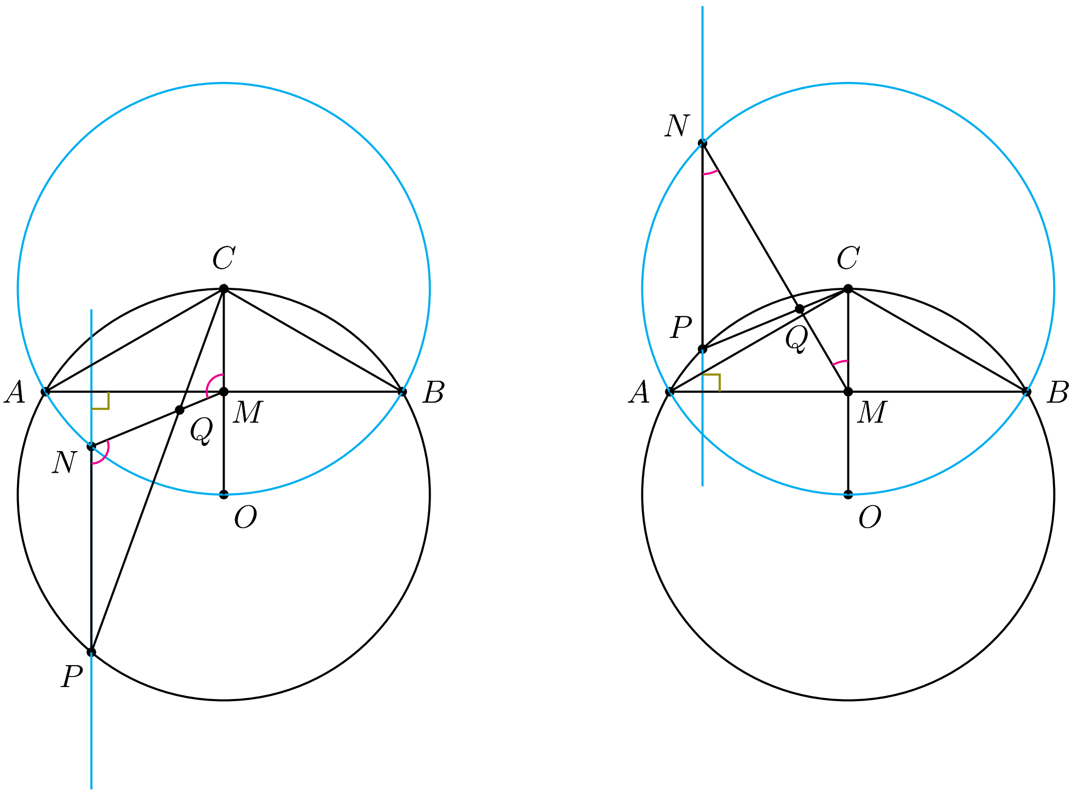
Problem 1 Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point on the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N.

Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

(Velina Ivanova, Bulgaria)

Solution Let O be the circumcenter of ABC. From the assumption that $\angle ACB = 120°$ it follows that M is the midpoint of CO.

Let ω denote the circle with center in C and radius CO. This circle in the image of the circumcircle of ABC through the translation that sends O to C. We claim that N lies on ω.



Let us consider the triangles QNP and QMC. The angles in Q are equal. Since NP is parallel to MC (both lines are perpendicular to AB), it turns out that $\angle QNP = \angle QMC$, and hence the two triangles are similar. Since $QP = 2QC$, it follows that

$$NP = 2MC = CO,$$

which proves that N lies on ω.

Comment The possible positions of N are all the points of ω with the exception of the two points lying on the line CO. Indeed, P does not lie on the line CO because otherwise the point N is not well-defined, and therefore also N does not lie on the same line.

Conversely, let N be any point on ω and not lying on the line CO. Let P be the corresponding point on the circumcircle of ABC, namely such that NP is parallel and equal to CO. Let Q be the intersection of CP and NM. As before, the triangles QNP and QMC are similar, and now from the relation $NP = 2MC$ we deduce that $QP = 2QC$. This proves that N can be obtained from P through the construction described in the statement of the problem.

Alternative solution Let M' denote the symmetric of M with respect to O.

Let us consider the quadrilateral $MM'PN$. The lines MM' and NP are parallel by construction. Also the lines PM' and NM are parallel (homothety from C with coefficient 3). It follows that $MM'PN$ is a parallelogram, and hence $PN = MM' = OC$.

Computational solution There are many computation approaches to this problem. For example, we can set Cartesian coordinates so that

$$A = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \qquad B = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \qquad C = (0,1), \qquad M = \left(0, \frac{1}{2}\right).$$

Setting $P = (a, b)$, we obtain that $Q = (a/3, (2 + b)/3)$. The equation of the line through P and perpendicular to AB is $x = a$. The equation of the line MQ (if $a \neq 0$) is

$$y - \frac{1}{2} = \frac{x}{a}\left(\frac{1}{2} + b\right).$$

The intersection of the two lines is therefore

$$N = (a, 1 + b) = P + (0, 1).$$

This shows that the map $P \to N$ in the translation by the vector $(0, 1)$. This result is independent of the position of P (provided that $a \neq 0$, because otherwise N is not well-defined).

When P lies on the circumcircle of ABC, with the exception of the two points with $a = 0$, then necessarily N lies on the translated circle (which is the circle with center in C and radius 1).

Problem 2 Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

(a) Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

(b) For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.

Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2$, $y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$).

(Mihail Baluna, Romania)

Solution Every integer $x \geq 2$ can be written as the telescopic product of $x - 1$ elements of A as

$$x = \left(1 + \frac{1}{x-1}\right) \cdot \left(1 + \frac{1}{x-2}\right) \cdot \ldots \cdot \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{1}\right),$$

which is enough to establish part (a). We now consider part (b). Notice that for any positive integer k we have

$$f(2^k + 1) \leq k + 1,$$

because $2^k + 1 = \left(1 + \frac{1}{2^k}\right) \cdot 2^k$ is a representation of $2^k + 1$ as a product of $k + 1$ elements of A. We claim that all the pairs (x, y) of the form

$$x = 5, \qquad y = \frac{2^{4k+2} + 1}{5}$$

satisfy the required inequality. Notice that y is an integer for any positive value of k, because $2^{4k+2} + 1 \equiv 16^k \cdot 4 + 1 \equiv 5 \equiv 0 \pmod 5$. Furthermore, $f(xy) = f(2^{4k+2} + 1) \leq 4k + 3$ (and $f(x) = f(2^2 + 1) \leq 3$) by the above. We now need some lower bounds on the values of f. Notice that no element of A exceeds 2, and therefore the product of at most k elements of A does not exceed 2^k: it follows that

$$f(n) \geq \lceil \log_2(n) \rceil, \tag{Q2.1}$$

and in particular that

$$f(5) = f(2^2 + 1) \geq \lceil \log_2(5) \rceil = 3.$$

We have thus proven $f(x) = f(5) = 3$. We want to show $f(xy) < f(x) + f(y)$, and since we know $f(xy) \leq 4k + 3$ and $f(x) = 3$ we are reduced to showing $f(y) > 4k$. Since $y > 2^{4k-1}$, from (Q2.1) we already know that $f(y) \geq 4k$, and hence we just need to exclude that $f(y) = 4k$. Let us assume that we can represent y in the form $a_1 \cdot \ldots \cdot a_{4k}$ with every a_i in A. At least one of the a_i is not 2 (otherwise the product would be a power of 2, while y is odd), and hence it is less than or equal to $3/2$. It follows that

$$a_1 \cdot \ldots \cdot a_{4k} \leq 2^{4k-1} \cdot \frac{3}{2} = 15 \cdot \frac{2^{4k-2}}{5} < \frac{2^{4k+2}}{5} < y,$$

which contradicts the fact that $a_1 \cdot \ldots \cdot a_{4k}$ is a representation of y.

Note. Using a similar approach one can also prove that all pairs of the form

$$\left(3, \frac{2^{2k+1}+1}{3}\right) \quad \text{and} \quad \left(11, \frac{2^{10k+5}+1}{11}\right)$$

satisfy the required inequality.

Second solution As in the previous solution we obtain the lower bound (Q2.1).

Now we claim that all the pairs of the form

$$x = 2^k + 1, \qquad y = 4^k - 2^k + 1$$

satisfy the required inequality when k is large enough. To begin with, it is easy to see that

$$2^k + 1 = \frac{2^k+1}{2^k} \cdot \underbrace{2 \cdot \ldots \cdot 2}_{k \text{ terms}} \qquad \text{and} \qquad 2^{3k} + 1 = \frac{2^{3k}+1}{2^{3k}} \cdot \underbrace{2 \cdot \ldots \cdot 2}_{3k \text{ terms}},$$

which shows that $f(2^k + 1) \leq k + 1$ and $f(2^{3k} + 1) \leq 3k + 1$. On the other hand, from (Q2.1) we deduce that the previous inequalities are actually equalities, and therefore

$$f(x) = k + 1 \qquad \text{and} \qquad f(xy) = 3k + 1.$$

Therefore, it remain to show that $f(y) > 2k$. Since $y > 2^{2k-1}$ (for $k \geq 1$), from (Q2.1) we already know that $f(y) \geq 2k$, and hence we just need to exclude that $f(y) = 2k$. Let us assume that we can represent y in the form $a_1 \cdot \ldots \cdot a_{2k}$. At least one of the factors is not 2, and hence it is less than or equal to 3/2. Thus when k is large enough it follows that

$$a_1 \cdot \ldots \cdot a_{2k} \leq 2^{2k-1} \cdot \frac{3}{2} = \frac{3}{4} \cdot 2^{2k} < 2^{2k} - 2^k < y,$$

which contradicts the fact that $a_1 \cdot \ldots \cdot a_{2k}$ is a representation of y.

Third solution Let's start by showing that $(x, y) = (7, 7)$ satisfies $f(xy) < f(x) + f(y)$. We have $f(7) \geq 4$ since 7 cannot be written as the product of 3 or fewer elements of A: indeed $2^3 > 7$, and any other product of at most three elements of A does not exceed $2^2 \cdot \frac{3}{2} = 6 < 7$. On the other hand, $f(49) \leq 7$ since $49 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot \frac{3}{2} \cdot \frac{49}{48}$.

Suppose by contradiction that there exist only finitely many pairs (x, y) that satisfy $f(xy) < f(x) + f(y)$. This implies that there exists M large enough so that whenever $a > M$ or $b > M$ holds we have $f(ab) = f(a)+f(b)$ (indeed, it is clear that the reverse inequality $f(ab) \leq f(a)+f(b)$ is always satisfied).

Now take any pair (x, y) that satisfies $f(xy) < f(x) + f(y)$ and let $n > M$ be any integer. We obtain

$$f(n) + f(xy) = f(nxy) = f(nx) + f(y) = f(n) + f(x) + f(y),$$

which contradicts $f(xy) < f(x) + f(y)$.

Problem 3 The n contestants of an EGMO are named $C_1, \ldots, C_n$. After the competition they queue in front of the restaurant according to the following rules.

- The Jury chooses the initial order of the contestants in the queue.
- Every minute, the Jury chooses an integer i with $1 \leq i \leq n$.
  - If contestant C_i has at least i other contestants in front of her, she pays one euro to the Jury and moves forward in the queue by exactly i positions.
  - If contestant C_i has fewer than i other contestants in front of her, the restaurant opens and the process ends.

(a) Prove that the process cannot continue indefinitely, regardless of the Jury's choices.

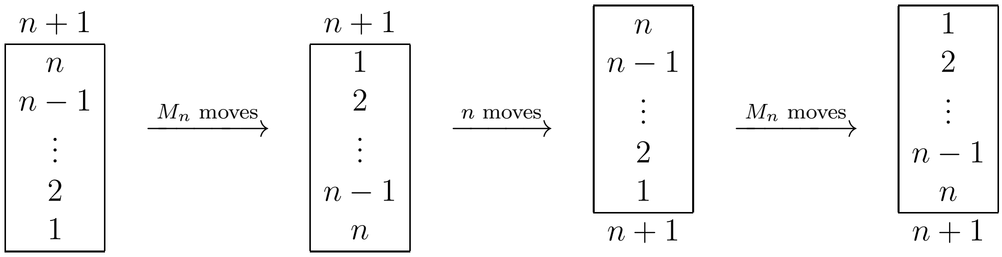
(b) Determine for every n the maximum number of euros that the Jury can collect by cunningly choosing the initial order and the sequence of moves.

(Hungary)

Solution The maximal number of euros is $2^n - n - 1$.

To begin with, we show that it is possible for the Jury to collect this number of euros. We argue by induction. Let us assume that the Jury can collect M_n euros in a configuration with n contestants. Then we show that the Jury can collect at least $2M_n + n$ moves in a configuration with $n+1$ contestants. Indeed, let us begin with all the contestants lined up in reverse order. In the first M_n moves the Jury keeps C_{n+1} in first position and reverses the order of the remaining contestants, then in the next n moves all contestants $C_1, \ldots, C_n$ (in this order) jump over C_{n+1} and end up in the first n positions of the line in reverse order, and finally in the last M_n moves the Jury rearranges the first n positions.

Since $M_1 = 0$ and $M_{n+1} \geq 2M_n + n$, an easy induction shows that $M_n \geq 2^n - n - 1$.



Let us show now that at most $2^n - n - 1$ moves are possible. To this end, let us identify a line of contestants with a permutation σ of $\{1, \ldots, n\}$. To each permutation we associate the set of *reverse pairs*

$$R(\sigma) := \{(i,j) : 1 \leq i < j \leq n \text{ and } \sigma(i) > \sigma(j)\},$$

and the nonnegative integer

$$W(\sigma) := \sum_{(i,j) \in R(\sigma)} 2^i,$$

which we call the *total weight* of the permutation. We claim that the total weight decreases after any move of the contestants. Indeed, let us assume that C_i moves forward in the queue, let σ be the permutation before the move, and let σ' denote the permutation after the move. Since C_i jumps over exactly i contestants, necessarily she jumps over at least one contestant C_j with index

$j > i$. This means that the pair (i, j) is reverse with respect to σ but not with respect to σ', and this yields a reduction of 2^i in the total weight. On the other hand, the move by C_i can create new reverse pairs of the form (k, i) with $k < i$, but their total contribution is at most

$$2^0 + 2^1 + \ldots + 2^{i-1} = 2^i - 1.$$

In conclusion, when passing from σ to σ', at least one term 2^i disappears from the computation of the total weight, and the sum of all the new terms that might have been created is at most $2^i - 1$. This shows that $W(\sigma') \leq W(\sigma) - 1$.

We conclude by observing that the maximum possible value of $W(\sigma)$ is realized when all pairs are reverse, in which case

$$W(\sigma) = \sum_{i=1}^{n} (i-1)2^i = 2^n - n - 1.$$

This proves that the number of moves is less than or equal to $2^n - n - 1$, and in particular it is finite.

Alternative solution As in the previous solution, the fundamental observation is again that, when a contestant C_i moves forward, necessarily she has to jump over at least one contestant C_j with $j > i$.

Let us show now that the process ends after a finite number of moves. Let us assume that this is not the case. Then at least one contestant moves infinitely many times. Let i_0 be the largest index such that C_{i_0} moves infinitely many times. Then necessarily C_{i_0} jumps infinitely many times over some fixed C_{j_0} with $j_0 > i_0$. On the other hand, we know that C_{j_0} makes only a finite number of moves, and therefore she can precede C_{i_0} in the line only a finite number of times, which is absurd.

In order to estimate from above the maximal number of moves, we show that the contestant C_i can make at most $2^{n-i} - 1$ moves. Indeed, let us argue by "backward extended induction". To begin with, we observe that the estimate is trivially true for C_n because she has no legal move.

Let us assume now that the estimate has been proved for C_i, C_{i+1}, ..., C_n, and let us prove it for C_{i-1}. When C_{i-1} moves, at least one contestant C_j with $j > i - 1$ must precede her in the line. The initial configuration can provide at most $n - i$ contestants with larger index in front of C_{i-1}, which means at most $n - i$ moves for C_{i-1}. All other moves are possible only if some contestant in the range C_i, C_{i+1}, ..., C_n jumps over C_{i-1} during her moves. As a consequence, the total number of moves of C_{i-1} is at most

$$n - i + \sum_{k=i}^{n} (2^{n-k} - 1) = 2^{n-i+1} - 1.$$

Summing over all indices we obtain that

$$\sum_{i=1}^{n} (2^{n-i} - 1) = 2^n - n - 1,$$

which gives an estimate for the total number of moves.

The same example of the first solution shows that this upper bound can actually be achieved.

Comment In every move of the example, the moving contestant jumps over exactly one contestant with larger index (and as a consequence over all contestants with smaller index).

Problem 4 A *domino* is a 1×2 or 2×1 tile.

Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap.

The *value* of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called *balanced* if there exists some $k \geq 1$ such that each row and each column has a value of k.

Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

(Merlijn Staps, The Netherlands)

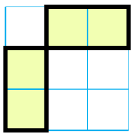
Solution The minimal number of dominoes required in a balanced configuration is $2n/3$ if n is a multiple of 3, and $2n$ otherwise.

In order to show that this number is necessary, we count in two different ways the number of elements of the set S of all pairs (ℓ, d), where ℓ is a row or a column of the board, and d is a domino that covers at least one cell of that row or column. On the one hand, since each row or column intersects the same number k of dominoes, the set S has $2nk$ elements. On the other hand, since each domino intersects 3 rows/columns, the set S has $3D$ elements, where D is the total number of dominoes on the board. This leads to the equality

$$2nk = 3D.$$

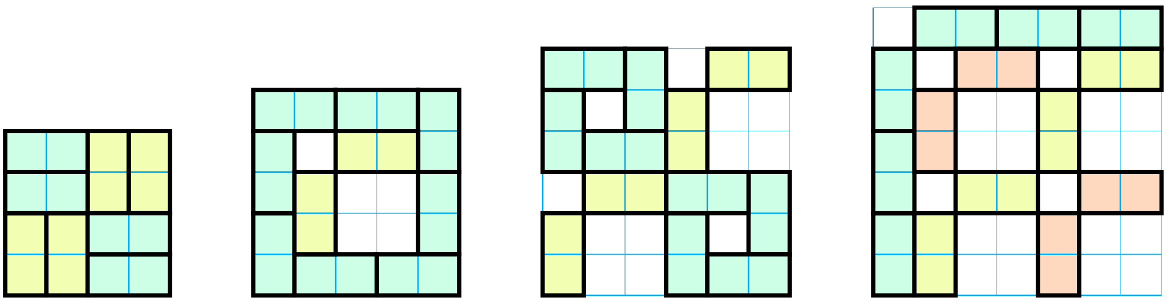
If n is a multiple of 3, from the trivial inequality $k \geq 1$ we obtain that $D \geq 2n/3$. If n is not a multiple of 3, then k is a multiple of 3, which means that $k \geq 3$ and hence $D \geq 2n$.

Now we need to exhibit a balanced configuration with this number of dominoes. The following diagram shows a balanced configuration with $n = 3$ and $k = 1$.



If n is any multiple of 3, we can obtain a balanced configuration with $k = 1$ by using $n/3$ of these 3×3 blocks along the principal diagonal of the board.

The following diagrams show balanced configurations with $k = 3$ and $n \in \{4, 5, 6, 7\}$.



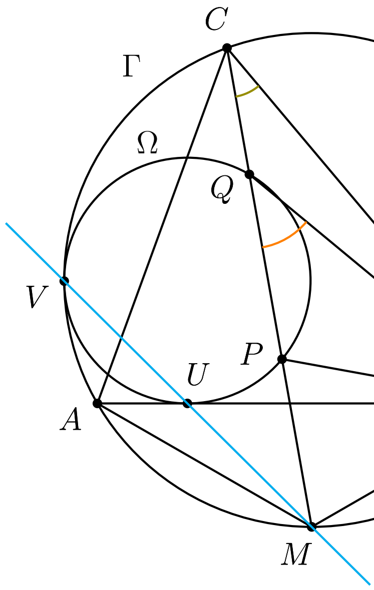
Any $n \geq 8$ can be written in the form $4A + r$ where A is a positive integer and $r \in \{4, 5, 6, 7\}$. Therefore, we can obtain a balanced configuration with $n \geq 8$ and $k = 3$ by using one block with size $r \times r$, and A blocks with size 4×4 along the principal diagonal of the board. In particular, this construction covers all the cases where n is not a multiple of 3.

Problem 5 Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q.

Prove that $\angle ABP = \angle QBC$.

(Dominika Regiec, Poland)

Solution 1 Let M be the midpoint of the arc AB that does not contain C, let V be the intersection of Ω and Γ, and let U be the intersection of Ω and AB.



The proof can be divided in two steps:

1. *Proving that* $MP \cdot MQ = MB^2$.

   It is well-known that V, U and M are collinear (indeed the homothety with center in V that sends Ω to Γ sends U to the point of Γ where the tangent to Γ is parallel to AB, and this point is M), and
   $$MV \cdot MU = MA^2 = MB^2.$$
   This follows from the similitude between the triangles $\triangle MAV$ and $\triangle MUA$. Alternatively, it is a consequence of the following well-known lemma: *Given a circle* Γ *with a chord* AB*, let* M *be the middle point of one of the two arcs* AB*. Take a line through* M *which intersects* Γ *again at* X *and* AB *at* Y*. Then* $MX \cdot MY$ *is independent of the choice of the line.*

   Computing the power of M with respect to Ω we obtain that
   $$MP \cdot MQ = MU \cdot MV = MB^2.$$

2. *Conclude the proof given that* $MP \cdot MQ = MB^2$.

   The relation $MP \cdot MQ = MB^2$ in turn implies that triangle $\triangle MBP$ is similar to triangle $\triangle MQB$, and in particular $\angle MBP = \angle MQB$. Keeping into account that $\angle MCB = \angle MBA$, we finally conclude that
   $$\angle QBC = \angle MQB - \angle MCB = \angle MBP - \angle MBA = \angle PBA,$$
   as required.

Solution 2 The second solution is in fact a different proof of the first part of Solution 1.

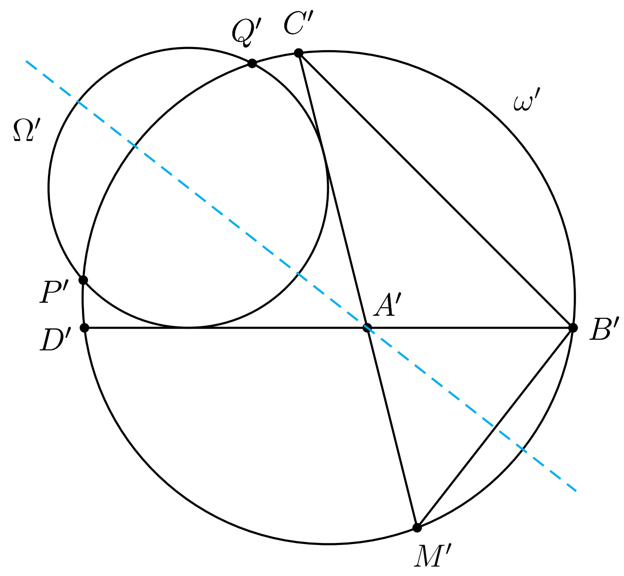
Let us consider the inversion with respect to circle with center M and radius $MA = MB$. This inversion switches AB and Γ, and fixes the line passing through M, U, V. As a consequence, it keeps Ω fixed, and therefore it switches P and Q. This is because they are the intersections between the fixed line MC and Ω, and the only fixed point on the segment MC is its intersection with the inversion circle (thus P and Q are switched). This implies that $MP \cdot MQ = MB^2$.

Solution 3 This solution is instead a different proof of the second step of Solution 1.

Let I and J be the incenter and the C-excenter of $\triangle ABC$ respectively. It is well-known that $MA = MI = MJ$, therefore the relation $MP \cdot MQ = MA^2$ implies that $(P, Q, I, J) = -1$.

Now observe that $\angle IBJ = 90°$, thus BI is the angle bisector of $\angle PBQ$ as it is well-known from the theory of harmonic pencils, and this leads easily to the conclusion.

Solution 4 Let D denote the intersection of AB and CM. Let us consider an inversion with respect to B, and let us use primes to denote corresponding points in the transformed diagram, with the gentlemen agreement that $B' = B$.



Since inversion preserves angles, it turns out that

$$\angle A'B'M' = \angle A'M'B' = \angle ACB,$$

and in particular triangle $A'B'M'$ is isosceles with basis $B'M'$.

The image of CM is the circumcircle of $B'C'M'$, which we denote by ω'. It follows that the centers of both ω' and the image Ω' of Ω lie on the perpendicular bisector of $B'M'$. Therefore, the whole transformed diagram is symmetric with respect to the perpendicular bisector of $B'M'$, and in particular the arcs $D'P'$ and $Q'C'$ of ω' are equal.

This is enough to conclude that $\angle D'B'P' = \angle Q'B'C'$, which implies the conclusion.

Problem 6

(a) Prove that for every real number t such that $0 < t < \frac{1}{2}$ there exists a positive integer n with the following property: for every set S of n positive integers there exist two different elements x and y of S, and a *non-negative* integer m (i.e. $m \geq 0$), such that

$$|x - my| \leq ty.$$

(b) Determine whether for every real number t such that $0 < t < \frac{1}{2}$ there exists an infinite set S of positive integers such that

$$|x - my| > ty$$

for every pair of different elements x and y of S and every *positive* integer m (i.e. $m > 0$).

(Merlijn Staps, The Netherlands)

Solution

Part (a) Let n be any positive integer such that

$$(1 + t)^{n-1} \geq \frac{1}{t} \tag{Q6.1}$$

(this inequality is actually true for every large enough n due to Bernoulli's inequality).

Let S be any set of n distinct positive integers, which we denote by

$$s_1 < s_2 < \ldots < s_n.$$

We distinguish two cases.

- If $s_{i+1} \leq (1 + t)s_i$ for some $i \in \{1, \ldots, n - 1\}$, then

  $$|s_{i+1} - s_i| = s_{i+1} - s_i \leq ts_i,$$

  and therefore the required inequality is satisfied with $x = s_{i+1}$, $y = s_i$, and $m = 1$.

- If $s_{i+1} > (1 + t)s_i$ for every $i \in \{1, \ldots, n - 1\}$, then by induction we obtain that

  $$s_n > (1 + t)^{n-1}s_1.$$

  As a consequence, from (Q6.1) it follows that

  $$|s_1| = s_1 < \frac{1}{(1 + t)^{n-1}} \cdot s_n \leq ts_n,$$

  and therefore the required inequality is satisfied with $x = s_1$, $y = s_n$, and $m = 0$.

Part (b) (Explicit formula) We claim that an infinite set with the required property exists. To this end, we rewrite the required condition in the form

$$\left|\frac{x}{y} - m\right| > t.$$

This is equivalent to saying that the distance between the ratio x/y and the set of positive integers is greater than t.

Now we construct an increasing sequence s_n of odd coprime positive integers satisfying

$$\frac{1}{2} - \frac{1}{2s_n} > t \qquad \forall n \geq 1, \tag{Q6.2}$$

and such that for every $j > i$ it turns out that

$$\frac{s_i}{s_j} < \frac{1}{2} \qquad \text{and} \qquad t < \left\{\frac{s_j}{s_i}\right\} < \frac{1}{2}, \tag{Q6.3}$$

where $\{\alpha\}$ denotes the fractional part of α. This is enough to show that the set $S := \{s_n : n \geq 1\}$ has the required property.

To this end, we consider the sequence defined recursively by

$$s_{n+1} = \frac{(s_1 \cdot \ldots \cdot s_n)^2 + 1}{2},$$

with s_1 large enough. An easy induction shows that this is an increasing sequence of odd positive integers. For every $i \in \{1, \ldots, n\}$ it turns out that

$$\frac{s_i}{s_{n+1}} \leq \frac{2}{s_i} \leq \frac{2}{s_1} < \frac{1}{2}$$

because s_1 is large enough, which proves the first relation in (Q6.3). Moreover, it turns out that

$$\frac{s_{n+1}}{s_i} = \frac{(s_1 \cdot \ldots \cdot s_n)^2}{2s_i} + \frac{1}{2s_i}.$$

The first term is a positive integer plus $1/2$, from which it follows that the distance of s_{n+1}/s_i from the positive integers is greater than or equal to

$$\frac{1}{2} - \frac{1}{2s_i} \geq \frac{1}{2} - \frac{1}{2s_1},$$

which is greater than t if s_1 is large enough. This proves the second relation in (Q6.3).

Part (b) (Arithmetic approach) We produce an increasing sequence s_n of odd and coprime positive integers that satisfies (Q6.3) every $j > i$. As in the previous solution, this is enough to conclude.

We argue by induction. To begin with, we choose s_1 to be any odd integer satisfying the inequality in (Q6.2). Let us assume now that $s_1, \ldots, s_n$ have already been chosen, and let us choose s_{n+1} in such a way that

$$s_{n+1} \equiv \frac{s_i - 1}{2} \pmod{s_i} \qquad \forall i \in \{1, \ldots, n\}.$$

We can solve this system because the previously chosen integers are odd and coprime. Moreover, any solution of this system is coprime with $s_1, \ldots, s_n$. Indeed, for every $1 \leq i \leq n$ it turns out that

$$s_{n+1} = \frac{s_i - 1}{2} + k_i s_i$$

for some positive integer k_i. Therefore, any prime p that divides both s_{n+1} and s_i divides also $(2k_i + 1)s_i - 2s_{n+1} = 1$, which is absurd. Finally, we observe that we can assume that s_{n+1} is odd and large enough. In this way we can guarantee that

$$\frac{s_i}{s_{n+1}} < \frac{1}{2} \qquad \forall i \in \{1, \ldots, n\},$$

which is the first requirement in (Q6.3), and

$$k_i + t < k_i + \frac{1}{2} - \frac{1}{2s_i} = \frac{s_{n+1}}{s_i} < k_i + \frac{1}{2} \qquad \forall i \in \{1, \ldots, n\},$$

which implies the second requirement in (Q6.3).

Part (b) (Algebraic approach) Again we produce an increasing sequence s_n of positive integers that satisfies (Q6.3) every $j > i$.

To this end, for every positive integer x, we define its *security region*

$$S(x) := \bigcup_{n \geq 1} \left((n+t)x, (n + \tfrac{1}{2})x\right).$$

The security region $S(x)$ is a periodic countable union of intervals of length $(\frac{1}{2} - t)x$, whose left-hand or right-hand endpoints form an arithmetic sequence. It has the property that

$$t < \left\{\frac{y}{x}\right\} < \frac{1}{2} \qquad \forall y \in S(x).$$

Now we prove by induction that we can choose a sequence s_n of positive integers satisfying (Q6.3) and in addition the fact that every interval of the security region $S(s_n)$ contains at least one interval of $S(s_{n-1})$.

To begin with, we choose s_1 large enough so that the length of the intervals of $S(s_1)$ is larger than 1. This guarantees that any interval of $S(s_1)$ contains at least a positive integer. Now let us choose a positive integer $s_2 \in S(s_1)$ that is large enough. This guarantees that s_1/s_2 is small enough, that the fractional part of s_2/s_1 is in $(t, 1/2)$, and that every interval of the security region $S(s_2)$ contains at least one interval of $S(s_1)$, and hence at least one positive integer.

Let us now assume that $s_1, \ldots, s_n$ have been already chosen with the required properties. We know that every interval of $S(s_n)$ contains at least one interval of $S(s_{n-1})$, which in turn contains an interval in $S(s_{n-2})$, and so on up to $S(s_1)$. As a consequence, we can choose a large enough positive integer s_{n+1} that lies in $S(s_k)$ for every $k \in \{1, \ldots, n\}$. Since s_{n+1} is large enough, we are sure that

$$\frac{s_k}{s_{n+1}} < t \qquad \forall k \in \{1, \ldots, n\}.$$

Moreover, we are sure also that all the intervals of $S(s_{n+1})$ are large enough, and therefore they contain at least one interval of $S(s_n)$, which in turn contains at least one interval of $S(s_{n-1})$, and so on. Finally, the condition

$$t < \left\{\frac{s_{n-1}}{s_n}\right\} < \frac{1}{2}$$

is guaranteed by the fact that s_{n+1} was chosen in an interval that is contained in $S(s_k)$ for every $k \in \{1, \ldots, n\}$. This completes the induction.