SHORT PROBLEMS: MARKING SCHEME

1. Several exoplanets have been observed in the Gliese 876 system $(\mathrm{M_G} = 0.33 \pm 0.03\mathrm{M_\odot})$ as given in the following table,

Gliese System	**Mass**	**Semi Major Axis (AU)**
Gliese 876 b	2.276 M_J	0.2083
Gliese 876 c	0.714 M_J	0.1296
Gliese 876 d	6.8 $M_\oplus$	0.0208
Gliese 876 e	15 $M_\oplus$	0.334

where $\mathrm{M_\odot}$ is mass of Sun, $\mathrm{M_J}$ is mass of Jupiter ($M_J = 1.89813 \times 10^{27}\ kg$), and $\mathrm{M_\oplus}$ is mass of Earth. Assume that all these planets revolve around Gliese 876 in the same direction. Two planets are said to be in resonant orbits if the synodic period of one planet with respect to the other planet is an integer multiple of the orbital period of the second planet.

Find if any of the exoplanets of Gliese 876 system may have resonant orbits.

Answer and Marking Scheme:

1 **Determine planet periods**

Synodic period of planet 1 and 2 (assume $P_1 < P_2$) is $\frac{1}{P_S} = \frac{1}{P_1} - \frac{1}{P_2} \rightarrow P_S = \frac{P_1 P_2}{P_2 - P_1}$

Resonant orbit happens if

$$P_S = mP_1 \rightarrow \frac{P_1 P_2}{P_2 - P_1} = mP_1 \rightarrow m = \frac{P_2}{P_2 - P_1} > 1, \text{integer}$$

Using the data of the Gliesean system we have

Gliese System	**Mass (kg)**	**Semi Major Axis (m)**	**Period (Earth, days)** $P = 2\pi\sqrt{\frac{a^3}{GM_G}}$
Gliese 876 star	6.64359×10^{29}		
Gliese 876 b	4.31938×10^{27}	3.1161×10^{10}	60.07568
Gliese 876 c	1.35564×10^{27}	1.9388×10^{10}	29.48304
Gliese 876 d	4.079×10^{25}	3.1116×10^{9}	1.8956
Gliese 876 e	8.7194×10^{25}	5.0011×10^{10}	122.1432

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2	**Calculate resonant orbit for each planet** From the results above we see that • Gliese 876 c and Gliese 876 b have resonant orbit $$m = \frac{P_b}{P_b - P_c} = 1.96 \approx 2$$ • Gliese 876 b and Gliese 876 e have resonant orbit $$m = \frac{P_e}{P_e - P_b} = 1.9679 \approx 2$$	**25** **25**

2. One satellite of a planet has an orbital period of 7 days, 3 hours, 43 minutes, and the semi major axis is 15.3 times the mean radius of the planet. The Moon has an orbital period of 27 days, 7 hours, 43 minutes and the semi major axis is 60.3 times the Earth's mean radius. Assume that the mass of the moon and the satellite is negligible compared to the mass of the planet. Calculate the ratio of the planet's mean density to that of the Earth.

Answer and Marking Scheme:

First Version

1	Kepler third Law for satelite and Moon $\frac{P_S^2}{a_S^3} = \frac{P_S^2}{(15.3\ R_P)^3} = \frac{4\pi^2}{GM_P};$ $\frac{P_M^2}{a_M^3} = \frac{P_M^2}{(60.3\ R_E)^3} = \frac{4\pi^2}{GM_E};$	**30**
2	Express mass in term of mass density: $\frac{P_S^2}{(15.3\ R_P)^3} = \frac{4\pi^2}{GM_P} = \frac{4\pi^2}{G\rho_P \frac{4}{3}\pi R_P^3}$ $\frac{P_M^2}{(60.3\ R_E)^3} = \frac{4\pi^2}{GM_E} = \frac{4\pi^2}{G\rho_E \frac{4}{3}\pi R_E^3}$	**30**
3	Evaluate ratio of mass-density: $\frac{\rho_E}{\rho_P} = \frac{60.3^3 \times P_S^2}{15.3^3 \times P_M^2}$ $\frac{\rho_E}{\rho_P} = 0.238\ \approx 0.24$	**30** **10**

Second Alternate Version

- *The period of the Satellite:* $P_S = 7\ days, 3\ hours, 43\ minutes = 0.0196\ years,$
- *Mean orbital radius of the Satellite :* $a_S = 15.3\ R_P$*;* R_P*: radius of the Planet.*
- *The period of the Moon:* $P_M = 27\ days, 7\ hours, 43\ minutes =\ 0.0749\ years,$
- *Mean orbital radius of the Moon :* $a_M = 60.3\ R_E$ *;* R_E*: radius of the Earth.*
- *Mass of the Sun (M)*
- *Kepler's Third Law :* $G(M+m) = 4\pi^2 \frac{a^3}{T^2}$ →
    - *Mass of the Planet:* m_P
    - *Radius of the Planet :* R_P

- *Semi major axis of the Planet = Mean orbital radius of the Plant:* a_P
- *Semi major axis of theSatellite = Mean orbital radius of theSatellite:* a_S
- *Semi major axis of the Moon = Mean orbital radius of the Moon:* a_M
- *Mass of the Earth :* m_E
- *Radius of the Earth:* R_E
- *Semi major axis of the Earth = Mean orbital radius of the Earth:* a_E

Planet – Satellite: $G(m_P + m_S) = 4\pi^2 \frac{a_S{}^3}{P_S{}^2}$, $m_P \gg m_S \rightarrow G(m_P) = 4\pi^2 \frac{a_S{}^3}{P_S{}^2}$ (1) (5)

The Earth–the Moon:$G(m_E + m_M) = 4\pi^2 \frac{a_M{}^3}{P_M{}^2}$, $m_E \gg m_M \rightarrow G(m_E) = 4\pi^2 \frac{a_M{}^3}{P_M{}^2}$ (2) (5)

Planet–Sun : $G(M + m_P) = 4\pi^2 \frac{a_P{}^3}{P_P{}^2}$, $M \gg m_P \rightarrow G(M) = 4\pi^2 \frac{a_P{}^3}{P_P{}^2}$ (3) (5)

Earth–Sun : $G(M + m_E) = 4\pi^2 \frac{a_E{}^3}{P_E{}^2}$, $M \gg m_E \rightarrow G(M) = 4\pi^2 \frac{a_E{}^3}{P_E{}^2}$ (4) (5)

From equation (1) and (3):

$$\frac{m_P}{M} = \frac{4\pi^2 \frac{a_S{}^3}{P_S{}^2}}{4\pi^2 \frac{a_P{}^3}{P_P{}^2}} = \frac{(a_S)^3}{(a_P)^3}\frac{(P_P)^2}{(P_S)^2} = \frac{(15.3R_P)^3}{(a_P)^3}\frac{(P_P)^2}{(P_S)^2} = \frac{(15.3R_P)^3}{(a_P)^3}\frac{(P_P)^2}{(0.0196)^2}$$ (20)

From equation (2) and (4):

$$\frac{m_E}{M} = \frac{4\pi^2 \frac{a_M{}^3}{P_M{}^2}}{4\pi^2 \frac{a_E{}^3}{P_E{}^2}} = \frac{(a_M)^3}{(a_E)^3}\frac{(P_E)^2}{(P_M)^2} = \frac{(60.3R_E)^3}{(a_E)^3}\frac{(P_E)^2}{(P_E)^2} = \frac{(60.3R_E)^3}{(a_E)^3}\frac{(P_E)^2}{(0.0749)^2} = \frac{(60.3R_E)^3}{(1)^3}\frac{(1)^2}{(0.0749)^2}$$ (20)

→

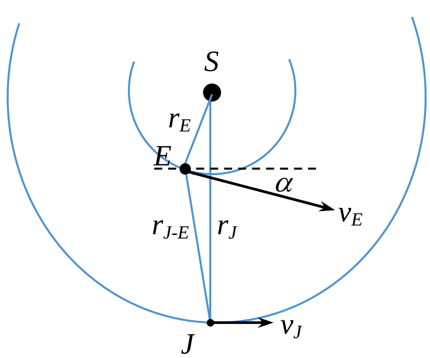
$$\frac{m_E}{m_P} = \frac{\frac{(15.3R_P)^3}{(a_P)^3}\frac{(P_P)^2}{(0.0196)^2}}{\frac{(60.3R_P)^3}{(1)^3}\frac{(1)^2}{(0.0749)^2}} = \frac{(15.3R_P)^3/(0.0196)^2}{(60.3R_E)^3/(0.0749)^2}$$ (20)

→

$$\frac{\rho_P}{\rho_E} = \frac{m_P/Vol_P}{m_E/Vol_E} = \frac{m_P/\frac{4}{3}\pi(R_P)^3}{m_E/\frac{4}{3}\pi(R_E)^3} = \frac{(15.3R_P)^3/(0.0196)^2}{(60.3R_E)^3/(0.0749)^2}\cdot\frac{\frac{4}{3}\pi(R_E)^3}{\frac{4}{3}\pi(R_P)^3} = \frac{\frac{(15.3)^3}{(0.0196)^2}}{\frac{(60.3)^3}{(0.0749)^2}} = 0.24$$ (20)

3. On 27 May 2015 at 02:18:49, the occultation of the star HIP 89931 (δ −Sgr) by the asteroid 1285 Julietta was observed from Borobudur temple, which was located at the center of the asteroid shadow path. It lasted for only 6.201 s. Assume that Earth's orbit is circular and the orbit of Julietta is on the ecliptic plane and revolves in the same direction as Earth. At the occultation, Julietta is near its aphelion. At the time of the occultation the distances of Julietta from the Sun and the Earth are 3.076 AU and 2.156 AU respectively. Find the approximate diameter of asteroid Julietta, if the semi major axis of Julietta is $a = 2.9914$ AU.

Answer and Marking Scheme:

1	**Calculate the Earth revolution speed** Earth revolution speed is $$v_E = \frac{2\pi\bar{r}_E}{T_E} = \frac{2\pi a_E}{T_E} = 29805\ m/s$$	20
2	**Calculate Julietta revolution speed** Julietta revolution is (since the given data is semimajor axis, the orbit of Julietta is ellipse) $$v_J = \sqrt{GM\left(\frac{2}{r_J} - \frac{1}{a}\right)} = 16742{,}9\ m/s$$	25
3	**Calculate the approximate dimension of Julietta**  From the geometry $$\cos\alpha = \frac{r_E^2 + r_J^2 - r_{J-E}^2}{2r_E r_J} = 0.995$$ Thus, Julietta's relative projected speed to Earth is $$v_{J,rel} = v_E - v_J\cos\alpha = 13984.16\ m/s$$ Hence, the approximate dimension of Julietta is $$l_J = v_{J,rel}t_{occil} = 86701.8\ m$$	40 15

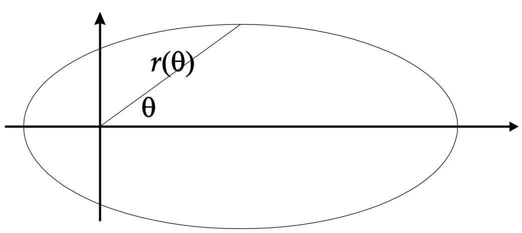
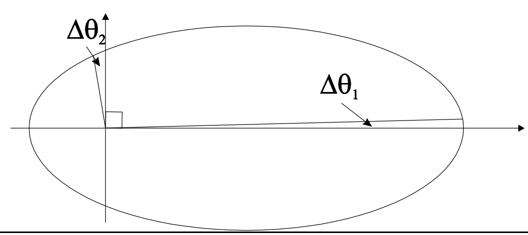
4. Let us assume that an observer using a hypothetical, far-infrared, Earth-sized telescope (wavelength range 20 to 640 μm) found a static and neutral supermassive black hole with a mass of 2.1 x 10^{10} $M_\odot$. Determine the maximum distance at which this black hole can be resolved by the observer.

Answer and Marking Scheme:

1	**Identify the relationship between the telescope's resolution and the angular size of black hole** This problem can be solved by using relationship between the telescope's resolution and the angular size of black holes which can be seen in this equation: $\phi_{telescope} \le \phi_{blackhole} \quad \Rightarrow \quad 1.2\frac{\lambda}{D_{telescope}} \le \frac{\text{Diameter of event horizon}}{\text{Observer Distance to galatic's the center}}$ $1.2\frac{\lambda}{D_{telescope}} \le \frac{2R_{blackhole}}{d}$	25
2	**Identify the relation for determining the distance** Then, it is assumed that the radius of black hole is the Schwarszchild radius; and the telescope is also Earth-sized telescope. Therefore: $1.2\frac{\lambda}{2R_{Earth}} \le \frac{2R_{Schwarzchild,BH}}{d}$ $d \le \frac{4\times R_{Schwarzchild,BH}\times R_{Earth}}{1.2\times\lambda} \quad \Rightarrow \quad d \le \frac{4\times M_{Black\ Hole}\times R_{Schwarchild,Sun}\times R_{Earth}}{1.2\times M_{Sun}\times\lambda}$ $d \le \frac{4\times 2.1\times 10^{10} M_{Sun}\times R_{Schwarchild,Sun}\times R_{Earth}}{1.2\times M_{Sun}\times\lambda} \Rightarrow \quad d \le \frac{8.4\times 10^{10}\times R_{Schwarchild,Sun}\times R_{Earth}}{1.2\times\lambda}$	50
3	**Calculating the maximum-observer distance** To find the maximum distance, then we should use the minimum λ. Therefore: $d_{\max} = \frac{8.4\times 10^{10}\times R_{Schwarchild,Sun}\times R_{Earth}}{1.2\times\lambda_{\min}} = \frac{8.4\times 10^{10}\times 2.95\times 10^{3}\times 6.3708\times 10^{6}}{1.2\times 20\times 10^{-6}} m = 6.6 \times 10^{25} m$	25

5. An observer is trying to determine an approximate value of the orbital eccentricity of a man-made satellite. When the satellite was at apogee, it was observed to have moved by $\Delta\theta_1 = 2'44''$ in a short time. When the radius vector connecting Earth and the satellite is perpendicular to the major axis (true anomaly is equal to 90°), within the same duration of time, it was observed to have moved by $\Delta\theta_2 = 21'17''$. Assume that the observer is located at the center of the Earth. Find an approximate value of the eccentricity of the satellite's orbit.

Answer and Marking Scheme:

1	**Identify expression for $r(\theta)$.** We know that $r(\theta)=\dfrac{a(1-e^2)}{1-e\cos\theta}$. Hence $$r(0)=\frac{a(1-e^2)}{1-e}=a(1+e)$$ $$r\left(\frac{\pi}{2}\right)=\frac{a(1-e^2)}{1-e\times 0}=a(1-e^2)$$   	30
2	**Estimate area of sectors using area of triangles.** Estimate the area of swept out sectors as the area of sectors of circles $$A(S_1)\approx 0.5\times\Delta\theta_1\times\left(r(0)\right)^2=\Delta\theta_1\left(a(1+e)\right)^2$$ $$A(S_2)\approx 0.5\times\Delta\theta_2\times\left(r\left(\tfrac{\pi}{2}\right)\right)^2=\Delta\theta_2\left(a(1-e^2)\right)^2$$	20
3	**Use Kepler's second law to find a relation between the ratio $\frac{\Delta\theta_1}{\Delta\theta_2}$ and the eccentricity e.** Use Kepler's second law to obtain that $$A(S_1)=A(S_2)$$ $$\Delta\theta_1\times(a(1+e))^2=\Delta\theta_2\times(a(1-e^2))^2$$ Thus, $$\frac{\Delta\theta_1}{\Delta\theta_2}=\left(\frac{1-e^2}{1+e}\right)^2=(1-e)^2=\frac{2'44''}{21'17''}\approx 0.12843$$	30
4	**Obtain an estimate value of the eccentricity.** Thus, the eccentricity is $e\approx 1-\sqrt{0.12843}=0.64$	20

6. At the start of every observation, a radio telescope is pointed at a point-source calibrator that has a known flux density of 21.86 Jy outside the Earth's atmosphere. However, on a certain date, the measured flux density of the calibrator source was 14.27 Jy. If the calibrator source was at an altitude of 35 degrees, estimate the zenith atmospheric optical depth, τ_z.

Answer and Marking Scheme:

1	Atmospheric optical depth τ_A can be determined from the measured flux, that is: $S_{meas} = \exp(-\tau_A)\, S_{real}$ $14.27 = \exp(-\tau_A) \times 21.86 \rightarrow \exp(-\tau_A) = 14.27/21.86 = 0.65$ Then $\tau_A = -\ln(0.65) = 0.43$	**50**
2	Now we have: $\tau_A = \tau_Z \sec z$ $\tau_z = \tau_A \cos z$ $= 0.43 \times \cos(90 - 35)° = 0.25$ where z is zenith angle.	**50**

7. A galaxy at the boundary of a galaxy cluster of radius 10 Mpc is expected to escape from the cluster if it has an initial velocity of at least 700 km/s relative to the center of the cluster. Calculate the density of the cluster.

Answer and Marking Scheme:

1	We have the total energy of the escaping galaxy: $E_T = E_K + E_P = \frac{mV^2}{2} - \frac{GMm}{R}$	**15**
2	Where *V* is velocity of galaxy and *R* radius of galaxy cluster Galaxy will escape from galaxy cluster if the total energy, $E_T = 0$	**10**
3	$E_T = 0 \rightarrow \frac{mV^2}{2} - \frac{GMm}{R} = 0$ $V^2 = \frac{2GM}{R}$	**15**
4	Escape velocity $V^2 = \frac{2G\left(\frac{4\pi}{3}R^3\rho\right)}{R}$	**20**
5	Density of cluster $\rho = \frac{3V^2}{8\pi G R^2}$	**15**
6	V=700 km/s R=10 Mpc $G=6.6378\times10^{-11} Nm^2kg$ $\rho=9.20121\times10^{-27} kg/m^3$	**25**

8. A strong continuum radio signal from a celestial body has been observed as a burst with a very short duration of 700 μs. The observed flux density at a frequency of 1660 MHz is measured to be 0.35 kJy. If the distance from the source is known to be 2.3 kpc, estimate the brightness temperature of this source.

Answer and Marking Scheme:

1	During 700 μs, the radio wave travels about $r = c\,t = 3 \times 10^{10} \times 700 \times 10^{-6} = 2.1 \times 10^{7}$ cm. The region from where the burst originates must be no larger than the distance that light can travel during the duration of the burst. So we estimate that r is the size of the source. ($r = 2.1 \times 10^{7}$ cm, $R = 2.3$ kpc) Flux density observed at 1660 MHz is $S_{1660MHz} = 0.35$ kJy $= 0.35 \times 10^{3} \times 10^{-23}$ ergs s^{-1} cm^{-2} Hz^{-1} $= 3.5 \times 10^{-21}$ ergs s^{-1} cm^{-2} Hz^{-1}	**10** **10**
2	The solid angle subtended by this source of radiation is $\Omega = \pi\,(r/R)^2 = 3.14\,(2.1 \times 10^{7}/2.3 \times 10^{3} \times 3.086 \times 10^{18})^2$ $= 2.75 \times 10^{-29}$ sr	**30**
3	The flux density is related to the total brightness by the relation: $S_{1660MHz} = B_{1660MHz}\,\Omega$ while at this frequency, the total brightness can be approximated from the Rayleigh-Jeans formula: $B_{1660MHz} = 2kT_b\nu^2/c^2$ where T_b is the brightness temperature. Then we have: $T_b = S_{1660MHz}c^2/2k\nu^2\Omega$ $T_b = [3.5 \times 10^{-21} \times (3 \times 10^{10})^2]/[2 \times (1.38 \times 10^{-16}) \times (1660 \times 10^{6})^2 \times (2.75 \times 10^{-29})]$ $= 1.5 \times 10^{26}$ K	**20** **20** **10**

9. Assume that the Sun is a perfect blackbody. Venus is also assumed to be a blackbody, with temperature T_V, and it is in thermal equillibrium (i.e. it is radiating about as much energy as it receives from the Sun) at its orbital distance of 0.72 AU. Suppose that at closest approach to Earth, Venus has an angular diameter of about 66 arcsec. What is the flux density of Venus at the closest approach to Earth as observed by a radio telescope at an observing frequency of 5 GHz?

Answer and Marking Scheme:

1	Solar luminosity is $L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4$ $= 4 \times 3.14 \times (6.96 \times 10^{10})^2 (5.67 \times 10^{-5})(5780)^4$ ergs/sec $= 3.9 \times 10^{33}$ ergs/sec	**10**
2	Flux of the Sun intercepted by Venus is $L_V = (A_{V(projected)}/A_{(sphere,0.72au)}) \times L_\odot$ Also, $L_V = 4\pi R_V^2 \sigma T_V^4$ So, $4\pi R_V^2 \sigma T_V^4 = (\pi R_V^2/4\pi d^2) \times L_\odot = (\pi R_V^2/4\pi d^2) \times 4\pi R_\odot^2 \sigma T_\odot^4$	**30**
3	Therefore, $T_V^4 = (R_\odot^2/4d^2) \times T_\odot^4 = [(6.957 \times 10^5 \text{ km})^2/4 \times (0.72 \times 149.6 \times 10^6)^2] \times (5780)^4$ $= (483998490000/46407499776000000) \times (5780)^4 = (4.84 \times 10^{11}/4.64 \times 10^{16}) \times (5780)^4$ $= 1.0429316216908190111241385226915e\text{-}5 \times 1114577187760656$ $= 1.04 \times 10^{-5} \times 1.11 \times 10^{15}$ $= 11624277939.308134327737156481455 = 1.1624 \times 10^{10}$ $T_V = 328.5$ K	**20**
4	The flux density of Venus at closest approach to Earth at an observing frequency of 5 GHz: (1″ = 4.84×10^{-6} rad) $S_{5GHz} = B_{5GHz}\,\Omega = (2kT_b\nu^2/c^2)\,\Omega$ ($\rightarrow T_b = T_V$) $= (2 \times 1.38 \times 10^{-16} \times 328.5 \times (5 \times 10^9)^2 \times \pi\,(66 \times 4.84 \times 10^{-6})^2)/(3 \times 10^{10})^2$ $= 8.07 \times 10^{-22}$ ergs s^{-1} cm^{-2} Hz^{-1} $= 80.7$ Jy	**40**

10. A molecular hydrogen cloud is known to have a temperature T = 115 K. The hydrogen atoms (assumed spherical) have (covalent) radius $r_H = 0.37\,x\,10^{-10}m$ and the separation centre-to-centre distance between the two atoms is $d_{H_2} = 0.74\,x\,10^{-10}m$. Assume that the molecules are in thermal equilibrium. Estimate the frequency at which they will radiate due to molecular rotational excitation.

Answer and Marking Scheme:

1	The molecules rotate with angular velocity ω and for spherical molecules, their moment of inertia is $$I_H = I_{cm} + m_H r_H^2$$ $$= \frac{2}{5} m_H^2 + m_H r_H^2 = \frac{7}{5} m_H r_H^2$$ For H_2 molecule, yield $$I_{H_2} = 2I_H = \frac{14}{5} m_H r_H^2\,,$$ Thus $$I = 6.401 \times 10^{-48}\ \mathrm{Kg\ m^2}$$	**30**
2	In thermal equilibrium(only 2 rotational degrees of freedom), we have: $$\frac{1}{2} I\omega^2 = kT$$	**30**
3	So we can estimate: $$\omega = \sqrt{2kT/I} = (2 \times 1.38 \times 10^{-16} \times 115/6{,}40 \times 10^{-48})^{1/2} = 2.23 \times 10^{13} s^{-1}$$ $$\nu = \frac{\omega}{2\pi} = 3.55 \times 10^{13}\mathrm{Hz}$$	**40**

11. The mass density of an object is inversely proportional to the radial distance from the center of the object with a factor of proportionality $\alpha = 5.0 \times 10^{13}\ kg/m^2$. If the escape velocity at the surface of the object is $v_0 = 1.5 \times 10^4\ m/s$, calculate the total mass of the object.

	Answer	
1	$\frac{1}{2} m{v_0}^2 - \frac{GmM}{R} = 0 \rightarrow R = \frac{2GM}{{v_0}^2}$	**20**
2	$\Delta m(r) = \rho 4\pi r^2 \Delta r = 4\pi\alpha r \Delta r \rightarrow M = 2\pi\alpha R^2$ $(dm(r) = \rho 4\pi r^2 dr = 4\pi\alpha r dr \rightarrow M = 2\pi\alpha R^2)$	**30**
3	$M = \alpha 2\pi R^2 = \alpha 2\pi \frac{4G^2M^2}{{v_0}^4} \rightarrow M = \frac{{v_0}^4}{8\pi\alpha G^2}$	**30**
4	$M = \frac{{v_0}^4}{8\pi\alpha G^2} = \frac{(1.5\times10^4\ ms^{-1})^4}{8\pi(5\times10^{13}\ kgm^{-2})(6.67\times10^{-11}\mathrm{kg^{-1}m^3s^{-2}})^2} = 9.1\times10^{21} kg$	**20**

12. A proton with a kinetic energy of $1GeV$ propagates out from the surface of the Sun towards the Earth. Neglecting the magnetic field of the Sun, calculate the travel time of the proton as seen from the Earth.

	Answer	
1	$E_0 = m_0c^2 = 941\ MeV$	**20**
2	$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{K}{E_0} + 1 = \frac{1000\ MeV}{941\ MeV} + 1 = 2.062$	**20**
3	$1 - \frac{v^2}{c^2} \approx \frac{1}{4} \rightarrow v \approx 0.87c$	**30**
4	$Elapse\ time\ = \frac{Earth - Sun\ distance}{speed} = \frac{1.50 \times 10^{11} m}{0.87 \times (3 \times 10^8)\ m/s}$ $= 9.6\ minutes$	**30**

13. Volcanic activity on Io, whose rotation period synchronizes with its orbital period, was proposed to be the result of tidal heating mainly from Jupiter. The resultant tidal force on a body is the difference in gravitational force experienced by the near and far sides of that body due to another body. Measurements of the surface distortion of Io via satellite radar altimeter mapping indicate that the surface rises and falls by up to 100m during one-half orbit. Only the surface layers will move by this amount. Interior layers within Io will move by a smaller amount, and thus we assume that on average the entire mass of Io is moved through 50m. Assume that Io is considered as two hemispheres each treated as a point mass. Calculate the *average power* of the tidal heating on Io.

Hint: you can use the following approximation $(1+x)^n \approx 1+nx$ for small *x*.

The mass of Io is $m_{Io} = 8.931938 \times 10^{22}\ kg$

The perijove distance is r_{peri} = 420000 km

The apojove distance is r_{apo} = 423400 km

The orbital period of Io is 152853 s

The radius of Io is R_{Io} = 1821.6 km.

Answer and Marking Scheme:

1	**Calculate the tidal force** The tidal force (assuming the point mass of hemisphere is located at $R_{Io}/2$): $$F_{\text{tidal}} = F(r - R_{Io}) - F(r + R_{Io})$$ $$= -\frac{GM_J \frac{m_{IO}}{2}}{\left(r - \frac{R_{Io}}{2}\right)^2} + \frac{GM_J \frac{m_{IO}}{2}}{\left(r + \frac{R_{Io}}{2}\right)^2} = -GM_J \frac{m_{IO}}{2}\left[\frac{1}{\left(r - \frac{R_{Io}}{2}\right)^2} - \frac{1}{\left(r + \frac{R_{Io}}{2}\right)^2}\right]$$ $$= -\frac{GM_J \frac{m_{IO}}{2}}{r^2}\left[\frac{\left(\frac{2R_{Io}}{r}\right)}{1 - \left(\frac{R_{Io}}{r}\right)^2}\right]$$	40

	The term $\left(\frac{R_{Io}}{r}\right)^2$ is too small and it can be ignored , so that $$F_{\text{tidal}} = -\frac{GM_J m_{Io} R_{Io}}{r^3}$$	10
2	**Calculate the difference of the tidal force** The difference in the tidal force experienced at perijove (closest to Jupiter) and apojove (furthest from Jupiter) $$\Delta F = \left\|F_{\text{tidal}}(r_{\text{peri}}) - F_{\text{tidal}}(r_{\text{apo}})\right\|$$ $$= GM_J m_{Io} R_{Io}\left[\frac{1}{r_{\text{peri}}^3} - \frac{1}{r_{\text{apo}}^3}\right] = 6.65\times10^{18}\ N$$	15 15
3	**Calculate the average power of work done** Hence, the average power of the work done on the rock during one-half orbit is $$\bar{P} = \frac{\Delta F\ d}{t_{Io}/2}$$ $$\bar{P} = 4.35\times10^{15} W$$	10 10

Alternate solution

If assume the mass of the hemisphere is located at the center of mass of a solid hemisphere, $3R_{Io}/8$, then

1	**Calculate the tidal force** The tidal force (assuming the point mass of hemisphere is located at $3R_{Io}/8$): $$F_{\text{tidal}} = F(r - R_{Io}) - F(r + R_{Io})$$ $$= -\frac{GM_J \frac{m_{IO}}{2}}{\left(r - \frac{3R_{Io}}{8}\right)^2} + \frac{GM_J \frac{m_{IO}}{2}}{\left(r + \frac{3R_{Io}}{8}\right)^2} = -GM_J\frac{m_{IO}}{2}\left[\frac{1}{\left(r - \frac{3R_{Io}}{8}\right)^2} - \frac{1}{\left(r + \frac{3R_{Io}}{8}\right)^2}\right]$$ $$= -\frac{GM_J \frac{m_{IO}}{2}}{r^2}\left[\frac{\left(\frac{3R_{Io}}{2r}\right)}{\left[1 - \left(\frac{6R_{Io}}{8r}\right)^2\right]^2}\right]$$	40

	The term $\left(\frac{R_{Io}}{r}\right)^2$ is too small and it can be ignored , so that $$F_{tidal} = -\frac{3GM_J m_{Io} R_{Io}}{4r^3}$$	10
2	**Calculate the difference of the tidal force** The difference in the tidal force experienced at perijove (closest to Jupiter) and apojove (furthest from Jupiter) $$\Delta F = \left\|F_{tidal}(r_{peri}) - F_{tidal}(r_{apo})\right\|$$ $$= \frac{3}{4} G M_J m_{Io} R_{Io} \left[\frac{1}{r_{peri}^3} - \frac{1}{r_{apo}^3}\right] = 4.97 \times 10^{18}\ N$$	15 15
3	**Calculate the average power of work done** Hence, the average power of the work done on the rock during one-half orbit is $$\bar{P} = \frac{\Delta F\ d}{t_{Io}/2}$$ $$\bar{P} = 3.25 \times 10^{15} W$$	10 10

14. Suppose we live in a static and infinitely large universe where the average density of stars is $n = 10^9\ \mathrm{Mpc}^{-3}$ and the average stellar radius is equal to the solar radius. Assume that standard Euclidean geometry holds true in this universe. How far, on average, could you see in any direction before your line of sight strikes a star? Please write your answer in Mpc.

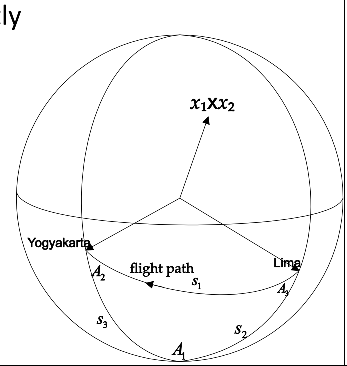
Answer and Marking Scheme:

1.	To decide how far one can see on average in a universe filled with spherical objects of radius *R*, it is simplest to think of a long cylinder along the line of sight. If an object is closer than *R* to the line of sight, then the line of sight intersects its surface. For a distance ℓ, the cylindrical volume which would contain such objects is $\pi R^2 \ell$. If the density of objects is *n*, then the volume that will on average contain one object is defined by $\pi R^2 \ell n = 1$ and the average distance to which we see before our vision is blocked is $$\ell = \frac{1}{\pi R^2 n}$$	**50**
2.	The average radius *R* in Mpc $$R = \frac{6.96 \times 10^8\ \mathrm{m}}{3.086\ \times\ 10^{22}\mathrm{m\ Mpc^{-1}}} =\ 2.3 \times 10^{-14}\ \mathrm{Mpc}$$	**25**
3.	Then $$\ell = \frac{1}{3.14\ (\ 2.3 \times 10^{-14}\ \mathrm{Mpc})^2 (10^9\ \mathrm{Mpc}^{-3}\ )} = 6.3 \times 10^{17}\ \mathrm{Mpc}$$	**25**

15. An airplane was flying from Lima, capital of Peru ($12^{o}2'S$ and $77°1'W$) to Yogyakarta ($7°47'S$ and $110^{o}26'E$), near the venue of the 9th IOAA. The airplane chooses the shortest flight path from Lima to Yogyakarta. Find the latitude of the southernmost point of the flight path.

Answer and Marking Scheme:

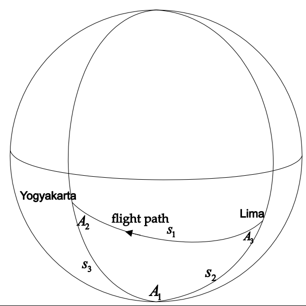
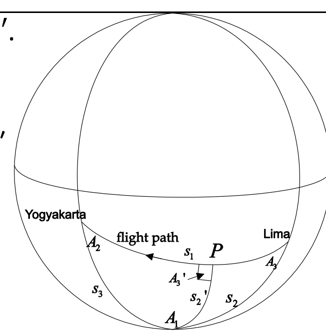
1st version

1	**Determine coordinates of Lima and Yogyakarta.** Assume that the equator is the $z = 0$ plane and the Earth is a unit sphere. The Cartesian coordinates of the Lima is $\boldsymbol{x_1} = (x_1, y_1, z_1)$ $= (\cos(-12^0 2')\cos(282°58'48), \cos(-12°2')\sin(282°58'48), \sin(-12^0 2'))$ $= (0.2197308726, -0.9530236794, -0.2084807188)$ The Cartesian coordinates of the Yogyakarta is $\boldsymbol{x_2} = (x_2, y_2, z_2) = (\cos(-7^0 47')\cos(110^0 26'), \cos(-7°47')\sin(110^0 26'), \sin(-7^0 47'))$ $= (-0.3459009563, 0.9284459898, -0.1354273699)$	**15**
2	**Determine the normal of the plane defined by Lima, Yogyakarta, and center of the Earth.** The normal of the plane containing the two places and the center of the Earth is $\boldsymbol{x_1} \times \boldsymbol{x_2} = (a, b, c) = (0.3226285776, 0.1018712545, -0.12564355210)$	**20**
3	**Find the equation of the (shortest flight) path from Lima to Yogyakarta.** Let the equation of the place be $ax + by + cz = 0$ or equivalently $y = -a'x - c'z$ where $a' = -3.16702272082062$ and $c' = 1.23335628599724$. For the sake of simplicity, from now let us denote a' and c' as a and c, respectively. Intersection of the Earth and the plane is the large circle $x^2 + (-ax - cz)^2 + z^2 = 1$ or equivalently $(1 + a^2)x^2 + 2acxz + (1 + c^2)z^2 - 1 = 0$ It is also the equation for the shortest flight path from Lima to Yogyakarta. 	**25**
4	**Determine the z-value of the southernmost point of the path (at the point, the discriminant of the equation** $(1 + a^2)x^2 + 2acmx + (1 + c^2)m^2 - 1 = 0$ **equals zero)**. The plane $z = m$, it intersects the large circle at at either at one or two points. The number of intersection point at the highest or at the lowest point is one, i.e. when the discriminant is zero. $(1 + a^2)x^2 + 2acmx + (1 + c^2)m^2 - 1 = 0$	**30**

	The discriminant is $D = (2acm)^2 - 4(1 + a^2)[(1 + c^2)m^2 - 1]$ $= 4 + 4a^2 - (4 + 4a^2 + 4c^2)m^2$ Therefore $D = 0$, if $m = \pm\sqrt{\frac{1+a^2}{1+a^2+c^2}}$ $=0.937444937497383$. Then choose $z = -0.937444937497383$	
5	**Determine the latitude of the southernmost point.** Thus, the latitude of the point is $\sin^{-1} z = 1.21521692653748 = 69.6268010651290° = 69°37'36''S$.	**10**

2nd version (using the Law of Cosines for sides)

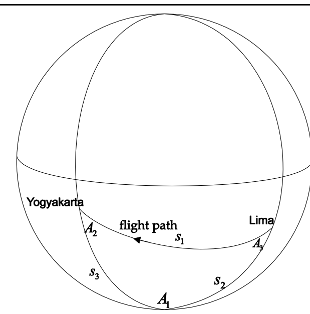
Answer and Marking Scheme:

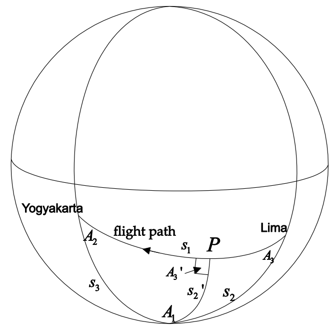
1	**Determine length of the sides of the triangle** $s_2 = 90° - 12°2' = 77°58' = 1.3608\, rad, s_3 = 90° - 7°47' = 82°13' = 1.4350\, rad; A_1 = (180° - 77°1') + (180° - 110°26') = 172°33' = 3.0116\, rad$ Using Law of Cosines for sides $\cos s_1 = (\cos s_2)(\cos s_3) + (\sin s_2)(\sin s_3)(\cos A_1)$ to obtain $s_1 = 2.7724$.rad  	**20**
2	**Determine the angles of the triangle** Then use $\cos s_2 = (\cos s_1)(\cos s_3) + (\sin s_1)(\sin s_3)(\cos A_2)$ to obtain $A_2 = 0.35921$ rad. Similarly, use Law of Cosines $\cos s_3 = (\cos s_1)(\cos s_2) + (\sin s_1)(\sin s_2)(\cos A_3)$ to get that $A_3 = 0.3642$ rad.	**30**
3	**Use Law of Cosines for sides to construct an equation in** $\cos s_2'$. If P is the southernmost point, then the angle A_3 is a right angle, $A_3 = \frac{\pi}{2}$, the Law of $\cos s_3 = \cos s_1' \cos s_2' + \sin s_1' \sin s_2' \cos A_3 = \cos s_1' \cos s_2'$ Thus, $\cos s_1' = \frac{\cos s_3}{\cos s_2'} = \frac{0.13538}{\cos s_2'}.$ Then substitute it to get  	**20**

	$\cos s_2' = \cos s_1' \cos s_3 + \sin s_1' \sin s_3 \cos A_2$ $= \frac{\cos s_3}{\cos s_2'} \cos s_3 + \sqrt{1 - \left(\frac{\cos s_3}{\cos s_2'}\right)^2} \sin s_3 \cos A_2$	
4	**Solve the equation** Let $x = \cos s_2'$. Then $x = \frac{0.018328}{x} + \sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ Multiply both sides by x to get $x^2 = 0.018328 + x\sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ $x^2 - 0.018328 = x\sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ $(x^2 - 0.018328)^2 = (x^2 - 0.018328) \times (0.92756)^2$ $x^4 - 0.89702x^2 + 0.016105 = 0$ which is a quadratic equation of x^2. The roots are ± 0.13538 and ± 0.93739.	20
5	**Determine the Latitude** Thus, $s_2' = \cos^{-1} 0.93739 = 0.35574 \text{ rad} = 20°23'$ or $s_2' = \cos^{-1} 0.13538 = 1.4350 \text{ rad} = 82°13'$ The latitude is $69°37'S$ or $7°47'$S. The second one is impossible because it higher then the latitude of Lima. Thus the answer is $69°37'S$	10

3rd version (using Napier's rule)

Answer and Marking Scheme:

1	**Determine length of the sides of the triangle** $s_2 = 90° - 12°2' = 77°58' = 1.3608\, rad, s_3 = 90° - 7°47' = 82°13' = 1.4350\, rad; A_1 = (180° - 77°1') + (180° - 110°26') = 172°33' = 3.0116\, rad$ Using Law of Cosines for sides $\cos s_1 = (\cos s_2)(\cos s_3) + (\sin s_2)(\sin s_3)(\cos A_1)$ to obtain $s_1 = 2.7724$.rad  	20
2	**Determine the angles of the triangle** Then use	30

	$$\cos s_2 = (\cos s_1)(\cos s_3) + (\sin s_1)(\sin s_3)(\cos A_2)$$ to obtain $A_2 = 0.35921$ rad. Similarly, use Law of Cosines $$\cos s_3 = (\cos s_1)(\cos s_2) + (\sin s_1)(\sin s_2)(\cos A_3)$$ to get that $A_3 = 0.3642$ rad.	
3	**Find the latitude of the point** Let P be the southernmost point on the flight path**.** Since the triangle has a right triangle at P, then use one of Napier's rules $$\sin s_2' = \sin s_3 \sin A_2$$ to get $s_2' = 0.35576\, rad = 20°23'$. Therefore, the latitude of P is $$90° - 20°23' = 69°37' S$$ 	**50**