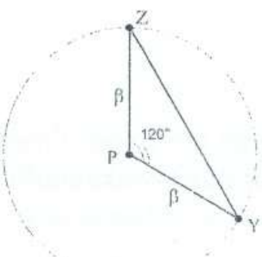
Theoretical round. Sketches for solutions
For preliminary work of Jury ONLY

Note for jury and team leaders. The proposed sketches are not full; the team leaders have to give more detailed explanations for students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

α-1. Observation of a star. The difference is 7^h59^m between the two observations. It is almost exactly 1/3 of a sidereal day (the exact value is $7^h58^m41^s$). During this time the star moves 120° of its daily path. And its zenith distance has changed by $\zeta = 90° - 87°12' = 2°48'$. This value is small, so it is possible to work with the approximation of plane geometry. If β is the angular distance from the star to the pole **P**, we may conclude (see figure) that during this time the star has moved from the zenith point **Z** to point **Y**, and changed the zenith distance by



$$\zeta = 2\beta \times \sin(120°/2) = 1.732 \times \beta,$$

therefore

$$\beta = \zeta / 1.732 = 2°48' / 1.732 = 1°37'.$$

Hence, the declination of the star is equal to

$$\delta = 90° - \beta = 90° - 1°37' = 88°23'.$$

Accordingly, this star culminates at zenith only at latitude $\varphi = \delta$, that is the latitude also equals 88°23'.

But it is not a complete solution. However, the value of 88°23' is only the absolute value of the unknown value. In principle, both the values: +88 ° 23 ' (Northern latitude, the star is near the North Pole) and -88°28' (Southern latitude, the star near the South Pole) can meet the condition of the more general problem. However, according to the condition of our problem the observations were made with the naked eye on 16 June, when polar day appears at such northern latitudes. Therefore, correct answer is only one: -**88°23'** (**88°23' South latitude**).

β-1. Solar radiation.

$$E = 4\pi R^2 \cdot A \cdot \tau, \quad E = mc^2, \quad X = (m/M) \cdot 100\ \%.$$

R – Earth-Sun distance, R = 149 600 000 000 m.
A – Solar constant, A = 1367 W/m^2.
$\tau = 1$ year $= 3.16 \cdot 10^7$ s.
c – speed of light, c = 299 800 000 m/s.
M – mass of the Sun, $M = 1.989 \cdot 10^{-30}$ kg.

$$X = (4\pi R^2 \cdot A \cdot \tau / c^2 / M) \cdot 100\ \%.$$

$$X = (4\pi A\tau(R/c)^2/M) \cdot 100\ \%.$$

$$X = (0.068 \cdot 10^{-12}) \cdot 100\ \% = 6.8 \cdot 10^{-12}\ \%.$$

-2. Planetarium. Image of a star 0^m in projection to the dome has diameter

$$L = \Gamma \cdot l_0,$$

where Γ – magnification coefficient of projecting system, $\Gamma = R/r$, r – distance from the hole in foil to the projecting lens. By the lens formula, $1/R + 1/r = 1/F$, but since $R \ll F$, one may use $\Gamma \approx R/F$.

$$L = \Gamma \cdot l_0 \approx (R/F) \cdot l_0 = 20 \cdot 0.1 \text{ mm} = 2 \text{ mm},$$

Accordingly, the area of the illuminated surface of the dome which gives the star 0^m is equal to

$$S_{m0} = \pi L^2/4 \approx 3.14 \text{ mm}^2.$$

After the slides-foils have been removed the whole dome will be illuminated, i.e. a hemisphere with a radius R = 5 m will shine, the area of the shining surface is

$$S = 2\pi R^2 \approx 157 \text{ m}^2.$$

Thus, the shining area will be

$$K = S / S_{m0} = 2\pi R^2 / \pi L^2/4 = 8(R/L)^2 = 8(F/l_0)^2 = 50\ 000\ 000 \text{ times larger.}$$

In terms of stellar magnitudes the difference is

$$\Delta\mathbf{m} = 2{,}5^m \lg K = 2{,}5^m \lg (8(F/l_0)^2) \approx 19{,}25^m.$$

Thus, the stellar magnitude of the dome will be $0^m - 19.25^m = -19.25^m$. It gives an illuminance 1000 times less, than the illuminance outdoors at clear sunny day, and in 400 times larger than illuminance from full Moon. It corresponds approximately to indoors illuminance at daytime. Of course, it is possible to read without any problem.

We should note that solution does not require the values of limiting stellar magnitude 6^m, number of projecting systems for hemisphere and the diameter of dome of planetarium.

β-2. Planetarium.

2.1. The visible angular size of the "stars" should be less than the resolution of an eye, that is the linear size (diameter) of images of these "stars" on the dome would not exceed $L_0 = \alpha \cdot R$, where α is the resolving power of a human eye in the dark (nearby $50'' \approx 2{,}5 \cdot 10^{-4}$ rad), R is the radius of the hall of the planetarium. In our case $L_0 = \alpha \cdot R \approx 2.5 \cdot 10^{-4} \cdot 5 \text{ m} \approx 1.25$ mm.

The size of the image of one star appearing on the dome with the optical system is defined by two parameters:

a) The first parameter is geometrical, it is defined by the optical increase of the size of a star at its display on the dome. If the size of a star on a slide is $l_0 = 0.1$ mm, the size of the image is calculated by the formula of increase of the objective (lens): $L = \Gamma \cdot l_0 = l_0 \cdot R/r$, where r is the distance from the mentioned hole in a foil to the projecting lens. According to the formula for a lens $1/R + 1/r = 1/F$. In our case the increasing factor should not exceed $\Gamma_0 = L_0/l_0$, , whence we find that the focal length of the system should be not less than

$$F = R/(\Gamma_0+1) = R/(L_0/l_0 + 1) \approx R \cdot l_0/L_0 = R \cdot l_0/(\alpha \cdot R) = l_0/\alpha = 0{,}1 \text{ мм} / 2{,}5 \cdot 10^{-4} \approx 40 \text{ cm}.$$

It is the first condition. The condition is quite feasible.

b) The second parameter is diffraction. It is defined by the size of the Airy disc, the angular size of a divergence of the rays from a dot source (the source being near to focus of a lens) is equal to λ/D, where λ is the wavelength (of order 500 nm or $5 \cdot 10^{-7}$ m), and D is the diameter of the lens of the projecting optical system. Thus the size of the image of a dot source on the dome of radius R is $R \cdot \lambda/D$. So we need a condition.

$$R \cdot \lambda/D \le \alpha \cdot R, \quad D \ge \lambda/\alpha \approx 5 \cdot 10^{-7} \text{ m} / 2{,}5 \cdot 10^{-4} \approx 2 \cdot 10^{-3} \text{ m} = 2 \text{ mm}.$$

So, the second condition: the diameter of the objective should be not less than 2 mm. The condition is quite feasible as well.

By the way, we may notice that this value is easy to point out at once: it is the diameter of a pupil exposure that corresponds to the resolution of a human eye in the conditions of day light.

Formal answer: two parameters:

1. Focal length of the system $F \ge l_0/\alpha \approx 40$ cm.

2. Diameter of the lens of the system $D \geq \lambda/\alpha \approx 2$ mm.

2.2. The image of a star 0^m in projection on the dome has a diameter

$$L = \Gamma \cdot l_0,$$

where Γ is the magnification coefficient of the projecting system, $\Gamma = R/r$, r is the distance from the hole in the foil to the projecting lens. By the lens formula, $1/R + 1/r = 1/F$, but since $R \ll F$, one may use $\Gamma \approx R/F$.

$$L = \Gamma \cdot l_0 \approx (R/F) \cdot l_0 = 12.5 \cdot 0.1 \text{ mm} = 1.25 \text{ mm},$$

Accordingly, the area of the illuminated surface of the dome which gives the star 0^m is equal to

$$S_{m0} = \pi L^2/4 \approx 1.23 \text{ mm}^2.$$

After slides-foils have been removed the whole dome will be illuminated, i.e. a hemisphere of radius $R = 5$ m will shine, the area of shining surface is

$$S = 2\pi R^2 \approx 157 \text{ m}^2.$$

Thus, the shining area will be

$$K = S / S_{m0} = 2\pi R^2 / \pi L^2/4 = 8(R/L)^2 = 8(F/l_0)^2 \approx 130\,000\,000 \text{ times larger.}$$

In terms of stellar magnitudes the difference is

$$\mathbf{\Delta m} = 2{,}5^m \lg K = 2{,}5^m \lg (8(F/l_0)^2) \approx 20{,}25^m.$$

Thus, the stellar magnitude of the dome will be $0^m - 20.25^m = -20.25^m$. It gives an illuminance 400 times less than the illuminance outdoors at clear sunny day, and 1000 times larger than the illuminance from the full Moon. It corresponds approximately to the indoors illuminance at daytime. Of course, it is possible to read without any problem.

We should note that the solution does not require the values of the limiting stellar magnitude 6^m, the number of projecting systems for the hemisphere and the diameter of the dome of the planetarium.

αβ-3. Sunrise on Mars. For an observer on the North Pole of Mars the sunrise happens once a year, in time close to the day of the martian vernal equinox. The solution of the problem is analogous to the solution about the sunrise on the Earth's poles, but we have to remember that some parameters are different on Mars and on Earth:

- α = the visible angular size of Sun.
- v = the speed of the Sun along the planet's ecliptic.
- ε = the inclination of the planet's equator plane to the plane of the planet's ecliptic.
- R = radius of the planet.

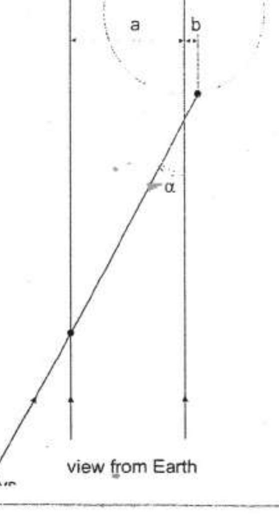
For the observer on the North Pole of Mars the angle between the ecliptic and the horizon is equal to $\varepsilon = 25.2°$. During the sunrise the Sun goes up to the angle equal to its apparent angular diameter, that is

$$\alpha = D/L,$$

where D is the diameter of the Sun, L is the distance from Mars to the Sun. During this time the Sun moves an angular distance $\varphi = \alpha/\sin\varepsilon$ on the ecliptic. Thus, the duration of the sunset equals to $\tau = \varphi/v = \alpha/(v \cdot \sin\varepsilon)$, where v is the speed of the Sun along the martial ecliptic.

Due to the large eccentricity of orbit of Mars both the distance L and speed v may vary significantly ($\pm$ e = 9,3% of mean value). Nevertheless we do not know a priory, and cannot understand from the tables, in which point of orbit Mars is situated at the moments of its vernal equinox. So we shall use mean values, equal to L = 228 million km and $v = 360°/687^d \approx 0.524°$ per day. Thus,

$$\tau = (1.392/228) \text{ rad} \times 180/\pi \text{ °/rad} / (360°/687^d) / \sin 25.2° \approx 1.57 \text{ days or } 37.6 \text{ hours.}$$



β-4. Photo of Jupiter. A picture of Jupiter. The main thing in this problem is to draw the scheme correctly. If **R** is the radius of Jupiter and **a** and **b** are the projections of the distances from the center of Jupiter to the moon (to the left) and to the shadow of the moon (to the right) the distance L from the center of the moon to the center of Jupiter can be calculated using formulas of rectangular triangle:

$$L^2 = [(a+b)/\mathrm{tg}\alpha + R]^2 + a^2,$$

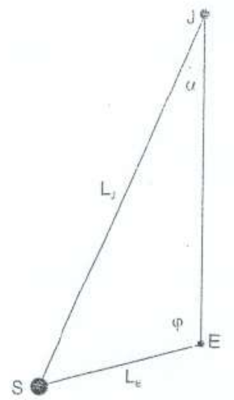
$$L/R = \{[(a+b)/R\mathrm{tg}\alpha + 1]^2 + (a/R)^2\}^{1/2},$$

Where α is the angle between the direction to the Sun (The Sun beams come a little from the left) and the direction to the Earth (from where the picture is made). We can estimate this angle in two ways.

Or we remember that the Capricorn is a zodiac constellation, and the center of this constellation (not zodiac sign, but zodiac constellation!) is passed by the Sun approximately in the end of January - the beginning of February. The angle between the Sun positions on October, 19th and, assume, on February, 3rd, is equal 3.6 months or 108°.

Or to start with that fact that the picture is made on October, 19th (a direct ascension of the Sun 26 days after an equinox we will estimate as $12^h + 26\cdot 4^m = 13^h 44^m$), Jupiter was in the middle of Capricorn constellation (not zodiac sign, but constellation!) (right ascension of this point we will estimate to 21^h), and both the Sun and Jupiter were close to the ecliptic, declination of both points nearby -10°-15°. This data allow us to estimate the angle between Sun and Jupiter at the celestial sphere as about the difference in declinatios, that is, $7^h 16^m$ or 109°.

From the second drawing, knowing this angle φ, and distances to the Sun from Earth L_E and from Jupiter L_J, it is possible to calculate the angle corner α necessary to us:



$$L_J \cdot \sin\alpha = L_E \cdot \sin\varphi, \quad \sin\alpha = (L_E/L_J)\cdot \sin\varphi.$$

$$\sin\alpha \approx (1/5.2)\cdot \sin 108° \approx (1/5.2)\cdot 0.95 \approx 0.183.$$

$$\alpha \approx 10.5°.$$

By measuring the values **R**, **a** and **b** (accordingly 48,5 mm, 59 mm and 18 mm on the author printed picture, on other scaled printed pictures other values may be obtained, but their absolute values are not important, important is only ratios that do not depend on scale) in the photo, and using the formula for a rectangular triangle, we can receive a value of L/R equal to 9,64.

From the given tables it is possible to find that for Io this ratio equals to 5.90, for Europa – 9.39, for Ganimede – 14.97, for Callisto – 26.33. It is evident, that our calculations are made with some accuracy, and the proximity of values 9.64 and 9.39 (1.5 % of accuracy) says that we see the moon Europa in the picture.

αβ-5. Jupiter disappeared. Let us find the most important parameters that we need for the solution. The orbital velocities of Jupiter and its moons are,

Jupiter: $T = 2\pi \cdot 778\,600\,000 / 4\,332.59 / 86400 = 13.069$ km/s,

and analogously:	Io:	17.33 km/s,
	Europa:	13.74 km/s,
	Ganimede:	10.88 km/s,
	Callisto:	8.20 km/s.

After disappearance of Jupiter the velocity of each of its former moons relative to the Sun will remain and will be equal to sum of vectors of the velocity of Jupiter relative to the Sun and the velocity of the moons relative to Jupiter.

5.1. A body can leave the Solar system if its total energy in the field of gravitation of the Sun would exceed zero:

$$-GMm/R + mv^2/2 \geq 0 \quad \text{or} \quad mv^2/2 \geq GMm/R.$$

But also we know that for a circular orbit

$$mv^2/2 = GMm/2R.$$

Therefore, the body can leave the Solar system if its full speed at the distance of the orbit of Jupiter is at least $2^{1/2}$ times larger than the orbital velocity of Jupiter, i.e. $2^{1/2}\cdot 13.07$ km/s = 18.48 km/s. As we see, for any moon such a velocity is possible. At least, it is reached in a configuration of local opposition of the moon (analog to a full moon at Earth). By using a diagram of geometrical summation of velocities it is possible to find a sector for every moon, from which the moon will leave the Solar system (if Jupiter disappears). The necessary drawing to this question should contain these sectors. Student have to find boundaries of the sectors geometrically from the diagram of geometrical summation of velocities or calculate the angles algebraically.

Answer: Io, Europa, Ganimede, Callisto, every moon in the case being in the sector shown at the drawing.

5.2. A body can fall into the Sun in the opposite situation, if the velocity relative to the Sun is strongly reduced, namely reduced to zero the orbital component of the velocity, then it will move to the Sun or from the Sun by a thin ellipse and finally fall into the Sun. Or, as an alternative we may keep the velocity large, but its vector should be directed strictly to the Sun, so that the point of perihelion of the (not closed) orbit appear in the Sun. By the same diagram of geometrical summation of velocities we may find that falling into the Sun is possible only for Io and Europe, and for both of them it would take place only in two points, sometime before the local conjunctions with Sun (analog to new moon at Earth) and sometimes after. The necessary drawing to this question should contain these points. The student has to find the points geometrically from the diagram of geometrical summation of velocities or calculate their positions algebraically.

Answer: Io and Europa in the case being in the points of orbit shown at the drawing.

Note: the drawings will be presented for jury sometime later.