Theoretical round. Sketches for solutions

Note for jury and team leaders. The proposed sketches are not full; the team leaders have to give more detailed explanations for students. But the correct solutions in the students' papers (enough for 8pts) may be shorter.

1α. Earth and Moon. Let us find the distance from the satellite to the Moon.

Let ρ, R and L are angular radii as seen from the satellite. (radii and distances to the Earth (E) and Moon (M)). From the equations

$$R_E = L_E \cdot \sin\rho_E \approx L_E \cdot \rho_E,$$

$$R_M = L_M \cdot \sin\rho_M \approx L_M \cdot \rho_M,$$

one can find

$$L_M / L_E = (R_M/R_E)\cdot(\rho_E/\rho_M)$$

or

$$L_E / L_M = (R_E/R_M)\cdot(\rho_M/\rho_E).$$

The close positions of the Earth and the Moon on the picture shows us that the satellite, the Earth and the Moon are approximately on one line and therefore we can approximately say

$$L_M = L_{EM} + L_E,$$

or

$$L_E = L_M - L_{EM},$$

where L_{EM} is the distance from the Earth to the Moon.

$$L_M\cdot(R_E/R_M)\cdot(\rho_M/\rho_E) = L_M - L_{EM},$$

$$L_M = L_{EM} / (1 - (R_E/R_M)\cdot(\rho_M/\rho_E)).$$

From the picture we see that for the observer on the satellite the phase of the Moon is full moon. If m_0 is the magnitude of the full moon for the observer on the Earth (the distance to the Moon is L_{EM}), the magnitude for the observer on the satellite is

$$m_1 = m_0 + 5\lg(L_M/L_{EM}) = m_0 - 5\lg(1 - (R_E/R_M)\cdot(\rho_M/\rho_E)).$$

We may measure values ρ_E and ρ_M on the picture and take values R_M, R_E and m_0 (and, by the way, L_{EM}) from the table of the Solar System,

$\rho_E/\rho_M \approx 80/13 \approx 6.15$, $L_{EM} = 384\,400$ km, $R_E/R_M = D_E/D_M = 12756 \text{ km} / 3475 \text{ km} = 3.67$, $m_0 = -12.7^m$.

$$m_1 = -12.7^m - 5^m \lg(1 - 3.67\cdot 0.16) = -12.7^m - 5^m \lg(0.41) \approx -12.7^m + 1.9^m \approx -10.8^m.$$

Note 1 for Jury members. Due to difference in printers the scales of the pictures were different in different copies of the pictures. Therefore the values of the lengths may be different from those written here, but the ratios, which are necessary for the solutions, of the lengths should be about the same. Nevertheless the problem is estimational. Measuring of ρ_E/ρ_M and correct calculations have to result to answers from -10.6^m to -11.0^m.

Note 2 for Jury members. By the way, the short approximate solution that also should be regarded as a full solution (the so called solution in integer values):

After using the formula $L_M = L_{EM} / (1 - (R_E/R_M)\cdot(\rho_M/\rho_E))$ (or an analogous formula) and calculations we may find that $L_M \approx 2.5\, L_{EM}$, enlarging distance every 2.5 times (more exactly 2.512) enlarges the magnitude on 2^m, so

$$m_1 = -12.7^m + 2^m \approx -10.7^m.$$

1β. Galaxy. A simple short solution that may be written by somebody is:

$$m_2 = m_1 + 5^m\cdot\lg(L_2/L_1) = 6.88^m + 15^m = 21.88^m.$$

This short solution would be correct, for example, for stellar magnitudes of a star for L_1 = 3 pc and L_2 = 3 kpc.

Note for jury: the above part of solution should render not more than 2 pts.

But the second distance done in the problem is of cosmological scale. So we have to take into account cosmological effects, red shift. According to the Hubble law the galaxy is moving away with a speed of

$$v = H\cdot L,$$

where L is distance to the galaxy, and H is the Hubble constant.

Note for jury: The value of the Hubble constant is not defined. Student may use any value in the interval 50–100 (km/s)/Mpc.

We will use the value H = (70 km/s)/Mpc. For L_1 = 3 Mpc this effect is not important (and the Galaxy can move with velocities larger than 210 km/s due to other reasons). But for L_2 = 3 Gpc the speed of the galaxy is about v_2 = 70 km/s)/Mpc × 3000 Mpc = 210 000 km/s, i.e. 0.7 of the light speed. So we have to take into account that every photon emitted by stars in the galaxy reach us with lower energy. So the visible stellar magnitude is larger due to this diminishing of the energy of all photons.

For every photon $$E_0 = h\nu_0 = hc/\lambda_0$$
and $$E_2 = h\nu_2 = hc/\lambda_2,$$
$$E_2/E_0 = \lambda_0/\lambda_2.$$

Since the ratios λ_0/λ_2 are equal for every photon, the ratio E_2/E_0 for the whole galaxy is equal to this ratio for any photon. Therefore we may conclude that Δm for the whole galaxy may be calculated as $\Delta m = -2^m.5 \lg(E_2/E_0) = -2^m.5 \lg(\lambda_0/\lambda_2)$.

There are two ways of solution presented later: in the classical theory and in the special theory of relativity. The solution according to the special theory of relativity is better but not all students know the formula of this theory *(1 pt difference recommended in the evaluation)*.

According to the classical Doppler effect,
$$\lambda_2 = \lambda_0(1+v_2/c),$$
and $$\Delta m = -2.5^m \lg(\lambda_0/\lambda_2) = 2.5^m \lg(\lambda_2/\lambda_0) = 2.5^m \lg(1+v_2/c).$$
For $v_2 = 0.7$ c $$\Delta m = 2.5^m \lg(1+0.7) = 2.5^m \lg(1.7) = 0.58^m.$$
So $$m_2 = m_1 + 5^m \cdot \lg(L_2/L_1) + 2.5^m \lg(1+HL_2/c) = 22.46^m.$$

According to the Doppler effect in special theory of relativity,
$$\lambda_2 = \lambda_0(1+v_2/c)/(1-(v_2/c)^2)^{1/2},$$
and $$\Delta m = -2.5^m \lg(\lambda_0/\lambda_2) = 2.5^m \lg(\lambda_2/\lambda_0) = 2.5^m \lg[(1+v_2/c)/(1-(v_2/c)^2)^{1/2}].$$
For $v_2 = 0.7$ c $$\Delta m = 2.5^m \lg[(1+0.7)/(1-0.7^2)^{1/2}] = 2.5^m \lg(2.38) = 0.94^m.$$
So $$m_2 = m_1 + 5^m \cdot \lg(L_2/L_1) + 2.5^m \lg(1+HL_2/c) = 22.82^m.$$

2α. Sidereal period. The smallest sidereal period is a period of rotation of a body around a body of the Solar System with the largest mean density inside the sphere of orbit,
$$T = 2\pi r/V = 2\pi r / (G M / r)^{1/2} = 2\pi r / (G\, 4/3\, \pi r^2 \rho)^{1/2} = (3\pi/G\rho)^{1/2}.$$
It is known that the densest bodies in our Solar System are located not far from the Sun. They are planets of the Earth group. We may find from the table of Solar System that the densest bodies are Earth and Mercury. The Earth has a little larger density but the orbit cannot lie near the planet surface. And the orbit around Mercury can lie just near the planet surface. As a result, the mean density inside the sphere of orbit around the Earth is smaller than the mean density inside the sphere of orbit around Mercury that is equal to the density of substance, i.e. 5430 kg/m^3.
$$T\text{ [around Mercury]} = (3\pi/G\rho)^{1/2} = 5100 \text{ sec} = 85 \text{ min}.$$
Note for jury. Some students will take into account only bodies rotating around the Sun. 1 pt difference recommended in the evaluation of a solutionusing such a way, the answer for the minimal T for bodies rotating around the Sun is about 10000 sec = 167 min.

Note for jury. It seems in this part of the problem that some students may find solutions that are unexpected for the jury (and correct!) solutions. Keep attention.

The largest sidereal period is a period of rotation around the Sun of a body with the maximum possible semiaxis. We may propose roughly that the largest possible semiaxis is less than a half distance to the nearest stars, i.e. about 0.6 pc ≈ 120 000 a.u.
$$T\text{ [in years]} = (a\text{ [in a.e.]})^{3/2},$$
$$T\text{ [maximum]} = (120000)^{3/2} \approx 40 \text{ mln. years}.$$
Note for jury. The estimation is very rough. The semiaxis used in solutions may be in the interval 100 000 – 200 000 a.u., so the correct answers are 30 – 90 mln. years.

2β. Space sail. Since the square of the sail is not varying the number of photons falling on the sail is reversly proportional to the square distance to the Sun and force of the light pressure is central, i.e. $F \sim 1/R^2$. It means that the probe will have an orbit according to the Kepler's laws but the "effectivity of gravitation" will be smaller. In other words the probe moves as in gravitational field of a Sun with reduced (from M_0 to M_1) mass. That is the orbit of the probe with the sail is an ellipse with perihelion at the Earths' orbit (distance to the Sun is R_0) and aphelion (since it takes half of rotation around the Sun to reach it) in the asteroid belt (distance to the Sun is R_1). By using the II Kepler law, the law of conservation of energy for the points of perihelion and aphelion, and also the equation for the circular orbit (without sail), one can find *(these procedures are standard and not displayed here)* that the new "effective mass" of the Sun is:
$$M_1 = M_0 \cdot (1 + R_0/R_1)/2 \approx 0.679\, M_0,$$
and the force of gravitation became less by the value
$$(G\Delta M)/GM_0 \approx (1 - R_0/R_1)/2 \approx 0.321.$$
Then we have to find a force of light pressure that leads to such an effect. The energy of 1 photon is $E = mc^2$, its momentum is $p = mc$, i.e. $p = E/c$ for every photon. The force of pressure is $\Delta P/\Delta t$ – the change of the moment of the system sail-probe due to all photons that fall to the sail in one time unit. According to the law of conservation of

momentum $\Delta P/\Delta t$ is equal to the sum of varying of moment of all photons that came to the sail: $\Delta P/\Delta t = \Sigma(\Delta p/\Delta t)$. Since the sail is of the mirror-type the photons are reflected and change the moment to the opposite value, that is the change of the momentum of each of them Δp $\Delta p = 2p_0 = 2E_0/c$. Thus,

$$F_P = \Delta P/\Delta t = (2\cdot\Sigma E / \Delta t) / c.$$

Taking into account that the energy of the photons that fall in a unit of time to the sail remote to a distance R from the Sun is:

$$\Delta E/\Delta t = A\cdot(R_0/R)^2\cdot S,$$

we can write,

$$F_P = 2AS(R_0/R)^2/c.$$

This value is equal to $(1 - R_0/R_1)/2$ part of gravitation, so:

$$(1 - R_0/R_1)\cdot GM_0m/2R^2 = 2AS(R_0/R)^2/c,$$
$$(1 - R_0/R_1)\cdot GM_0m = 4ASR_0^2/c,$$
$$S = (1 - R_0/R_1)\cdot(GM_0/R_0^2)\cdot(mc/4A).$$

It is easy to find (by using equations of the circular orbit of Earth rotation around the Sun) that $GM_0 = 4\pi^2R_0^3/T_0^2$, where T_0 is a sidereal period of the Earth. Thus,

$$S = (1 - R_0/R_1)\cdot(\pi^2R_0/T_0^2)\cdot(mc/A).$$
$$S = 0.643\cdot(\pi^2\cdot 1{,}5\cdot 10^{11}/10^{15})\cdot(10^3\cdot 3\cdot 10^8/1.37\cdot 10^3)\ m^2 \approx 2{,}1\cdot 10^5\ m^2 \approx 0.2\ km^2.$$

Not very much.

Then let us estimate the thickness **d** of this sail. The mass of the sail is $M_\Pi = \rho\cdot d\cdot S$, $d = M_\Pi / \rho\cdot S$. It is reasonable to propose that the sail has mirror characteristics due to the metallic substance. It may give us the interval of possible density of the substance ρ: from $2\cdot 10^3$ kg/m^3 till $2\cdot 10^4$ kg/m^3. Let us take the density of aluminum $\rho = 2.7\cdot 10^3$ kg/m^3 for an estimate. It is evident that the mass of the sail is a part of the mentioned 1 ton of the probe-system. Let us take a half of this value for an estimation, $M_\Pi = 5\cdot 10^2$ kg. So:

$$d = 5\cdot 10^2\ kg / (2.7\cdot 10^3\ kg/m^3\cdot 2\cdot 10^5\ m^2) \approx 0.9\cdot 10^{-6}\ m = 0.9\ mkm.$$

This value can be reached by the modern techniques.

Note for jury: The correct answers are in interval from 0.1 till 1.5 mkm.

3α. Mars set. From the table of Solar System we may find the distance to Mars in the Great opposition

$$S = a_M(1-e_M) - a_E = 57.1\ mln.\ km.$$

and the angular size of the disk of Mars,

in radians $\alpha = D_M / S = 6794 / 57\,100\,000 = 1.19\cdot 10^{-4}$ rad,

in arcminutes $\alpha = 1.19\cdot 10^{-4}$ rad $\cdot$ 3438 arcmin/rad = 0.41 arcmin.

Mars is near the ecliptic, this part of the sky is moving with the angular speed

$$\omega = 360\cdot 60\ arcmin / 86164\ sec \approx 0.25\ arcmin / sec.$$

Taking into account that the latitude of the Simeiz is $\varphi \approx 45°$, the duration of Mars disk set is

$$\tau = \alpha / (\omega\cdot\sin\varphi) \approx 2.3\ sec.$$

Refraction and the height of the observer above the sea level do not affect on this value. They vary the time of the Marsset but not its duration. The speed of moving the Mars along ecliptic (of order of 1" in minute) also almost does not affect on this value since this is too negligible in comparison with the days' moving of the sky (15' in minute).

Note for jury: It is proposed that the students have to know the latitude of the town-host of the Olympiad.

3β. Alcohol in Universe. The problem, of course, estimational. The result may give us only very approximate value. The cloud will be stable and not dissolve in the case the average speed of molecules of alcohol, atomic weight of C_2H_5OH is $\mathbf{m} = 2\cdot 12 + 6\cdot 1 + 1\cdot 16 = 46$, so molar weight $\mu = 0.046$ kg/mol, **R** is universal gas constant and **T** is absolute temperature (i.e. in Kelvins),

$$V_{av} = (3RT/\mu)^{1/2}$$

not exceeds the II cosmic speed (the parabolic speed in other words) for the bodies at the edges of the cloud. **M** is mass of the cloud and **r** is its radius (half of mentioned 463 billion kilometres),

$$V_{II} = (2GM/r)^{1/2}.$$

So

$$3RT/\mu < 2GM/r.$$
$$M = 4/3\ \pi r^3\rho = 4/3\ \pi r^3\ n\cdot m = 4/3\ \pi r^3\ n\cdot\mu/N_A,$$

where **n** is concentration of molecules (for approximation it may be used value $n = 10$ atoms/mm^3 = 10^{10} m^{-3} or, that is more correct, this value should be deleted by the number of atoms in alcohol molecule, i.e. 9, and $n = 1.1\cdot 10^9$ m^{-3}) and $N_A = 6.022\cdot 10^{23}$ is Avogadro number.

$$3RT/\mu < 8/3\ G\ \pi r^2\ n\cdot\mu/N_A.$$
$$T < 8\pi/9\ G\cdot r^2\cdot n\cdot\mu^2 / RN_A.$$

Calculations of $8\pi/9\ G\cdot r^2\cdot n\cdot\mu^2 / RN_A$ give us 42.2 K for n = 10 molecules/mm^3 and 4.7 K for 10 atoms/mm^3, but since it is only rough approximation one cannot use an accuracy of more than 1 digit, i.e. 40 or 5 Kelvins (or, according to past standards "40 degrees of Kelvin" or "5 degrees of Kelvin").

Note for jury: It is proposed that the students have to know by hard the fundamental constants as G, R or N_A and also the atomic weights of H, C and O since these constants and these 3 elements are fundamental.

Photo. At first, the character of illumination of the exhaust trail from an airplane shows us that the photo was aken around the sunrise or sunset.

Second, we see the illuminated area on the right side of moon. For the northern hemisphere it means that the phase of Moon is rising, i.e. the image was shot around sunset. Also it means that the new moon was a short time ago (and s not to be during the next few days). That is, the solar eclipse already took place. (Answer «**Было-Was**».)

Wee see the illuminated area on the moon with relative size b/R about 2 mm / 52 mm (measurement of author).

$$b/R = 1 - \cos\varphi, \qquad \varphi = \arccos(1 - b/R) \approx 15.9°.$$

Or (easier to calculate without calculator):

$$\varphi \approx (2b/R)^{1/2} = (0.077)^{1/2} \approx 0.277 \text{ rad} \approx 15.9°.$$

Taking into account that the relative angular speed of the moon relative to the sun is

$$360° / 29.53^d, \text{ i.e. } \approx 12.2° \text{ per day}$$

we may estimate the period since the new moon (solar eclipse) as

$$15.9° / 12.2° \approx 1.30^d \approx 31^h.$$

One should take into account that the main inaccuracy in the solution is due to the inaccuracy in measuring the size of the illuminate area. While taking it 25% narrower or 25% wider (the real inaccuracy while using the picture) the calculations result to 28^h or 34^h respectively. Nevertheless, since the photo has been shot near sunset, this interval allows us to say that the solar eclipse definitely took place in daytime ([midday plus 2 – minus 4] hours) on the previous day. So it was seen in Simeiz in the case the eclipse zone covered Simeiz. (Answer «**Может быть-May be**».)

Visit. At first we have to find the minimum distance D between two spacecrafts. From the equation

$$R_E^2 + L^2 = (R_E + h)^2$$

where R_E is the Earth's radius, we may find the distance L to the visible horizon from the top of the spacecraft,

$$L = (2R_Eh)^{1/2} \approx 11.3 \text{ km}.$$

The minimum distance between two spacecrafts equals to double of this distance, i.e.

$$D = 2L = (8R_Eh)^{1/2} \approx 22.6 \text{ km}.$$

The navigation devices of the spacecrafts of vituloids should stay at the circle with latitude exactly equal to the declination of the Polar Star, i.e. $\varphi = 89°\ 16'$. The length of this circle is

$$S = 2\pi\sin(90° - \varphi)R_E = 512 \text{ km}.$$

So an estimate of the possible number of vituloids spacecrafts is the largest integer number not exceeding

$$S/D = 2\pi\sin(90° - \varphi)R_E / (8R_Eh)^{1/2} \approx 22.7.$$

Thus,

$$N_V = [S/D] = [2\pi\sin(90° - \varphi)\cdot R_E / (8R_Eh)^{1/2}] = 22.$$

[S/D] means the largest integer number not exceeding S/D.

(For a more accurate solution we have to measure the distance not along the circle but directly. The correcting coefficient would be $\sin\alpha/\alpha$, α is half the angle between meridians of two nearest spacecrafts. The coefficient is too near to 1, $\alpha \approx 8.2°$, $k = 0.997$, so the integer value [S/D] is still equal to 22.)

And the navigation devices of the spacecrafts of crocodilods should stay not at the circle but in whole belt near equator. The length of this belt is about the equator length, i.e. $A = 2\pi R_E = 40076$ km, and the width of the belt is twice the value $\sin(90°-\varphi)R_E$,

$$B = 2\sin(90° - \varphi)R_E = 163.3 \text{ km}.$$

Been placed in triangular net, there may stay $[B/(D(3)^{1/2}/2)]+1$ lines with [A/D] spacecrafts in every line; or [B/D]+1 lines with $[A/(D(3)^{1/2}/2)]$ spacecrafts in every line.

$$[B/D] = [2\sin(90° - \varphi)\cdot R_E / (8R_Eh)^{1/2}] = [\sin(90° - \varphi)\cdot(R_E/2h)^{1/2}] = 7,$$
$$[B/(D(3)^{1/2}/2)] = 8,$$
$$[A/D] = [2\pi R_E / (8R_Eh)^{1/2}] = [\pi(R_E/2h)^{1/2}] = 1774,$$
$$[A/(D(3)^{1/2}/2)] = 2048.$$

Calculations give:

$$([B/(D(3)^{1/2}/2)]+1) \times [A/D] = 9 \times 1774 = 15\,966,$$
$$([B/D]+1) \times [A/(D(3)^{1/2}/2)] = 8 \times 2048 = 16\,384.$$

The second number is larger so it should be taken as the answer.

Note: It is not written in the text at what horizon should be the Polar Star: real or mathematical. But the answer does not depend on it. It may only move the belt but not change (as first approximation) its square and so not change the number of spacecrafts.

e for jury: The important difference between these two situations should be understood by students and accordingly evaluated: cecrafts of vituloids placed in line and spacecrafts of crocodiloids placed in area.

Answers. 22 spacecrafts of vituloids and 16 384 spacecrafts of crocodiloids.